Palestine Journal of Mathematics

Vol 13(1)(2024) , 361–370

Exact solution to a general tumor growth model on time scales

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MSC 2010 Classifications: Primary 26A33; Secondary 34A12.

Keywords and phrases: Dynamic equations on time scales, Cobb-Douglas production, Norton Massagué model.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract. *In the current study, we provide a general Norton-Massagué tumor growth model on time scales that is nonlinear and of first order. The Cobb-Douglas production function is defined on time scales after which it is used to provide the solution for the model. There are specific examples mentioned.*

1 Introduction

Mathematical models are powerful tools that are often used to describe real-world problems, illuminating different scientific and technical disciplines [17, 15, 12, 1]. In the literature, we find several mathematical models of tumor growth that have been proposed, each with its own details and parameters [2]. We mention, among them, a plethora of macroscopic tumor growth models [4]. In 1960, Bertalanffy derived the equation that can be used to describe a tumor growth process

$$\frac{dV}{dt} = aV^{\frac{2}{3}}(t) - bV(t) \tag{1.1}$$

where a and b are proportionality constants. The solution to the equation (1.1) is the following form:

$$V(t) = \left[\frac{a}{b} - (\frac{a}{b} - c)exp(\frac{-bt}{3})\right]^3, where\ c \in \mathbb{R}.$$

In 2005, L. Norton introduced the model.

$$\frac{dV}{dt} = aV^{\frac{d}{3}}(t) - bV(t) \tag{1.2}$$

$$V(t_0) = V_0, \tag{1.3}$$

where $d > 0$. *We point out that the Norton-Massagué equation (1.2)-(1.3) may be solved in closed form, namely,*

$$V(t) = V(0)\left[\frac{a}{b}V(0)^{\frac{d}{3}-1} + e^{(b(\frac{d}{3}-1)t)}(1 - \frac{a}{b}V(0)^{\frac{b}{3}-1})\right]^{\frac{3}{d-3}}.$$

In 2006, L. Norton and J. Massagué [10] introduced the general model

$$\frac{dV(t)}{dt} = aV^{\alpha}(t) - bV(t) \tag{1.4}$$

where $0 < \alpha < 1$ *and a, b are constants of anabolism (growth) and catabolism (death), respectively. In this article, we will focus on solving the general Norton-Massagué model on arbitrary time scales* $\mathbb{T}$

$$V^{\Delta}(t) = a(t)V^{\alpha}(t) - b(t)V(t) \tag{1.5}$$

where $0 < \alpha < 1$ *and* $a(t) > 0, b(t) > 0$. *To our knowledge, this is the first study that considers the resolution of the so-called general Norton-Massagué tumor growth model on time scales. The study of dynamical systems on time scales is today an active field of research. Recently, Martin Bohner et al [7] studied solow models on time scales. Motivated by the work as mentioned above, we study the general tumor growth model on time scales.*

2 Preliminaries

A time scale $\mathbb{T}$ *is an arbitrary nonempty closed subset of the real numbers* $\mathbb{R}$. *For* $t \in \mathbb{T}$, *we define the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by* $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ *and the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by* $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. *Then, one defines the graininess function* $\mu : \mathbb{T} \to [0, +\infty[$ *by* $\mu(t) = \sigma(t) - t$. *If* $\sigma(t) > t$, *then we say that* t *is right-scattered; if* $\rho(t) < t$, *then* t *is left-scattered. Moreover, if* $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$, *then* t *is called right-dense; if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$, *then* t *is called left-dense. If* $\mathbb{T}$ *has a left-scattered maximum* m, *then we define* $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$; *otherwise* $\mathbb{T}^\kappa = \mathbb{T}$. *If* $f : \mathbb{T} \to \mathbb{R}$, *then* $f^\sigma : \mathbb{T} \to \mathbb{R}$ *is given by* $f^\sigma(t) = f(\sigma(t))$ *for all* $t \in \mathbb{T}$.

Definition 2.1 (The Hilger derivative [8]). Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. We define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t (it means, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the Hilger (or delta) derivative of f at t.

We denote the set of rd-continuous functions by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. *Next,* f *is considered regressive since*

$$1 + \mu(t)f(t) \neq 0 \textit{ for all } t \in \mathbb{T}$$

holds. The set of all regressive functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. *We also define the set* $\mathcal{R}^+$ *of all positively regressive elements by*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \quad \textit{for all} \quad t \in \mathbb{T}\}.$$

Let now p *and* $q \in \mathcal{R}$. *We define the circle plus addition* $\oplus$ *on* $\mathcal{R}$ *by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \textit{for all} \quad t \in \mathbb{T},$$

and the circle minus subtraction $\ominus$ *on* $\mathcal{R}$ *by*

$$(p \ominus q)(t) := \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \textit{for all} \quad t \in \mathbb{T}.$$

We put $\mathcal{R}(\alpha) = \begin{cases} \mathcal{R} & \textit{if } \alpha \in \mathbb{N}, \\ \mathcal{R}^+ & \textit{if } \alpha \in \mathbb{R} \setminus \mathbb{N}, \end{cases}$
for $\alpha \in \mathbb{R}$ *and* $p \in \mathcal{R}(\alpha)$. *We define:*

$$(\alpha \odot p)(t) := \alpha p(t) \int_0^1 (1 + \mu(t)p(t)\tau)^{\alpha - 1} d\tau. \tag{2.1}$$

The time scales exponential function $e_p(., t_0)$ *is defined for* $p \in \mathcal{R}$ *and* $t_0 \in \mathbb{T}$ *as the unique solution of the initial value problem*

$$y^\Delta = p(t)y \textit{ and } y(t_0) = 1 \textit{ on } \mathbb{T}.$$

We have

$$e_p(., t_0)e_q(., t_0) = e_{p \oplus q}(., t_0) \quad \textit{and} \quad \frac{e_p(., t_0)}{e_q(., t_0)} = e_{p \ominus q}(., t_0).$$

If $\alpha \in \mathbb{R}$ and $p \in \mathcal{R}(\alpha)$, then

$$e_{\alpha \odot p} = e_p^\alpha.$$

Let $\alpha \in \mathbb{R} \setminus \{1\}$ and $g \in C_{rd}$. We say that see([8])

$$x^\Delta = [q \ominus (\frac{1}{\alpha - 1} \odot (gx^{\alpha-1}))]x, \tag{2.2}$$

is a Bernoulli equation on time scales.

3 General tumor growth model on time scales

In this section, we consider the general Norton-Massagué tumor growth model on time scales

$$V^\Delta(t) = a(t)V^\alpha(t) - b(t)V(t), \tag{3.1}$$

$$V(t_0) = V_0, \tag{3.2}$$

where $0 < \alpha < 1$ and $a, b \in C_{rd}$ with $a(t) > 0, b(t) > 0$ for all $t \in \mathbb{T}$.
Let $f(x) = x^\alpha, w(t) = (\frac{1}{\alpha-1} \odot \frac{bg}{a})(t)$ and $g(t) = (1-\alpha)a(t)$.

Definition 3.1. [7]we define the generalized Cobb-Douglas production function on time scales by $f(x) = x\tilde{f}(x)$. Provided that

$$\tilde{f}(x) := \frac{b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1}))(t)}{a(t)} \tag{3.3}$$

is independent of $t \in \mathbb{T}$.

Lemma 3.2. *If $\mu(t) = 0$ at t then*

$$\frac{b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1})))(t)}{a(t)} = x^{\alpha-1}. \tag{3.4}$$

Proof. Suppose $\mu(t) = 0$ at t then we have

$$\begin{aligned}
\frac{b(t) + \left(w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1}))\right)(t)}{a(t)} &= \frac{b(t) + w(t) - \frac{g(t)x^{\alpha-1}}{\alpha-1}}{a(t)} \\
&= \frac{b(t) + \frac{b(t)g(t)}{(\alpha-1)a(t)} - \frac{g(t)x^{\alpha-1}}{\alpha-1}}{a(t)} \\
&= \frac{b(t) - b(t) + a(t)x^{\alpha-1}}{a(t)} \\
&= x^{\alpha-1}.
\end{aligned}$$

The proof is complete. □

Example 3.3. If $\mathbb{T} = \mathbb{R}$, then $\tilde{f}(x) = x^{\alpha-1}$, and thus (3.3) holds. Hence the Cobb-Douglas production function is defined and equals

$$f(x) = x\tilde{f}(x) = xx^{\alpha-1} = x^\alpha.$$

Lemma 3.4. *Let $t \in \mathbb{T}$. If $\mu(t) > 0$, suppose $A = b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1})))(t)$ then*

$$A = \frac{1}{\mu(t)} \left\{ b(t)\mu(t) - 1 + \left(\frac{1 + (1-\alpha)\mu(t)a(t)x^{\alpha-1}}{1 + (1-\alpha)\mu(t)b(t)} \right)^{\frac{1}{1-\alpha}} \right\} \tag{3.5}$$

Proof. Let $\mu(t) > 0$ at $t \in \mathbb{T}$ then, we have

$$\begin{aligned}
\frac{1}{\alpha-1} \odot (gx^{\alpha-1})(t) &= \frac{1}{\alpha-1}(gx^{\alpha-1})(t) \int_0^1 (1+\mu(t)g(t)x^{\alpha-1}\tau)^{\frac{1}{\alpha-1}-1} d\tau \\
&= \frac{1}{\alpha-1} gx^{\alpha-1} \frac{\left(1+\mu(t)(gx^{\alpha-1})(t)\tau\right)^{\frac{1}{\alpha-1}}]_0^1}{\frac{1}{\alpha-1}\mu(t)gx^{\alpha-1}} \\
&= \frac{1}{\mu(t)} \left((1+\mu(t)gx^{\alpha-1}\right)^{\frac{1}{\alpha-1}} - 1).
\end{aligned}$$

Again, we have

$$\begin{aligned}
w(t) &= \frac{1}{\alpha-1} \odot \frac{(bg)(t)}{a(t)} \\
&= \frac{1}{\alpha-1} \odot (b(t)(1-\alpha)) \\
&= \frac{1}{\alpha-1} b(t)(1-\alpha) \int_0^1 (1+\mu(t)b(t)(1-\alpha)\tau)^{\frac{1}{\alpha-1}-1} d\tau \\
&= b(t)(1-\alpha) \left[\frac{(1+\mu(t)b(t)(1-\alpha)\tau)^{\frac{1}{\alpha-1}}}{\mu(t)b(t)(1-\alpha)} \right]_0^1 \\
&= \frac{1}{\mu(t)} \left[(1+\mu(t)b(t)(1-\alpha))^{\frac{1}{\alpha-1}} - 1 \right];
\end{aligned}$$

hence

$$\begin{aligned}
w \ominus \left(\frac{1}{\alpha-1} \odot (gx^{\alpha-1}) \right) &= \frac{w - \frac{(1+\mu((gx)^{\alpha-1})^{\frac{1}{\alpha-1}}-1}{\mu}}{1+\mu\frac{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}-1}{\mu}} \\
&= \frac{w - (1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}} - 1}{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}} \\
&= \frac{\frac{(1+\mu b(1-\alpha))^{\frac{1}{\alpha-1}}-1}{\mu} - \frac{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}-1}{\mu}}{(1+\mu gx^{\alpha-1})^{\frac{1}{\alpha-1}}} \\
&= \frac{1}{\mu} \left\{ -1 + \left(\frac{1+\mu b(1-\alpha)}{1+\mu(1-\alpha)ax^{\alpha-1}} \right)^{\frac{1}{\alpha-1}} \right\} \\
&= \frac{1}{\mu} \left\{ -1 + \left(\frac{1+\mu(1-\alpha)ax^{\alpha-1}}{1+\mu b(1-\alpha)} \right)^{\frac{1}{1-\alpha}} \right\}.
\end{aligned}$$

Which ends the proof. □

We assume the following additional hypothesis:
$H(1)$*: Let* $a(t) > 0$ *and* $b(t) > 0$*, for all* $t \in \mathbb{T}$ *and suppose* $\tilde{a} = a(t)\mu(t)$*,* $\tilde{b} = b(t)\mu(t)$ *are independents of* $t \in \mathbb{T}$

Theorem 3.5. *Let* $\mu(t) > 0$ *assume* $H(1)$ *hold then, (3.3) hold and the Cobb-Douglas production is defined by:*

$$f(x) = \frac{x}{\tilde{a}} \left\{ \tilde{b} - 1 + \left(\frac{1+(1-\alpha)\tilde{a}x^{\alpha-1}}{1+(1-\alpha)\tilde{b}} \right)^{\frac{1}{1-\alpha}} \right\}.$$

Proof. In view of lemma 3.4, we obtain

$$\begin{aligned}\frac{b(t)+\Big(w\ominus(\frac{1}{\alpha-1}\odot(gx^{\alpha-1}))\Big)(t)}{a(t)} &= \frac{1}{\mu(t)a(t)}\Big\{b(t)\mu(t)-1\\ &\quad+\Big(\frac{(1+(1-\alpha)\mu(t)a(t)x^{\alpha-1})}{1+(1-\alpha)\mu(t)b(t)}\Big)^{\frac{1}{1-\alpha}}\Big\}\\ &= \frac{1}{\tilde a}\Big\{\tilde b-1+\Big(\frac{1+(1-\alpha)\tilde a x^{\alpha-1}}{1+(1-\alpha)\tilde b}\Big)^{\frac{1}{1-\alpha}}\Big\}\end{aligned}$$

is independent of t and therefore equals $\tilde f(x)$ hence, $f(x)=x\tilde f(x)$ □

Example 3.6. Assuming $\mathbb{T}=\mathbb{Z}$ and that a and b are constants then, the cobb-Douglas production function is defined and equals

$$f(x)=\frac{x}{a}\{b-1+(\frac{1+(1-\alpha)ax^{\alpha-1}}{1+(1-\alpha)b})^{\frac{1}{1-\alpha}}\}.$$

Example 3.7. Assuming $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$ and If $\tilde a(t):=(q-1)ta(t)$ and $\tilde b:=(q-1)tb(t)$ are independent of $t\in\mathbb{T}$ then, the cobb-Douglas production function is defined and equals

$$f(x)=\frac{x}{\tilde a}\Big\{\tilde b-1+(\frac{1+(1-\alpha)\tilde a x^{\alpha-1}}{1+(1-\alpha)\tilde b})^{\frac{1}{1-\alpha}}\Big\}.$$

Theorem 3.8. *Suppose $H(1)$ and (3.3) holds. Let f be defined by $f(x)=x\tilde f(x)$ then, (3.1) holds if and only if*

$$V^{\Delta}(t)=\{w\ominus(\frac{1}{\alpha-1}\odot(gV^{\alpha-1}))\}(t)V(t), \tag{3.6}$$

where $g=(1-\alpha)a$ and $w=\frac{1}{\alpha-1}\odot\frac{bg}{a}$.

Remark 3.9. The general Norton-Massagué equation is a special case of a Bernoulli equation on time scales.

Corollary 3.10. *Suppose $(\alpha-1)\in\mathbb{R}-\{0\}$, $w=\frac{1}{\alpha-1}\odot\frac{bg}{a}\in\mathcal{R}(\alpha-1)$ and $g=(1-\alpha)a\in C_{rd}$. Let $V_0\neq 0$ if,*

$$\frac{1}{V_0^{\alpha-1}}+\int_0^t e_w^{\alpha-1}(\tau,0)g(\tau)\Delta\tau>0 \quad \textit{for all} \quad t\in\mathbb{T}$$

then

$$V(t)=\frac{e_w(t,0)}{\Big[\frac{1}{V_0^{\alpha-1}}+\int_0^t e_w^{\alpha-1}(\tau,0)g(\tau)\Delta\tau\Big]^{\frac{1}{\alpha-1}}} \tag{3.7}$$

solves the general Norton-Massagué equation (3.1)-(3.2).

Theorem 3.11. *Assume that $a(t)>0, b(t)>0$ for all $t\in\mathbb{T}$ and $\lambda:=\dfrac{a(t)}{b(t)}$ is independent of $t\in\mathbb{T}$. If we define $p\in\mathcal{R}$ by*

$$p(t)=(1-\alpha)b(t)\textit{for all } t\in\mathbb{T}, \tag{3.8}$$

then the solution of (3.1)-(3.2) is given by

$$V(t)=\left\{\lambda+\frac{V_0^{1-\alpha}-\lambda}{e_p(t,t_0)}\right\}^{\dfrac{1}{1-\alpha}} \quad \textit{for all } t\in\mathbb{T} \tag{3.9}$$

provided that $\lambda+\dfrac{V_0^{1-\alpha}-\lambda}{e_p(t,t_0)}>0$.

Proof. Suppose V solves (3.1) such that $V(t_0) = V_0$. Let $\tilde{x} := V^{\alpha-1}$ by [[8], theorem 2.37], we have:

$$\begin{aligned} \frac{\tilde{x}^\Delta}{\tilde{x}} &= (\alpha-1)\odot\frac{V^\Delta}{V} \\ &= (\alpha-1)\odot\{\omega\ominus[\frac{1}{\alpha-1}\odot(gV^{\alpha-1})]\} \\ &= [(\alpha-1)\odot\omega]\ominus(gV^{\alpha-1}) \\ &= (b(\alpha-1)\ominus(gV^{\alpha-1})); \end{aligned}$$

so

$$\tilde{x}^\Delta = (p\ominus(g\tilde{x}))\tilde{x}.$$

Let $z := \frac{1}{\tilde{x}}$, we get

$$\begin{aligned} z^\Delta &= (\frac{1}{\tilde{x}})^\Delta \\ &= -\frac{(\tilde{x})^\Delta}{\tilde{x}\tilde{x}^\sigma} \\ &= -(p\ominus(g\tilde{x}))z^\sigma \\ &= \frac{g\tilde{x}-p}{1+\mu g\tilde{x}}z^\sigma, \end{aligned}$$

hence

$$(1+\mu g\tilde{x})z^\Delta = g\tilde{x}z^\sigma - pz^\sigma;$$

it means

$$z^\Delta + \mu z^\Delta g\tilde{x} = g\tilde{x}z^\sigma - pz^\sigma.$$

Using the simple useful formula

$$z^\Delta + g\tilde{x}(z^\sigma - z) = g\tilde{x}z^\sigma - pz^\sigma.$$

Seen that,

$$\begin{aligned} g &= (1-\alpha)a \\ &= (1-\alpha)b\lambda \\ &= \lambda p, \end{aligned}$$

we get

$$z^\Delta = -pz^\sigma + g \tag{3.10}$$

the variation of constants in the formula [[9] Theorem 2.74]. The solution to 3.10 is given by

$$\begin{aligned} z(t) &= z_0 e_{\ominus p}(t,t_0) + \int_{t_0}^t g(\tau)e_{\ominus p}(t,\tau)\Delta\tau \\ &= z_0 e_{\ominus p}(t,t_0) + \int_{t_0}^t \lambda p(\tau)e_p(\tau,t)\Delta\tau \\ &= z_0 e_{\ominus p}(t,t_0) + \lambda\int_{t_0}^t p(\tau)e_p(\tau,t)\Delta\tau \\ &= z_0 e_{\ominus p}(t,t_0) + \lambda e_p(\tau,t)\mid_{t_0}^t \\ &= z_0 e_{\ominus p}(t,t_0) + \lambda(1-e_{\ominus p}(t,t_0)). \end{aligned}$$

We have $z_0 = V_0^{1-\alpha}$ as well as $V(t) = \frac{1}{z(t)^{\frac{1}{\alpha-1}}}$ which shows (3.9). Conversely, V given by (3.9) is easily seen to be a solution to (3.1)-(3.2) □

Example 3.12. Let $\mathbb{T} = m\mathbb{Z}$ with $m > 0$, considering the following equation

$$V^{\Delta}(t) = \sqrt{2}V^{\frac{2}{3}}(t) - 5V(t), \quad t \in \mathbb{T} \tag{3.11}$$

$$V(0) = 1 \tag{3.12}$$

here $b(t) = 5, a(t) = \sqrt{2}, \alpha = \frac{2}{3}, g(t) = (1-\alpha)a(t) = (1-\frac{2}{3})\sqrt{2} = \frac{\sqrt{2}}{3}$ and $w(t) = \left(\frac{1}{\alpha-1} \odot \frac{bg}{a}\right)(t)$.
We taking

$$\begin{aligned} w(t) &= \frac{1}{\frac{2}{3}-1} \odot \frac{5\sqrt{2}}{3\sqrt{2}} \\ &= (-3) \odot \frac{5}{3} \\ &= (-3)\frac{5}{3}\int_0^1 (1+\mu(t)\frac{5}{3}\tau)^{-3-1}d\tau \\ &= \frac{1}{m}\left((1+\frac{5}{3}m)^{-3} - 1\right) \end{aligned}$$

also, $1 + \mu(t)w(t) = (1+\frac{5}{3}m)^{-3} > 0$ hence $w \in \mathcal{R}^+$. In addition

$$w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1}))(t) = \frac{1}{m}\frac{((1+\frac{5}{3}m)^{-3}-1) - \frac{\sqrt{2}m}{3}x^{-\frac{1}{3}}}{1+m\frac{\sqrt{2}}{3}x^{-\frac{1}{3}}};$$

so

$$\begin{aligned} \tilde{f}(x) &= \frac{b(t) + (w \ominus (\frac{1}{\alpha-1} \odot (gx^{\alpha-1})))(t)}{a(t)} \\ &= \frac{1}{\sqrt{2}m}\left\{\frac{5m + (1+\frac{5}{3}m)^{-3} - \frac{\sqrt{2}}{3}mx^{-\frac{1}{3}} - 1}{1+\frac{\sqrt{2}}{3}mx^{-\frac{1}{3}}}\right\} \end{aligned}$$

is independent of t. Let $\lambda(t) = \frac{a(t)}{b(t)} = \frac{\sqrt{2}}{5}$ and according to theorem 3.11 the solution of equation (3.11)-(3.12) is

$$V(t) = \left\{\frac{\sqrt{2}}{5} + \frac{1-\frac{\sqrt{2}}{5}}{e_{\frac{5}{3}}(t,0)}\right\}^3$$

seen that

$$\begin{aligned} e_{\frac{5}{3}}(t,o) &= exp\left(\int_0^t \zeta_{\mu(t)}(\frac{5}{3})\Delta t\right) \\ &= exp\sum_{\tau\in[0,t]_{\mathbb{T}}} \frac{1}{m}log(1+\frac{5}{3}m)\tau \\ &= (1+\frac{5}{3}m)^{\frac{t}{2m}(t+1)} \end{aligned}$$

where

$$\begin{aligned} \sum_{\tau\in I}\tau &= m(0+1+2+..+\frac{t}{m}) \\ &= \frac{m}{2}(\frac{t}{m}+1)(\frac{t}{m}) \\ &= \frac{t}{2}(\frac{t}{m}+1) \end{aligned}$$

here $I = \{0, m, 2m, 3m, ..., t\}$, we get

$$V(t) = \left\{ \frac{\sqrt{2}}{5} + \frac{1 - \frac{\sqrt{2}}{5}}{(1 + \frac{5}{3}m)^{\frac{t}{2m}(t+1)}} \right\}^3. \tag{3.13}$$

It is clear that $\tilde{b}(t) = mb(t) = 5m > 0$ and $\tilde{a}(t) = ma(t) = \sqrt{2}m > 0$ are independent of t. On the other hand, according to theorem 3.8 the system(3.11)-(3.12) equivalent to system(3.14)-(3.15)

$$V^{\Delta}(t) = (w \ominus (\frac{1}{\alpha - 1} \odot (gV^{\alpha-1})))(t)N(t), \quad t \in \mathbb{T} \tag{3.14}$$

$$V(0) = 1 \tag{3.15}$$

we have

$$1 + \frac{\sqrt{2}}{3}\int_0^t e_w^{\alpha-1}(\tau, 0)d\tau > 0 \quad \text{for all} \quad t \in \mathbb{T}$$

according to theorem 3.10, the general solution of Norton-Massagué equation (3.11)-(3.12) gives

$$\begin{aligned} V(t) &= \frac{e_w(t,0)}{\left[\frac{1}{V_0^{\alpha-1}} + \int_0^t e_w^{\alpha-1}(\tau,0)g(\tau)d\tau\right]^{\frac{1}{\alpha-1}}} \\ &= \frac{e_w(t,0)}{\left[1 + \frac{\sqrt{2}}{3}\int_0^t e_w^{-3}(\tau,0)d\tau\right]^{-3}} \end{aligned}$$

where

$$e_w(t,0) = exp\left(\int_0^t \zeta_{\mu(\tau)}(w(\tau))\Delta\tau\right) \tag{3.16}$$

since $\mathbb{T} = m\mathbb{Z}$ with $\mu(t) = m$

$$\begin{aligned} e_w(t,0) &= exp\left(\int_0^t \zeta_{\mu(\tau)}(w(\tau))\Delta\tau\right) \\ &= exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m} + 1)log(1 + mw)\right) \end{aligned}$$

where $\zeta_{\mu(\tau)}(w(\tau)) = \frac{1}{\mu(\tau)}log(1 + \mu(\tau)w(\tau))$ and

$$\sum_{\tau\in[0,t]_{\mathbb{T}}} \tau = \frac{t}{2}(\frac{t}{m} + 1)$$

therefore

$$\begin{aligned} V(t) &= \frac{exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m}+1)log(1+mw)\right)}{\left[1 + \frac{\sqrt{2}}{3}\int_0^t exp^{\frac{-1}{3}}(\frac{1}{m}\frac{\tau}{2}(\frac{\tau}{m}+1)log(1+mw))\Delta\tau\right]^{-3}} \\ &= \frac{exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m}+1)log(1+\frac{5}{3}m)\right)}{\left[1 + \frac{\sqrt{2}}{3}\int_0^t exp^{\frac{-1}{3}}(\frac{1}{m}\frac{\tau}{2}(\frac{\tau}{m}+1)log(1+\frac{5}{3}m))\Delta\tau\right]^{-3}} \\ &= \frac{exp\left(\frac{1}{m}\frac{t}{2}(\frac{t}{m}+1)log(1+\frac{5}{3}m)\right)}{\left[1 + \frac{\sqrt{2}}{3}\sum_{\tau=0}^t exp(\frac{\tau}{-6m}(\frac{\tau}{m}+1)log(1+\frac{5}{3}m))\right]^{-3}} \\ &= \frac{(1+\frac{5}{3}m)^{\frac{t}{2m}(\frac{t}{m}+1)}}{\left[1 + \frac{\sqrt{2}}{3}\sum_{\rho=0}^{\frac{t}{m}}(1+\frac{5}{3}m)^{\frac{\rho(\rho+1)}{-6}}\right]^{-3}} \end{aligned}$$

it means

$$V(t) = (1 + \frac{5}{3}m)^{\frac{t}{2m}(\frac{t}{m}+1)} \left[1 + \frac{\sqrt{2}}{3}\sum_{\rho=0}^{\frac{t}{m}}(1 + \frac{5}{3}m)^{\frac{\rho(\rho+1)}{-6}}\right]^3 \quad (3.17)$$

.

Conclusion

A new system of general nonlinear first-order Norton-Massagué tumor growth on time scales has been introduced. Using a resolved Cobb-Douglas production function on time scales to solve the proposed system and obtain the desired results generalize the continuous and discrete spaces. The results presented can be used to assess the solvability of some different classes of problems in the literature.

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Received: May 29,2023.
Accepted: December 10, 2023.