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Characterizations and representations of the core inverse involving idempotents

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Abstract In this paper, we have studied the concept of some results on the core invertibility of products and differences of idempotents. Furthermore, some idempotents conditions for the core inverse of sums, differences and products of idempotents are established.

1 Introduction

The core inverse for a complex matrix was introduced by Baksalary and Trenkler [1] in 2010. Let A be a $n \times n$ complex matrix and $P_{R(A)}$ be the orthogonal projector onto $R(A)$. A $n \times n$ complex matrix $A^{\#}$ satisfying $AA^{\#} = P_{R(A)}$ and $R(A^{\#}) \subseteq R(A)$ is the core inverse of A. A complex matrix has core inverse if and only if it is core invertible, and the core inverse is unique when it exists .

Let H be a complex Hilbert space. Denote by $B(H)$ the Banach algebra of all bounded linear operators on H. $R(T)$ and $N(T)$ represent the range and the null space of T, respectively. The core inverse reduces to the standard inverse, i.e.,$T^{\#} = T^{-1}$. If T is core invertible, then the spectral idempotent T^{π} is given by $T^{\pi} = I - TT^{\#}$. The operator matrix form of T with respect to the space decomposition $H = N(T^{\pi}) \oplus R(T^{\pi})$ is given by $T = T_1 \oplus 0$, where T_1 is invertible.

For an operator $P \in B(H)$ is said to be idempotent if $P^2 = P$. Denote by $\bar{P} = I - P$. $\bar{P}$ is an idempotent if P is an idempotent. If P and Q are idempotents, then we can consider the matrix representations of P and Q associated with the space decomposition $H = R(P) \oplus N(P)$. We have

$$P = I \oplus 0 \ \ and \ \ Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \tag{1.1}$$

Since $Q^2 = Q$, we obtain that

$$\begin{pmatrix} Q_1^2 + Q_2Q_3 & Q_1Q_2 + Q_2Q_4 \\ Q_3Q_1 + Q_4Q_3 & Q_3Q_2 + Q_4^2 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \tag{1.2}$$

The question of the invertibility of $P - Q$, where P and Q are idempotents on a Hilbert space H , is of great interest in operator theory as it is connected with the question of when the space $H = R(P) \oplus R(Q)$ where $R(P)$, $R(Q)$ denotes the Ranges space of P and Q respectively, with the existence of an idemotent operator.

In 2015, Jacek Mieliniczuk [5] investigated C-inverse of a core matrix. Weighted core-EP inverse of an operator between Hilbert spaces was established by Dijana Mosic [3] in 2017. In 2018, Sanzhang Xu et al. [7] developed the concept of new characterization of the CMP inverse of matrices. Three limit representation of the core - EP inverse was studied by Mengmang Zhou et al. [6] in 2018. In 2019, Huihui Zho and Qing - Wen Wang [4] investigated Weighted pseudo core inverses in rings. core invertibility of triangular matrices over a ring was developed by Sanzhang Xu [8] in 2019. In 2019, Yuanyuan Ke et al. [10] extended the core inverse of a product and 2×2 matrices.

Definition 1.1. [1] A matrix A is Hermitian if $A^* = A$, and A is called an idempotent if $A^2 = A$. A Hermitian idempotent is said to be a projection.

Definition 1.2. [1] The core inverse of $A \in M_{n\times n}(\mathbb{C})$ is the matrix $X \in M_{n\times n}(\mathbb{C})$ which satisfies

(1) $AXA = A$ (2) $XAX = X$ (3) $(AX)^* = AX$ (6) $XA^2 = A$ (7) $AX^2 = X$

The matrix X is unique if it exist and is denoted by $A^{\#}$.

Definition 1.3. [1] A matrix A is said to be core EP if $AA^{\#} = A^{\#}A$.

Definition 1.4. [1] A matrix A is said to be core invertible if $AA^{\#} = A^{\#}A = I$.

2 Core invertibility of Some expressions depending on two idempotents

In this section, we have investigated the core invertibility of some expressions depending on two idempotents.

Lemma 2.1. *If A is an idempotent, then $A^{\#} = A$. If A and B are core invertible and $AB = BA$, then $(AB)^{\#} = B^{\#}A^{\#} = A^{\#}B^{\#}$, $A^{\#}B = BA^{\#}$ and $B^{\#}A = AB^{\#}$. If AB and BA are core invertible , then*

$$(AB)^{\#} = A[(BA)^{\#}]^2B. \tag{2.1}$$

Proof. Evidently, if A satisfies $A^2 = A$, then A is core invertible and $A^{\#} = A$. If $A \in B(H)$ is core invertible, then A can be written as $A = A_0 \oplus 0$ with respect to the space decomposition $H = N(A^\pi) \oplus R(A^\pi)$, where A_0 is invertible. Since B is core invertible and $AB = BA$, we deduce that $B = B_0 \oplus B_{00}$, where B_0 and B_{00} are core invertible and $A_0B_0 = B_0A_0$.
In a similar way we conclude that $A_0 = A_1 \oplus A_2$, $B_0 = B_1 \oplus 0$, where $A_i(i = 1, 2), B_1$ are invertible with $A_1B_1 = B_1A_1$.
Now we have,

$$A = A_1 \oplus A_2 \oplus 0, \quad A^{\#} = A_1^{-1} \oplus A_2^{-1} \oplus 0, \quad B = B_1 \oplus 0 \oplus B_{00}, \quad B^{\#} = B_1^{-1} \oplus 0 \oplus B_{00}^{\#}. \tag{2.2}$$

Thus, $(AB)^{\#} = (A_1B_1)^{-1} \oplus 0 \oplus 0 = B^{\#}A^{\#} = A^{\#}B^{\#}$.
And $A^{\#}B = (A_1^{-1}B_1) \oplus 0 \oplus 0 = (B_1A_1^{-1}) \oplus 0 \oplus 0 = BA^{\#}$
Similarly, $B^{\#}A = AB^{\#}$.
To prove (2.1), Let $X = A[(BA)^{\#}]^2B, \quad A = AB$.
Clearly,
(1) $AXA = ABA[(BA)^{\#}]^2BAB$
$= ABA(BA)^{\#}(BA)^{\#}BAB$
$= ABA(BA)^{\#}B$
$= AB$
$= A$
(2) $XAX = A[(BA)^{\#}]^2BABA[(BA)^{\#}]^2B$
$= A[(BA)^{\#}]^2BABA(BA)^{\#}(BA)^{\#}B$
$= A[(BA)^{\#}]^2B$
$= X$
(3) $(AX)^* = (A(BA)[(BA)^{\#}]^2B)^*$
$= (A(BA)(BA)^{\#}(BA)^{\#}B)^*$
$= (A(BA)A^{\#}B^{\#}(BA)^{\#}B)^*$
$= (ABB^{\#}(BA)^{\#}B)^*$
$= (AA^{\#}B^{\#}B)^*$
$= (I)^* = I$
$= AX$
(6) $XA^2 = A[(BA)^{\#}]^2BABAB$
$= A[(BA)^{\#}]^2BABAB$
$= A(BA)^{\#}(BA)^{\#}BABAB$
$= AB$
$= A$

(7) $AX^2 = ABA[(BA)^{\#}]^2BA[(BA)^{\#}]^2B$
$= ABA(BA)^{\#}(BA)^{\#}BA[(BA)^{\#}]^2B$
$= A[(BA)^{\#}]^2B$
$= X$
Then
$(AB)^3X = (AB)^2ABX = (AB)^2A(BA)^{\#}B = A(BA)^2(BA)^{\#}B = (AB)^2$.
Since, AB is core invertible,
we have $(AB)^{\#} = A[(BA)^{\#}]^2B$. □

Lemma 2.2. *Let $M \in B(H \oplus K)$ have the operator matrix form $M = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$. Then M is core invertible if and only if AB and BA are core invertible and $(AB)^{\pi}A = 0$, $B(AB)^{\pi} = 0$. In this case, $M^{\#} = \begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A(BA)^{\#} \\ (BA)^{\#}B & 0 \end{pmatrix}$*

Proof. Assume that M is core invertible. Then $M^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$ is core invertible and therefore, AB and BA are core invertible.
Moreover, the core invertibility of M implies that $R(M) = R(M^2)$,
i.e., $R(AB) = R(A)$ and $R(BA) = R(B)$.
The equality $AB(AB)^{\#}AB = AB$ implies that $AB(AB)^{\#}x = x$ for all vectors $x \in R(AB)$. Thus $AB(AB)^{\#}x = x$ for all $x \in R(A)$ and therefore, $AB(AB)^{\#}A = A$.
Note that, M is core invertible if and only if M^* is core invertible.
Similarly, we obtain $B^*A^*(B^*A^*)^{\#} = B^*$ Since M^* is core invertible
therefore, $B(AB)^{\#}(AB) = B$.
Assume that AB and BA are core invertible and $(AB)^{\pi}A = 0$, $B(AB)^{\pi} = 0$.
Let $X = \begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix}$.
From Lemma 2.1 $B(AB)^{\#}A = (BA)^{\#}BA$ we get

$$MX = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} AB(AB)^{\#} & 0 \\ 0 & B(AB)^{\#}A \end{pmatrix}$$
$$MX = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} 0 & A(BA)^{\#} \\ (BA)^{\#}B & 0 \end{pmatrix}$$
$$= \begin{pmatrix} A(BA)^{\#}B & 0 \\ 0 & BA(BA)^{\#} \end{pmatrix}$$
$$XM = \begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$
$$= \begin{pmatrix} (AB)^{\#}AB & 0 \\ 0 & B(AB)^{\#}A \end{pmatrix}$$
$$XM = \begin{pmatrix} 0 & A(BA)^{\#} \\ (BA)^{\#}B & 0 \end{pmatrix}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$
$$= \begin{pmatrix} A(BA)^{\#}B & 0 \\ 0 & (BA)^{\#}BA \end{pmatrix}$$
(1)$MXM = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} A(BA)^{\#}B & 0 \\ 0 & (BA)^{\#}BA \end{pmatrix}$
$$= \begin{pmatrix} 0 & A(BA)^{\#}BA \\ BA(BA)^{\#}B & 0 \end{pmatrix}$$

$= \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$
$= M.$
(2)$XMX = \begin{pmatrix} A(BA)^{\#}B & 0 \\ 0 & (BA)^{\#}BA \end{pmatrix} \begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix}$
$= \begin{pmatrix} 0 & A(BA)^{\#}B(AB)^{\#}A \\ (BA)^{\#}BAB(AB)^{\#} & 0 \end{pmatrix}$
$= \begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix}$
$= X.$
(3)$(MX)^* = \begin{pmatrix} (AB)^{\#}AB & 0 \\ 0 & B(AB)^{\#}A \end{pmatrix}^*$
$= \begin{pmatrix} ((AB)^{\#}AB)^* & 0 \\ 0 & (B(AB)^{\#}A)^* \end{pmatrix}$
$= \begin{pmatrix} (AB)^{\#}AB & 0 \\ 0 & B(AB)^{\#}A \end{pmatrix}$
$= MX.$
(6)$XM^2 = \begin{pmatrix} A(BA)^{\#}B & 0 \\ 0 & (BA)^{\#}BA \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$
$= \begin{pmatrix} 0 & A(BA)^{\#}BA \\ (BA)^{\#}BAB & 0 \end{pmatrix}$
$= \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$
$= M.$
(7)$MX^2 = \begin{pmatrix} (AB)^{\#}AB & 0 \\ 0 & B(AB)^{\#}A \end{pmatrix} \begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix}$
$= \begin{pmatrix} 0 & (AB)^{\#}AB(AB)^{\#}A \\ B(AB)^{\#}AB(AB)^{\#} & 0 \end{pmatrix}$
$= \begin{pmatrix} 0 & (AB)^{\#}A \\ B(AB)^{\#} & 0 \end{pmatrix}$
$= X.$
Hence M is core invertible and $M^{\#} = X$. □

Theorem 2.3. *Let P and Q be idempotents given by (1.1).*
(i) $P - Q$ is core invertible if and only if $I - Q_1$ and Q_4 are core invertible and

$$(I-Q_1)^{\#}Q_2 = Q_2Q_4^{\#},\ \ Q_2Q_4^{\pi} = (I-Q_1)^{\pi}Q_2 = 0, Q_3(I-Q_1)^{\#} = Q_4^{\#}Q_3,\ \ Q_4^{\pi}Q_3 = Q_3(I-Q_1)^{\pi} = 0. \tag{2.3}$$

In this case,

$$(P-Q)^{\#} = \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix}. \tag{2.4}$$

(ii)$\bar{P} - Q$ is core invertible if and only if Q_1 and $I - Q_4$ are core invertible and

$$(I-Q_4)^{\#}Q_3 = Q_3Q_1^{\#},\ \ Q_3Q_1^{\pi} = (I-Q_4)^{\pi}Q_3 = 0, Q_2(I-Q_4)^{\#} = Q_1^{\#}Q_2,\ \ Q_1^{\pi}Q_2 = Q_2(I-Q_4)^{\pi} = 0. \tag{2.5}$$

In this case,

$$(\bar{P} - Q)^{\#} = \begin{pmatrix} -Q_1^{\#}Q_1 & -Q_1^{\#}Q_2 \\ -(I-Q_4)^{\#}Q_3 & (I-Q_4)^{\#}(I-Q_4) \end{pmatrix} \tag{2.6}$$

(iii)If $P-Q$ and $\bar{P}-Q$ are core invertible, then $PQ-QP$ is core invertible and

$$(PQ-QP)^{\#} = \begin{pmatrix} 0 & -Q_1^{\#}(I-Q_1)^{\#}Q_2 \\ Q_3Q_1^{\#}(I-Q_1)^{\#} & 0 \end{pmatrix}. \tag{2.7}$$

Proof. (i) Assume that $P-Q$ is core invertible. Then $(P-Q)^2$ is core invertible.
Since, $(P-Q)^2 = P+Q-PQ-QP$, by using representation (1.1),
Let P and Q are idempotents then we can consider the matrix representations of P and Q associated with the space decomposition $H = R(P) \oplus N(P)$. We have
$P = I \oplus 0$ and $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$
Since, $Q^2 = Q$, we obtain that
$\begin{pmatrix} Q_1^2+Q_2Q_3 & Q_1Q_2+Q_2Q_4 \\ Q_3Q_1+Q_4Q_3 & Q_3Q_2+Q_4^2 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$
$(P-Q)^2 = P+Q-PQ-QP$
$= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} - \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$
$= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} - \begin{pmatrix} Q_1 & Q_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} Q_1 & 0 \\ Q_3 & 0 \end{pmatrix}$
$= \begin{pmatrix} I+Q_1-Q_1-Q_1 & Q_2-Q_2 \\ Q_3-Q_3 & Q_4-0 \end{pmatrix}$
$= \begin{pmatrix} I-Q_1 & 0 \\ 0 & Q_4 \end{pmatrix}$
we get $(P-Q)^2 = (I-Q_1) \oplus Q_4$.
Hence $I-Q_1, Q_4$ are core invertible and
$(P-Q)^{\#} = [(P-Q)^2]^{\#}(P-Q) = (P-Q)[(P-Q)^2]^{\#}$
$= \begin{pmatrix} I-Q_1 & 0 \\ 0 & Q_4 \end{pmatrix}^{\#} \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix} = \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}\begin{pmatrix} I-Q_1 & 0 \\ 0 & Q_4 \end{pmatrix}^{\#}$
$= \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix} = \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & -Q_2Q_4^{\#} \\ -Q_3(I-Q_1)^{\#} & -Q_4Q_4^{\#} \end{pmatrix}$
Comparing the two sides of the above equation, we have

$$(I-Q_1)^{\#}Q_2 = Q_2Q_4^{\#} and Q_3(I-Q_1)^{\#} = Q_4^{\#}Q_3. \tag{2.8}$$

From $(P-Q)(P-Q)^{\#}(P-Q) = (P-Q)$ and
$(P-Q)(P-Q)^{\#} = (P-Q)^2[(P-Q)^2]^{\#}$
$(P-Q)(P-Q)^{\#} = \begin{pmatrix} (I-Q_1) & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}\begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & -Q_2Q_4^{\#} \\ -Q_3(I-Q_1)^{\#} & -Q_4Q_4^{\#} \end{pmatrix}$
$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$
$A_{11} = (I-Q_1)(I-Q_1)(I-Q_1)^{\#} + Q_2Q_3(I-Q_1)^{\#}$
$= I-Q_1+Q_2Q_4^{\#}Q_3$
$= I-Q_1+(I-Q_1)^{\#}Q_2Q_3$
$= (I-Q_1)(I-Q_1)^{\#}(I-Q_1)+(I-Q_1)^{\#}(I-Q_1)Q_1$
$= (I-Q_1)(I-Q_1)^{\#}$
$A_{12} = -(I-Q_1)Q_2Q_4^{\#} + Q_2Q_4Q_4^{\#}$
$= -(I-Q_1)Q_2Q_4^{\#} + Q_2$
$= -(I-Q_1)(I-Q_1)^{\#}Q_2 + Q_2$
$= -Q_2+Q_2$
$= 0$
$A_{21} = -Q_3(I-Q_1)(I-Q_1)^{\#} + Q_4Q_3(I-Q_1)^{\#}$

$= -Q_3 + Q_4Q_4^{\#}Q_3$
$= 0$
$A_{22} = Q_3Q_2Q_4^{\#} + Q_4Q_4Q_4^{\#}$
$= Q_3(I-Q_1)^{\#}Q_2 + Q_4$
$= Q_3Q_2Q_4^{\#} + Q_4$
$= (I-Q_4)Q_4Q_4^{\#} + Q_4$
$= Q_4Q_4^{\#} - Q_4Q_4Q_4^{\#} + Q_4$
$= Q_4Q_4^{\#}$
$(P-Q)(P-Q)^{\#} = (P-Q)^2[(P-Q)^2]^{\#} = \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}$
From $(P-Q)(P-Q)^{\#}(P-Q) = (P-Q)$ and
$(P-Q)(P-Q)^{\#} = (P-Q)^2[(P-Q)^2]^{\#} = (I-Q_1)(I-Q_1)^{\#} \oplus Q_4Q_4^{\#}$,
we get
$\begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix} \begin{pmatrix} (I-Q_1) & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix} = \begin{pmatrix} (I-Q_1) & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}.$
Hence,
$Q_2 = (I-Q_1)(I-Q_1)^{\#}Q_2,\ Q_3 = Q_4Q_4^{\#}Q_3.$
The first equality of (2.8) yields $(I-Q_1)(I-Q_1)^{\#}Q_2 = (I-Q_1)Q_2Q_4^{\#}$.
Moreover, (1.2) leads to $(I-Q_1)Q_2Q_4^{\#} = Q_2Q_4Q_4^{\#}$.
The second equality of (2.8) yields $Q_4Q_4^{\#}Q_3 = Q_4Q_3(^{\#}I - Q_1)$.
Moreover, (1.2) leads to $Q_4Q_3(I-Q_1)^{\#} = (Q_3 - Q_3Q_1)(I-Q_1)^{\#} = Q_3(I-Q_1)(I-Q_1)^{\#}$.
Assume that $I-Q_1, Q_4$ are core invertible and expressions in (2.3) hold. Let us denote by X the right side of (2.4). Then
$(P-Q)X = \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix} \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}$
because $(I-Q_1)(I-Q_1)^{\#}Q_2 = Q_2 = Q_2Q_4^{\#}Q_4$
and $Q_3(I-Q_1)(I-Q_1)^{\#} = Q_3 = Q_4Q_4^{\#}Q_3$ in view of (2.3),
$I-Q_1 + Q_2Q_4^{\#}Q_3 = I-Q_1 + (I-Q_1)^{\#}Q_2Q_3 = (I-Q_1)(I-Q_1)^{\#}(I-Q_1) + (I-Q_1)^{\#}(I-Q_1)Q_1 = (I-Q_1)^{\#}(I-Q_1)$ and $Q_3(I-Q_1)^{\#}Q_2 + Q_4 = Q_3Q_2Q_4^{\#} + Q_4 = Q_4Q_4^{\#}$ in view of (2.3) and (1.2). In a similar way we get
$X(P-Q) = \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix} \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}$
because $(I-Q_1)^{\#}(I-Q_1)Q_2 = Q_2 = Q_2Q_4^{\#}Q_4 = (I-Q_1)^{\#}Q_2Q_4$ and $Q_4^{\#}Q_3(I-Q_1) = Q_3(I-Q_1)^{\#}(I-Q_1) = Q_3 = Q_4Q_4^{\#}Q_3$ in view of (2.3), $I-Q_1 + (I-Q_1)^{\#}Q_2Q_3 = I-Q_1 + (I-Q_1)^{\#}(I-Q_1)Q_1 = (I-Q_1)^{\#}(I-Q_1)$ and $Q_4^{\#}Q_3Q_2 + Q_4 = Q_4^{\#}Q_4(I-Q_1) + Q_4 = Q_4Q_4^{\#}$ in view of (2.3) and (1.2). So
$X(P-Q) = \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}$

$$(P-Q)X = \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix} \tag{2.9}$$

Now,
$(1)(P-Q)X(P-Q) = \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix} \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}$
$= \begin{pmatrix} I-Q_1((I-Q_1)(I-Q_1)^{\#}) + 0 & 0 + (-Q_2)(Q_4Q_4^{\#}) \\ -Q_3((I-Q_1)(I-Q_1)^{\#}) + 0 & 0 + (-Q_4)(Q_4Q_4^{\#}) \end{pmatrix}$

$= \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}$
$= (P-Q)$

(2) $X(P-Q)X = \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix} \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1)((I-Q_1)(I-Q_1)^{\#})+0 & 0+(-(I-Q_1)^{\#}Q_2)(Q_4Q_4^{\#}) \\ -Q_4^{\#}Q_3((I-Q_1)(I-Q_1)^{\#})+0 & 0+(-Q_4^{\#}Q_4)(Q_4Q_4^{\#}) \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix}$
$= X$

(3) $((P-Q)X)^* = \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}^*$
$= \begin{pmatrix} ((I-Q_1)(I-Q_1)^{\#})^* & 0 \\ 0 & (Q_4Q_4^{\#})^* \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix}$
$= ((P-Q)X)$

(6) $X(P-Q)^2 = \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix} \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#}(I-Q_1)+0 & (I-Q_1)(I-Q_1)^{\#}(-Q_2)+0 \\ 0+Q_4Q_4^{\#}(Q_3) & 0+Q_4Q_4^{\#}(-Q_4) \end{pmatrix}$
$= \begin{pmatrix} I-Q_1 & -Q_2 \\ -Q_3 & -Q_4 \end{pmatrix}$
$= (P-Q)$

(7) $(P-Q)X^2 = \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#} & 0 \\ 0 & Q_4Q_4^{\#} \end{pmatrix} \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)(I-Q_1)^{\#}((I-Q_1)^{\#}(I-Q_1)+0 & (I-Q_1)(I-Q_1)^{\#}(-(I-Q_1)^{\#}Q_2)+0 \\ 0+Q_4Q_4^{\#}(-Q_4^{\#}Q_3) & 0+Q_4Q_4^{\#}(-Q_4^{\#}Q_4) \end{pmatrix}$
$= \begin{pmatrix} (I-Q_1)^{\#}(I-Q_1) & -(I-Q_1)^{\#}Q_2 \\ -Q_4^{\#}Q_3 & -Q_4^{\#}Q_4 \end{pmatrix}$
$= X$

Hence $P-Q$ is core invertible and $(P-Q)^{\#}$ has the representation (2.4)

(ii)Observe that $\bar{P}-Q = I-P-Q = -[P-(I-Q)]$. Thus the core invertibility of $\bar{P}-Q$ is equivalent to the core invertibility of $P-(I-Q)$.
If P and Q are represented as in (1.1), then $I-Q$ has an obvious representation by means of (1.1), and we can apply item (i) in an evident manner.

(iii)If $P-Q$ and $\bar{P}-Q$ are core invertible, by (1.2) and Lemma 2.1,
$(Q_2Q_3)^{\#} = Q_1^{\#}(I-Q_1)^{\#} = (I-Q_1)^{\#}Q_1^{\#}, \quad (Q_3Q_2)^{\#} = Q_4^{\#}(I-Q_4)^{\#} = (I-Q_4)^{\#}Q_4^{\#}$
By (2.3) and (2.5),
$(Q_2Q_3)(Q_2Q_3)^{\#}Q_2 = (I-Q_1)(I-Q_1)^{\#}Q_1Q_1^{\#}Q_2 = (I-Q_1)(I-Q_1)^{\#}Q_2 = Q_2$
and
$(Q_3Q_2)(Q_3Q_2)^{\#}Q_3 = (I-Q_4)(I-Q_4)^{\#}Q_4Q_4^{\#}Q_3 = (I-Q_4)(I-Q_4)^{\#}Q_3 = Q_3.$
Hence, Q_2Q_3, Q_3Q_2 are core invertible, $(Q_2Q_3)^{\pi}Q_2 = 0$ and $(Q_3Q_2)^{\pi}Q_3 = 0$.

$PQ-QP = \begin{pmatrix} 0 & Q_2 \\ -Q_3 & 0 \end{pmatrix}$ is core invertible and (2.7) follows directly by Lemma 2.2. □

3 Conclusion remarks

We some results on the core invertibility of products and differences of idempotents are studied in this paper.

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