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Edge irregular reflexive labeling for corona product of path and star

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Abstract Concerning a graph G in this research, we express a total k-labeling φ, which represents a combination of an edge labeling given by $\varphi_e(x) \to \{1, 2, \ldots, k_e\}$ as well as a vertex labeling given by $\varphi_v(x) \to \{0, 2, \ldots, 2k_v\}$. Here, $\varphi(x) = \varphi_v(x)$ when $x \in V(G)$, while $\varphi(x) = \varphi_e(x)$ when $x \in E(G)$, in which $k = \max\{k_e, 2k_v\}$. Moreover, the total k-labeling φ is known as an edge irregular reflexive k-labeling with respect to G, provided that every edge weights differ. The edge weight represents the sum of the edge label as well as its two end-vertex labels corresponding to it. The smallest value obtained for k provided that such labelling exists refers to *reflexive edge strength* with respect to G. This research examines the edge irregular reflexive labelling with regard to the corona product of a path and star graph, determining its reflexive edge strength.

1 Introduction

All of the graphs that are taken into account in this study have finite, simple, as well as undirected edge and vertex sets, which are denoted by $E(G)$ and $V(G)$, accordingly. By "labeling," we refer to the mapping of graph elements set to a set of labels, which are positive integers. Here, the labeling is known as vertex labeling (or edge labeling), provided that the vertex set (or edge set) is the domain. It is referred to as total labeling provided that the domain is both edge and vertex sets, represented by $V(G) \cup E(G)$. As a consequence of the Pigeonhole principle, in a simple graph is impossible to have a distinct degree for each vertex. However, it is possible in the multigraph.

Graphs are essential to decision-making software, computational linguistics, coding theory, as well as network path determination in computer science. In fifth-generation computers, the parallel processors' interconnection network is represented as a complete graph K_n, taking into account vertices as n processors as well as edges as the link that exists between them [1]. Recent studies have shown that adding an additional time dimension to many graph features and issues, such as graph labelling, causes them to change drastically and become noticeably more challenging. This condition spurs researchers to investigate new graphs, their labeling, and the requirement for developing algorithms for increasingly diverse forms of graphs.

Thus, Chartrand et al. [2] suggested a labeling problem by establishing an edge k-labeling $\delta : E(G) \to \{1, 2, \ldots, k\}$ with respect to a graph G, provided that the vertex weight $w_\delta(x) \neq w_\delta(y)$ for all vertices $x, y \in V(G)$ when $x \neq y$, in which $w_\delta(x) = \sum \delta(xy)$ is employed over all vertices y adjacent to x. The labeling is known as *irregular assignment*. On the other hand, the *irregularity strength* with respect to G, denoted by $s(G)$, resembles the minimum k in which G possesses an irregular assignment employing at most k labels. Therefore, if the set of the number of parallel edges joining adjacent vertices of G is $\{1, 2, \ldots, k\}$, then k is the minimal value such that the vertex degrees of G are distinct. This topic belongs to irregularity in graphs, which nowadays is being studied extensively and for more details, please refer to book [3].

Bača et al. [4] described a graph G has a total k-labeling which given by $\rho : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$. The total k-labeling is defined as an edge irregular total k-labeling if every two different edges xy and $x'y' \in G$, either one meets the condition of $wt(xy) \neq wt(x'y')$, in which $wt(xy) = \rho(x) + \rho(xy) + \rho(y)$. Meanwhile, the total k-labeling is called a vertex irregular total k-labeling if for every two distinct vertices x and $y \in G$, either one meets the condition of $wt(x) \neq wt(y)$, where $wt(x) = \rho(x) + \sum_{xy \in E(G)} \rho(xy)$. The minimum value of k for which the graph G has an edge irregular total k-labeling or a vertex irregular total k-labeling is called the total edge irregularity strength of the graph G, $tes(G)$ or the total vertex irregularity strength of the graph G, $tvs(G)$, respectively. Some other study results can be referred to [5, 6]. Please refer to [7] for a detailed analysis of graph labelings.

Inspired by the natural irregular multigraph problems [2] and irregular total labeling [4], by enabling for the vertex labels to be represented as loops, Tanna et al. [8] subsequently integrated both issues. They observed that (i) the vertex labels are even positive integers, signifying that 2 has been added to the vertex degree, while (ii) vertex label 0 is acceptable in constituting a loopless vertex.

Therefore, they expressed a total k-labeling φ as combining an edge labeling given by $\varphi_e : E(G) \to \{1, 2, \ldots, k_e\}$ as well as a vertex labeling given by $\varphi_v : V(G) \to \{0, 2, \ldots, 2k_v\}$. Here, the labeling of φ represents a total k-labeling of the graph G provided that $\varphi(x) = \varphi_v(x)$ when $x \in V(G)$, while $\varphi(x) = \varphi_e(x)$ when $x \in E(G)$, in which $k = \max\{k_e, 2k_v\}$. Moreover, the total k-labeling φ is known as an edge irregular reflexive k-labeling with respect to G if for every two distinct edges xy, given by $x'y'$ of G, either one meets the requirement of $wt(xy) \neq wt(x'y')$, in which $wt(xy) = \varphi_v(x) + \varphi_e(xy) + \varphi_v(y)$. Furthermore, the smallest value of k in which such labeling occurs is known as *reflexive edge strength* with respect to the graph G, expressed as $res(G)$. Recently, Yoong et al. [9, 10] investigated the edge irregular reflexive labeling of antiprism, corona product of cycle with path as well as convex polytopes, including several plane graphs classes. For more findings on reflexive edge strength of such graphs, refer [11, 12, 13, 14, 15, 16, 17, 18].

Moreover, Bača et al. [19] proposed a conjecture of the reflexive edge strength of the graph G as follows.

Conjecture 1.1. [19] Any graph G having maximum degree $\Delta(G)$ meets the condition:

$$res(G) = \max\left\{ \left\lfloor \frac{\Delta + 2}{2} \right\rfloor, \left\lceil \frac{|E(G)|}{3} \right\rceil + r \right\},$$

in which $r = 1$ when $|E(G)| \equiv 2, 3 \pmod 6$, and $r = 0$ otherwise.

As a continuation of the findings from the previous study, we add more evidence to the Conjecture 1.1 by calculating the precise value with respect to the reflexive edge strength of the corona product of the path and star.

2 Reflexive edge strength of corona product of path and star

The lemma [8] given below is essential.

Lemma 2.1. [8] *For any graph G,*

$$\text{res}(G) \geq \begin{cases} \left\lceil \frac{|E(G)|}{3} \right\rceil, & \text{if } |E(G)| \not\equiv 2, 3 \pmod 6, \\ \left\lceil \frac{|E(G)|}{3} \right\rceil + 1, & \text{if } |E(G)| \equiv 2, 3 \pmod 6. \end{cases}$$

We denote P_n and $K_{1,m}$ be a path and star with n and $m+1$ vertices graph, respectively. The corona product of path and star, expressed by $P_n \odot K_{1,m}$ is gained from a copy of P_n as well as n copies of $K_{1,m}$ by combining the i^{th} vertex with respect to P_n to every vertex in the i^{th} copy of $K_{1,m}$.

Let the vertex as well as edge sets with respect to $P_n \odot K_{1,m}$ be expressed as $V(P_n \odot K_{1,m}) = \{x_i, y_i^j, z_i : 1 \leq i \leq n, 1 \leq j \leq m\}$ as well as $E(P_n \odot K_{1,m}) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i z_i, x_i y_i^j, y_i^j z_i : 1 \leq i \leq n, 1 \leq j \leq m\}$, accordingly. Here, the number of edges of $P_n \odot K_{1,m}$, expressed by $|E(P_n \odot K_{1,m})|$, is $2n(m+1)-1$.

According to Lemma 2.1, we obtain

$$res(P_n \odot K_{1,m}) \geq k = \begin{cases} \left\lceil \frac{2n(m+1)-1}{3} \right\rceil, & \text{if } n(m+1) \not\equiv 2 \,(\mathrm{mod}\, 3), \\ \left\lceil \frac{2n(m+1)-1}{3} \right\rceil + 1, & \text{if } n(m+1) \equiv 2 \,(\mathrm{mod}\, 3). \end{cases} \tag{2.1}$$

Lemma 2.2. *For $n \geq 2$ and $m \equiv 0\,(mod\,3)$,*

$$res(P_n \odot K_{1,m}) = \begin{cases} \left\lceil \frac{2n(m+1)-1}{3} \right\rceil, & \textit{if } n(m+1) \not\equiv 2 \,(\textit{mod}\, 3), \\ \left\lceil \frac{2n(m+1)-1}{3} \right\rceil + 1, & \textit{if } n(m+1) \equiv 2 \,(\textit{mod}\, 3). \end{cases}$$

Proof. By referring to equation (2.1), we now prove that k refers to an upper bound with respect to $res(P_n \odot K_{1,m})$. Moreover, we also define a total k-labeling φ of $P_n \odot K_{1,m}$ given below.

$$\varphi(x_1) = \varphi(y_1^j) = 0, \ \text{ where } 1 \leq j \leq m.$$

$$\varphi(z_1) = \frac{2m}{3}.$$

$$\varphi(x_2) = \varphi(z_2) = \frac{4m}{3} + 2.$$

$$\varphi(y_2^j) = \begin{cases} \frac{2m}{3} - 1 + j, & \text{if odd } j \leq \frac{2m}{3} + 1, \\ \frac{2m}{3} - 2 + j, & \text{if } m = 3, j = 2 \text{ or } m \geq 6, \text{ even } j \leq \frac{2m}{3} + 2, \\ \frac{4m}{3} + 2, & \text{if } m \geq 9, \frac{2m}{3} + 3 \leq j \leq m. \end{cases}$$

For $i \geq 3$ as well as $1 \leq j \leq m$,

$$\varphi(x_i) = \varphi(y_i^j) = \varphi(z_i) = \begin{cases} \frac{2i(m+1)+2}{3}, & \text{if } i \equiv 2 \,(\mathrm{mod}\, 3), \\ \left\lceil \frac{2i(m+1)-2}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

The edge labels are as follows.

$$\varphi(x_i x_{i+1}) = \frac{2m}{3}, \ \text{ where } i = 1, 2.$$

$$\varphi(x_i z_i) = \begin{cases} \frac{4m}{3} + 1, & \text{if } i = 1, \\ \frac{4m}{3} - 1, & \text{if } i = 2. \end{cases}$$

$$\varphi(x_1 y_1^j) = j, \ \text{ where } 1 \leq j \leq m.$$

$$\varphi(y_1^j z_1) = \frac{m}{3} + j, \ \text{ where } 1 \leq j \leq m.$$

$$\varphi(x_2 y_2^j) = \begin{cases} 1, & \text{if odd } j \leq \frac{2m}{3} + 1, \\ 2, & \text{if } m = 3, j = 2 \text{ or } m \geq 6, \text{ even } j \leq \frac{2m}{3} + 2, \\ j - \frac{2m}{3} - 2, & \text{if } m \geq 9, \frac{2m}{3} + 3 \leq j \leq m. \end{cases}$$

$$\varphi(y_2^j z_2) = \begin{cases} m + 1, & \text{if odd } j \leq \frac{2m}{3} + 1, \\ m + 2, & \text{if } m = 3, j = 2 \text{ or } m \geq 6, \text{ even } j \leq \frac{2m}{3} + 2, \\ \frac{m}{3} - 2 + j, & \text{if } m \geq 9, \frac{2m}{3} + 3 \leq j \leq m. \end{cases}$$

For $i \geq 3$ and $1 \leq j \leq m$,

$$\varphi(x_i x_{i+1}) = \frac{2m(i-1)}{3} + 2\left\lceil \frac{i-2}{3} \right\rceil.$$

$$\varphi(x_i z_i) = \frac{2i(m+3)}{3} - 1 - 4\left\lceil \frac{i-1}{3} \right\rceil.$$

$$\varphi(x_i y_i^j) = \frac{2i(m+3)}{3} - 2m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j.$$

$$\varphi(y_i^j z_i) = \frac{2i(m+3)}{3} - m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j.$$

It is evident that the highest vertex label is $k = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil + 1$ when $n \equiv 2 \pmod 3$, on the contrary, $k = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil$. Furthermore, the maximum edge label can be written as $k = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil$ when $n \equiv 1 \pmod 3$, which is greater compared to all vertex labels under the labeling φ. Therefore, labeling φ denotes a total k-labeling of $P_n \odot K_{1,m}$. Subsequently, we demonstrate that the edge weights with respect to $P_n \odot K_{1,m}$ differ under the total k-labeling φ.

$wt_\varphi(x_i x_{i+1}) = \varphi(x_i) + \varphi(x_i x_{i+1}) + \varphi(x_{i+1})$.

$$wt_\varphi(x_1 x_2) = 0 + \frac{2m}{3} + \frac{4m}{3} + 2 = 2m + 2.$$

$$\begin{aligned} wt_\varphi(x_2 x_3) &= \frac{4m}{3} + 2 + \frac{2m}{3} + \left\lceil \frac{2(i+1)(m+1)-2}{3} \right\rceil = 2m + 2 + \frac{6(m+1)}{3} \\ &= 4m + 4. \end{aligned}$$

For $i \equiv 0 \pmod 3$,

$$\begin{aligned} wt_\varphi(x_i x_{i+1}) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2m(i-1)}{3} + 2\left\lceil \frac{i-2}{3} \right\rceil + \left\lceil \frac{2(i+1)(m+1)-2}{3} \right\rceil \\ &= \frac{2i(m+1)}{3} + \frac{2m(i-1)}{3} + \frac{2i}{3} + \frac{2(i+1)(m+1)-2}{3} \\ &= 2im + 2i. \end{aligned}$$

For $i \equiv 1 \pmod 3$,

$$\begin{aligned} wt_\varphi(x_i x_{i+1}) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2m(i-1)}{3} + 2\left\lceil \frac{i-2}{3} \right\rceil + \frac{2(i+1)(m+1)+2}{3} \\ &= \frac{2i(m+1)-2}{3} + \frac{2m(i-1)}{3} + \frac{2(i-1)}{3} + \frac{2(i+1)(m+1)+2}{3} \\ &= 2im + 2i. \end{aligned}$$

For $i \equiv 2 \pmod 3$,

$$\begin{aligned} wt_\varphi(x_i x_{i+1}) &= \frac{2i(m+1)+2}{3} + \frac{2m(i-1)}{3} + 2\left\lceil \frac{i-2}{3} \right\rceil + \left\lceil \frac{2(i+1)(m+1)-2}{3} \right\rceil \\ &= \frac{2i(m+1)+2}{3} + \frac{2m(i-1)}{3} + \frac{2(i-2)}{3} + \frac{2(i+1)(m+1)}{3} \\ &= 2im + 2i. \end{aligned}$$

$wt_\varphi(x_i z_i) = \varphi(x_i) + \varphi(x_i z_i) + \varphi(z_i)$.

$$wt_\varphi(x_1 z_1) = 0 + \frac{4m}{3} + 1 + \frac{2m}{3} = 2m + 1.$$

$$wt_\varphi(x_2z_2) = \frac{4m}{3} + 2 + \frac{4m}{3} - 1 + \frac{4m}{3} + 2 = 4m + 3.$$

For $i \equiv 0 \pmod 3$,

$$\begin{aligned} wt_\varphi(x_iz_i) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2i(m+3)}{3} - 1 - 4\left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{2i(m+1)-2}{3} \right\rceil \\ &= \frac{4i(m+1)}{3} + \frac{2i(m+3)}{3} - 1 - \frac{4i}{3} = 2im + 2i - 1. \end{aligned}$$

For $i \equiv 1 \pmod 3$,

$$\begin{aligned} wt_\varphi(x_iz_i) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2i(m+3)}{3} - 1 - 4\left\lceil \frac{i-1}{3} \right\rceil + \left\lceil \frac{2i(m+1)-2}{3} \right\rceil \\ &= \frac{4i(m+1)-4}{3} + \frac{2i(m+3)}{3} - 1 - \frac{4(i-1)}{3} = 2im + 2i - 1. \end{aligned}$$

For $i \equiv 2 \pmod 3$,

$$\begin{aligned} wt_\varphi(x_iz_i) &= \frac{2i(m+1)+2}{3} + \frac{2i(m+3)}{3} - 1 - 4\left\lceil \frac{i-1}{3} \right\rceil + \frac{2i(m+1)+2}{3} \\ &= \frac{4i(m+1)+4}{3} + \frac{2i(m+3)}{3} - 1 - \frac{4(i+1)}{3} = 2im + 2i - 1. \end{aligned}$$

$wt_\varphi(x_iy_i^j) = \varphi(x_i) + \varphi(x_iy_i^j) + \varphi(y_i^j).$

For $1 \le j \le m$,

$$wt_\varphi(x_1y_1^j) = 0 + j + 0 = j.$$

For odd $j \le \frac{2m}{3} + 1$,

$$wt_\varphi(x_2y_2^j) = \frac{4m}{3} + 2 + 1 + \frac{2m}{3} - 1 + j = 2m + 2 + j.$$

For $m = 3$ and $j = 2$ or $m \ge 6$ and even $j \le \frac{2m}{3} + 2$,

$$wt_\varphi(x_2y_2^j) = \frac{4m}{3} + 2 + 2 + \frac{2m}{3} - 2 + j = 2m + 2 + j.$$

For $m \ge 9$ and $\frac{2m}{3} + 3 \le j \le m$,

$$wt_\varphi(x_2y_2^j) = \frac{4m}{3} + 2 + j - \frac{2m}{3} - 2 + \frac{4m}{3} + 2 = 2m + 2 + j.$$

For $i \equiv 0 \pmod 3$ with $1 \le j \le m$,

$$\begin{aligned} wt_\varphi(x_iy_i^j) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2i(m+3)}{3} - 2m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j + \left\lceil \frac{2i(m+1)-2}{3} \right\rceil \\ &= \frac{4i(m+1)}{3} + \frac{2i(m+3)}{3} - 2m - 2 - \frac{4i}{3} + j \\ &= 2im + 2i - 2m - 2 + j. \end{aligned}$$

For $i \equiv 1 \pmod 3$ with $1 \le j \le m$,

$$\begin{aligned} wt_\varphi(x_iy_i^j) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2i(m+3)}{3} - 2m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j + \left\lceil \frac{2i(m+1)-2}{3} \right\rceil \\ &= \frac{4i(m+1)-4}{3} + \frac{2i(m+3)}{3} - 2m - 2 - \frac{4(i-1)}{3} + j \\ &= 2im + 2i - 2m - 2 + j. \end{aligned}$$

For $i \equiv 2 \pmod 3$ with $1 \leq j \leq m$,

$$\begin{aligned}
wt_\varphi(x_i y_i^j) &= \frac{2i(m+1)+2}{3} + \frac{2i(m+3)}{3} - 2m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j + \frac{2i(m+1)+2}{3} \\
&= \frac{4i(m+1)+4}{3} + \frac{2i(m+3)}{3} - 2m - 2 - \frac{4(i+1)}{3} + j \\
&= 2im + 2i - 2m - 2 + j.
\end{aligned}$$

$wt_\varphi(y_i^j z_i) = \varphi(y_i^j) + \varphi(y_i^j z_i) + \varphi(z_i)$.

For $1 \leq j \leq m$,

$$wt_\varphi(y_1^j z_1) = 0 + \frac{m}{3} + j + \frac{2m}{3} = m + j.$$

For odd $j \leq \frac{2m}{3} + 1$,

$$wt_\varphi(y_2^j z_2) = \frac{2m}{3} - 1 + j + m + 1 + \frac{4m}{3} + 2 = 3m + 2 + j.$$

For $m = 3$ and $j = 2$ or $m \geq 6$ and even $j \leq \frac{2m}{3} + 2$,

$$wt_\varphi(y_2^j z_2) = \frac{2m}{3} - 2 + j + m + 2 + \frac{4m}{3} + 2 = 3m + 2 + j.$$

For $m \geq 9$ and $\frac{2m}{3} + 3 \leq j \leq m$,

$$wt_\varphi(y_2^j z_2) = \frac{4m}{3} + 2 + \frac{m}{3} - 2 + j + \frac{4m}{3} + 2 = 3m + 2 + j.$$

For $i \equiv 0 \pmod 3$ with $1 \leq j \leq m$,

$$\begin{aligned}
wt_\varphi(y_i^j z_i) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2i(m+3)}{3} - m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j + \left\lceil \frac{2i(m+1)-2}{3} \right\rceil \\
&= \frac{4i(m+1)}{3} + \frac{2i(m+3)}{3} - m - 2 - \frac{4i}{3} + j \\
&= 2im + 2i - m - 2 + j.
\end{aligned}$$

For $i \equiv 1 \pmod 3$ with $1 \leq j \leq m$,

$$\begin{aligned}
wt_\varphi(y_i^j z_i) &= \left\lceil \frac{2i(m+1)-2}{3} \right\rceil + \frac{2i(m+3)}{3} - m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j + \left\lceil \frac{2i(m+1)-2}{3} \right\rceil \\
&= \frac{4i(m+1)-4}{3} + \frac{2i(m+3)}{3} - m - 2 - \frac{4(i-1)}{3} + j \\
&= 2im + 2i - m - 2 + j.
\end{aligned}$$

For $i \equiv 2 \pmod 3$ with $1 \leq j \leq m$,

$$\begin{aligned}
wt_\varphi(y_i^j z_i) &= \frac{2i(m+1)+2}{3} + \frac{2i(m+3)}{3} - m - 2 - 4\left\lceil \frac{i-1}{3} \right\rceil + j + \frac{2i(m+1)+2}{3} \\
&= \frac{4i(m+1)+4}{3} + \frac{2i(m+3)}{3} - m - 2 - \frac{4(i+1)}{3} + j \\
&= 2im + 2i - m - 2 + j.
\end{aligned}$$

The edge weights can be easily checked to make sure they are distinct integers in $\{1, 2, \ldots, 2n(m+1) - 1\}$ (refer Table 1). Hence, the total k-labeling φ refers to an edge irregular reflexive k-labeling of $P_n \odot K_{1,m}$. The lemma is hence proven. □

An instance of the obtained edge irregular reflexive 19-labeling with respect to $P_4 \odot K_{1,6}$ is demonstrated in Figure 1.

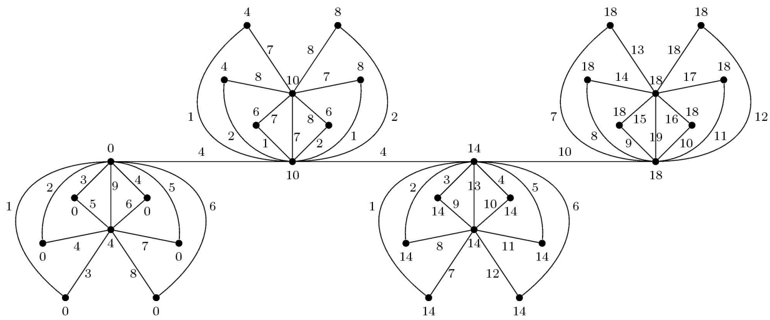

Figure 1. The edge irregular reflexive 19-labeling with respect to $P_4 \odot K_{1,6}$.

Table 1. The summary of all edge weights with respect to $P_n \odot K_{1,m}$, in which $n \geq 2$ and $m \geq 3$.

	Edge weights		$n \geq 2$ and $m \geq 3$
	$wt_\varphi(x_1x_2)$		$2m+2$
	$wt_\varphi(x_2x_3)$		$4m+4$
	$wt_\varphi(x_ix_{i+1})$	where $3 \leq i \leq n-1$	$2im+2i$
	$wt_\varphi(x_1z_1)$		$2m+1$
	$wt_\varphi(x_2z_2)$		$4m+3$
	$wt_\varphi(x_iz_i)$	where $3 \leq i \leq n$	$2im+2i-1$
$1 \leq j \leq m$	$wt_\varphi(x_1y_1^j)$		j
	$wt_\varphi(x_2y_2^j)$		$2m+2+j$
	$wt_\varphi(x_iy_i^j)$	where $3 \leq i \leq n$	$2im+2i-2m-2+j$
	$wt_\varphi(y_1^jz_1)$		$m+j$
	$wt_\varphi(y_2^jz_2)$		$3m+2+j$
	$wt_\varphi(y_i^jz_i)$	where $3 \leq i \leq n$	$2im+2i-m-2+j$

Lemma 2.3. *For $n \geq 2$ and $m \equiv 1\,(mod\,3)$,*

$$res(P_n \odot K_{1,m}) = \begin{cases} \left\lceil \frac{2n(m+1)-1}{3} \right\rceil, & \text{if } n(m+1) \not\equiv 2\,(mod\,3), \\ \left\lceil \frac{2n(m+1)-1}{3} \right\rceil + 1, & \text{if } n(m+1) \equiv 2\,(mod\,3). \end{cases}$$

Proof. As per equation (2.1), we now prove that k resembles an upper bound for $res(P_n \odot K_{1,m})$. We express a total k-labeling φ with respect to $P_n \odot K_{1,m}$ as given below.

$$\varphi(x_1) = \varphi(y_1^j) = 0, \ \text{where } 1 \leq j \leq m.$$

$$\varphi(z_1) = \frac{2(m+2)}{3}.$$

$$\varphi(x_2) = \varphi(z_2) = \frac{2(2m+1)}{3}.$$

$$\varphi(y_2^j) = \begin{cases} \frac{2(m+2)}{3} - 1 + j, & \text{if odd } j \leq \frac{2m-5}{3}, \\ \frac{2(m+2)}{3} - 2 + j, & \text{if even } j \leq \frac{2(m-1)}{3}, \\ \frac{2(2m+1)}{3}, & \text{if } \frac{2m+1}{3} \leq j \leq m. \end{cases}$$

Let $i \geq 3$ and $1 \leq j \leq m$, we now have the following

$$\varphi(x_i) = \varphi(y_i^j) = \varphi(z_i) = \begin{cases} \frac{2i(m+1)+2}{3}, & \text{if } i \equiv 1 \pmod 3, \\ \left\lceil \frac{2i(m+1)-2}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Next, the edge labels are defined as given below.

$$\varphi(x_i x_{i+1}) = \frac{2(m+2)}{3}, \quad \text{where } i = 1, 2.$$

$$\varphi(x_i z_i) = \begin{cases} \frac{4m-1}{3}, & \text{if } i = 1, \\ \frac{4m+5}{3}, & \text{if } i = 2. \end{cases}$$

$$\varphi(x_1 y_1^j) = j, \quad \text{where } 1 \leq j \leq m.$$

$$\varphi(y_1^j z_1) = \frac{m-4}{3} + j, \quad \text{where } 1 \leq j \leq m.$$

$$\varphi(x_2 y_2^j) = \begin{cases} 1, & \text{if odd } j \leq \frac{2m-5}{3}, \\ 2, & \text{if even } j \leq \frac{2(m-1)}{3}, \\ j - \frac{2(m-1)}{3}, & \text{if } \frac{2m+1}{3} \leq j \leq m. \end{cases}$$

$$\varphi(y_2^j z_2) = \begin{cases} m+1, & \text{if odd } j \leq \frac{2m-5}{3}, \\ m+2, & \text{if even } j \leq \frac{2(m-1)}{3}, \\ \frac{m+2}{3} + j, & \text{if } \frac{2m+1}{3} \leq j \leq m. \end{cases}$$

For $i \geq 3$ and $1 \leq j \leq m$,

$$\varphi(x_i x_{i+1}) = \frac{2(m+2)(i-1)}{3} - 2\left\lceil \frac{i-2}{3} \right\rceil.$$

$$\varphi(x_i z_i) = \frac{2i(m-1)}{3} + 3 + 4\left\lceil \frac{i-4}{3} \right\rceil.$$

$$\varphi(x_i y_i^j) = \frac{2(m-1)(i-3)}{3} + 4\left\lceil \frac{i-4}{3} \right\rceil + j.$$

$$\varphi(y_i^j z_i) = \frac{2i(m-1)}{3} - m + 2 + 4\left\lceil \frac{i-4}{3} \right\rceil + j.$$

It is clear that the maximum vertex label is $k = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil + 1$ for $n \equiv 1 \pmod 3$. In comparison, $k = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil$. Furthermore, the maximum edge label is $k = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil$ when $n \equiv 2 \pmod 3$, which is greater compared to all vertex labels under the labeling φ. Therefore, labeling φ represents a total k-labeling with respect to $P_n \odot K_{1,m}$. By employing the approach similar to the proof of Lemma 2.2, we may prove that the edge weights of $P_n \odot K_{1,m}$ differs under the total k-labeling φ, in which the edge weights are in $\{1, 2, \ldots, 2n(m+1) - 1\}$ representing distinct integers. This aligns with the edge weights of Lemma 2.2 as shown in Table 1. Therefore, the total k-labeling φ represents an edge irregular reflexive k-labeling with respect to $P_n \odot K_{1,m}$. The lemma has now been proven. □

Figure 2 illustrates the respective edge irregular reflexive 17-labeling with respect to $P_5 \odot K_{1,4}$.

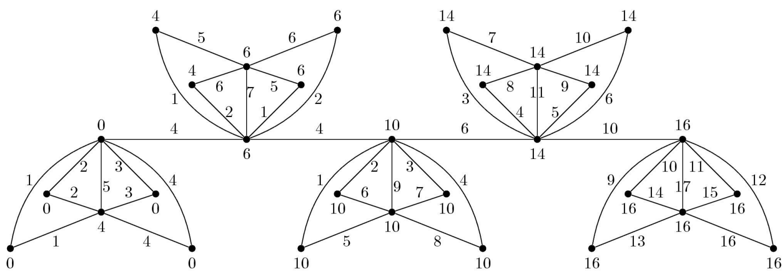

Figure 2. The edge irregular reflexive 17-labeling with respect to $P_5 \odot K_{1,4}$.

Lemma 2.4. *For $n \geq 2$ and $m \equiv 2\,(mod\,3)$, $res(P_n \odot K_{1,m}) = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil$ if $n(m+1) \equiv 0\,(mod\,3)$.*

Proof. By referring to equation (2.1), we prove that k resembles an upper bound for $res(P_n \odot K_{1,m})$. We express a total k-labeling φ with respect to $P_n \odot K_{1,m}$ given below.

$$\varphi(x_1) = \varphi(y_1^j) = 0, \text{ where } 1 \leq j \leq m.$$

$$\varphi(z_1) = \frac{2(m+1)}{3}.$$

$$\varphi(x_2) = \varphi(z_2) = \frac{4(m+1)}{3}.$$

$$\varphi(y_2^j) = \begin{cases} \frac{2(m+1)}{3} - 1 + j, & \text{if odd } j \leq \frac{2m-1}{3}, \\ \frac{2(m+1)}{3} - 2 + j, & \text{if even } j \leq \frac{2(m+1)}{3}, \\ \frac{4(m+1)}{3}, & \text{if } \frac{2m+5}{3} \leq j \leq m. \end{cases}$$

$$\varphi(x_i) = \varphi(y_i^j) = \varphi(z_i) = \frac{2i(m+1)}{3}, \text{ where } i \geq 3, \text{ and } 1 \leq j \leq m.$$

Next, the edges labels are given below.

$$\varphi(x_i x_{i+1}) = \frac{2(m+1)}{3}, \text{ where } i = 1, 2.$$

$$\varphi(x_i z_i) = \frac{4m+1}{3}, \text{ where } i = 1, 2.$$

$$\varphi(x_1 y_1^j) = j, \text{ where } 1 \leq j \leq m.$$

$$\varphi(y_1^j z_1) = \frac{m-2}{3} + j, \text{ where } 1 \leq j \leq m.$$

$$\varphi(x_2 y_2^j) = \begin{cases} 1, & \text{if odd } j \leq \frac{2m-1}{3}, \\ 2, & \text{if even } j \leq \frac{2(m+1)}{3}, \\ j - \frac{2(m+1)}{3}, & \text{if } \frac{2m+5}{3} \leq j \leq m. \end{cases}$$

$$\varphi(y_2^j z_2) = \begin{cases} m+1, & \text{if odd } j \leq \frac{2m-1}{3}, \\ m+2, & \text{if even } j \leq \frac{2(m+1)}{3}, \\ \frac{m-2}{3} + j, & \text{if } \frac{2m+5}{3} \leq j \leq m. \end{cases}$$

For $i \geq 3$ and $1 \leq j \leq m$,

$$\varphi(x_i x_{i+1}) = \frac{2(m+1)(i-1)}{3}.$$

$$\varphi(x_i z_i) = \frac{2i(m+1)}{3} - 1.$$

$$\varphi(x_i y_i^j) = \frac{2i(m+1)}{3} - 2m - 2 + j.$$
$$\varphi(y_i^j z_i) = \frac{2i(m+1)}{3} - m - 2 + j.$$

It is evident that the maximum vertex label is given by $k = \left\lceil \frac{2n(m+1)-1}{3} \right\rceil$, which is greater compared to all edge labels under the labeling φ. Therefore, labeling φ refers to a total k-labeling of $P_n \odot K_{1,m}$. By employing the same approach with the proof of Lemma 2.2, the edge weights with respect to $P_n \odot K_{1,m}$ can be easily verified and are distinct integers in $\{1, 2, \ldots, 2n(m+1)-1\}$ under the total k-labeling φ, which is consistent with the edge weights of Lemmas 2.2 and 2.3, see Table 1. Therefore, the total k-labeling φ denotes an edge irregular reflexive k-labeling of $P_n \odot K_{1,m}$. The lemma is proven. □

The obtained edge irregular reflexive 12-labeling with respect to $P_3 \odot K_{1,5}$ is demonstrated in Figure 3.

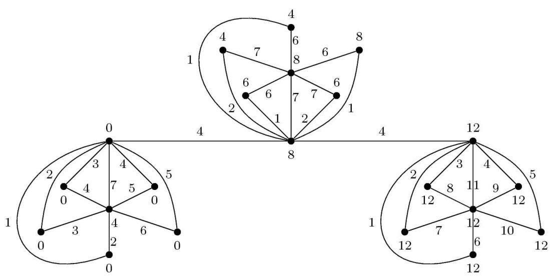

Figure 3. The edge irregular reflexive 12-labeling with respect to $P_3 \odot K_{1,5}$.

Now, we present the main result as given below.

Theorem 2.5. *For any integer $n \geq 2$ as well as $m \geq 3$,*

$$res(P_n \odot K_{1,m}) = \begin{cases} \left\lceil \frac{2n(m+1)-1}{3} \right\rceil, & \text{if } n(m+1) \not\equiv 2 \,(mod\, 3), \\ \left\lceil \frac{2n(m+1)-1}{3} \right\rceil + 1, & \text{if } n(m+1) \equiv 2 \,(mod\, 3). \end{cases}$$

Proof. The evidence follows directly from Lemmas 2.2, 2.3 and 2.4. □

3 Conclusion

This research studies the existence with respect to edge irregular reflexive k-labeling of corona product of path and star. We were able to precisely calculate the reflexive edge strength of this graph, denoted as $P_n \odot K_{1,m}$, where $n \geq 2$ and $m \geq 3$. Moreover, this result provided further support to Conjecture 1.1.

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