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Common neighborhood spectrum and energy of commuting graphs of finite rings

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Abstract The commuting graph of a finite non-commutative ring R with center $Z(R)$ is a simple undirected graph whose vertex set is $R\setminus Z(R)$ and two distinct vertices x, y are adjacent if and only if $xy = yx$. In this paper, we compute the common neighborhood spectrum and energy of commuting graphs of some classes of finite rings. Our computations show that commuting graphs of the rings considered in this paper are CN-integral but not CN-hyperenergetic.

1 Introduction

Let $\mathcal{G}$ be a simple graph whose vertex set is $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$. For $i \neq j$, consider the set $C(v_i, v_j) = \{v_k : k \neq i, j \text{ and } v_k \text{ is adjacent to both } v_i \text{ and } v_j\}$ is called the common neighborhood of v_i and v_j. Let $CN(\mathcal{G})$ be the common neighborhood matrix of $\mathcal{G}$. We write $CN(\mathcal{G})(v_i, v_j)$ to denote the (i, j)th entry of $CN(\mathcal{G})$ and

$$CN(\mathcal{G})(v_i, v_j) = \begin{cases} 0, & \text{if } i = j \\ |C(v_i, v_j)|, & \text{if } i \neq j. \end{cases}$$

The set of all the eigenvalues of $CN(\mathcal{G})$, denoted by CN-spec$(\mathcal{G})$, is called the common neighborhood spectrum (in short CN-spectrum) of $\mathcal{G}$. A graph $\mathcal{G}$ is called CN-integral if CN-spec$(\mathcal{G})$ contains only integers. If $\alpha_1, \alpha_2, \dots, \alpha_k$ are the eigenvalues of $CN(\mathcal{G})$ with multiplicities $a_1, a_2, \dots, a_k$ respectively then we write CN-spec$(\mathcal{G}) = \{\alpha_1^{a_1}, \alpha_2^{a_2}, \dots, \alpha_k^{a_k}\}$. The common neighborhood energy (abbreviated as CN-energy) of a graph $\mathcal{G}$ is given by

$$E_{cn}(\mathcal{G}) = \sum_{i=1}^{k} a_i |\alpha_i|.$$

It is well-known that

$$\text{CN-spec}(K_n) = \left\{(-(n-2))^{n-1}, ((n-1)(n-2))^1\right\}$$

and hence

$$E_{cn}(K_n) = 2(n-1)(n-2), \tag{1.1}$$

where K_n is the complete graph on n vertices. We also have the following useful result.

Theorem 1.1. ([20, Theorem 2.3] and [25, Theorem 2.3]) *Let* $\mathcal{G} = l_1 K_{m_1} \sqcup l_2 K_{m_2} \sqcup \cdots \sqcup l_k K_{m_k}$, *where* $l_i K_{m_i} = K_{m_i} \sqcup \cdots \sqcup K_{m_i}$ *(l_i-times) for* $1 \leq i \leq k$. *Then*

$$\text{CN-spec}(\mathcal{G}) = \Big\{(-(m_1-2))^{l_1(m_1-1)}, ((m_1-1)(m_1-2))^{l_1}, \dots, (-(m_k-2))^{l_k(m_k-1)}, ((m_k-1)(m_k-2))^{l_k}\Big\}$$

and $E_{cn}(\mathcal{G}) = 2\sum_{i=1}^{k} l_i(m_i-1)(m_i-2)$.

A graph $\mathcal{G}$ is called CN-hyperenergetic if $E_{cn}(\mathcal{G}) > E_{cn}(K_{|V(\mathcal{G})|})$. In 2011, the notion of CN-energy of a graph was introduced by Alwardi, Soner and Gutman [3]. Various properties of $E_{cn}(\mathcal{G})$ can be found in [3, 4]. However, CN-spectrum and CN-energy of algebraic graphs are yet to be explored. So far, only commuting graphs and commuting conjugacy class graphs of some finite groups are considered in [20, 25] and [21] to compute their CN-spectrum and CN-energy respectively. However, there are many graphs defined on finite groups (see [10]). Let R be a non-commutative ring with center $Z(R)$. The commuting graph of R, denoted by $\Gamma_c(R)$, is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two distinct vertices x, y are adjacent if and only if $xy = yx$. In recent years, many mathematicians have considered commuting graphs (and generalized commuting graphs) of non-commutative rings and studied various graph theoretic aspects (see [1, 2, 5, 6, 14, 15, 18, 19, 23, 24, 26, 27]). More graphs defined on commutative rings can be found in [7].

In this paper, we compute the CN-spectrum and CN-energy of commuting graphs of some classes of finite rings. We show that the commuting graph of a finite CC-ring is CN-integral. We also show that the commuting graph of a finite ring whose central factor is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, for any prime p, is CN-integral but not CN-hyperenergetic. As a consequence of this result it is shown that commuting graphs of non-commutative rings of orders p^2 and p^3 for any prime p are CN-integral but not CN-hyperenergetic. We shall also show that commuting graphs of non-commutative rings of orders pq, p^2q, p^3q, p^4 and p^5 for any two primes p and q (considered in [27, 28]) are CN-integral but not CN-hyperenergetic.

For any element r of a ring R, the set $C_R(r) = \{s \in R : rs = sr\}$ is called the centralizer of r in R. Let $|\text{Cent}(R)| = |\{C_R(r) : r \in R\}|$, that is the number of distinct centralizers in R. A ring R is called n-centralizer ring if $|\text{Cent}(R)| = n$. This class of rings is studied in [11, 12, 16]. As a consequence of our results, we show that commuting graphs of $4, 5$-centralizer finite rings are CN-integral but not CN-hyperenergetic. Further, we show that the commuting graph of a finite $(p+2)$-centralizer ring of order p^k is CN-integral but not CN-hyperenergetic for any prime p. We conclude this paper by computing CN-spectrum and CN-energy of commuting graphs of finite rings with some specific commuting probabilities. Recall that, the commuting probability of a ring R is the probability that a randomly chosen pair of elements of R commute (see [22]).

A non-commutative ring R is called a CC-ring if all the centralizers of its non-central elements are commutative. We conclude this section with the following two useful theorems regarding CC-rings from [15].

Theorem 1.2. *Let R be a finite CC-ring with distinct centralizers $S_1, S_2, \dots, S_n$ of non-central elements of R. Then $\Gamma_c(R) = \sqcup_{i=1}^{n} K_{|S_i|-|Z(R)|}$.*

Theorem 1.3. *Let R be a finite ring such that the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then $\Gamma_c(R) = (p+1)K_{(p-1)|Z(R)|}$.*

2 CN-spectrum and CN-energy

In [18], Erfanian et al. computed the diameter of the complement of $\Gamma_c(R)$ and showed that the clique number and chromatic number of the complement of $\Gamma_c(R)$ are same for a CC-ring R. Also, the spectrum and genus of $\Gamma_c(R)$ were computed in [15] recently. In the following theorem we compute CN-spectrum and CN-energy of $\Gamma_c(R)$ for a finite CC-ring R.

Theorem 2.1. *Let R be a finite CC-ring with distinct centralizers $S_1, S_2, \dots, S_n$ of non-central elements of R. Then* CN-spec$(\Gamma_c(R))$ *is given by the set*

$$\Big\{(-(|S_1|-|Z(R)|-2))^{|S_1|-|Z(R)|-1}, ((|S_1|-|Z(R)|-1)(|S_1|-|Z(R)|-2))^1, \dots,$$
$$(-(|S_n|-|Z(R)|-2))^{|S_n|-|Z(R)|-1}, ((|S_n|-|Z(R)|-1)(|S_n|-|Z(R)|-2))^1\Big\}$$

and $E_{cn}(\Gamma_c(R)) = 2\sum_{i=1}^{n}(|S_i|-|Z(R)|-1)(|S_i|-|Z(R)|-2)$.

Proof. By Theorem 1.2, we have $\Gamma_c(R) = \sqcup_{i=1}^{n} K_{|S_i|-|Z(R)|}$. Hence, the result follows from Theorem 1.1 considering $k = n, l_i = 1$ and $m_i = |S_i| - |Z(R)|$ for $1 \le i \le n$. □

Corollary 2.2. *Let R be a finite CC-ring and A be any finite commutative ring. Then* CN-spec$(\Gamma_c(R\times A))$ *is given by the set*

$$\Big\{(-((|S_1|-|Z(R)|)|A|-2))^{(|S_1|-|Z(R)|)|A|-1}, \\ (((|S_1|-|Z(R)|)|A|-1)((|S_1|-|Z(R)|)|A|-2))^1,\dots, \\ (-((|S_n|-|Z(R)|)|A|-2))^{(|S_n|-|Z(R)|)|A|-1}, \\ (((|S_n|-|Z(R)|)|A|-1)((|S_n|-|Z(R)|)|A|-2))^1\Big\}$$

and $E_{cn}(\Gamma_c(R\times A)) = 2\sum_{i=1}^{n}((|S_i|-|Z(R)|)|A|-1)((|S_i|-|Z(R)|)|A|-2)$, *where* $S_1,\dots,S_n$ *are the distinct centralizers of non-central elements of* R.

Proof. Note that $Z(R\times A) = Z(R)\times A$ and $S_1\times A, S_2\times A,\dots,S_n\times A$ are the distinct centralizers of non-central elements of $R\times A$, where $S_1,\dots,S_n$ are the distinct centralizers of non-central elements of R. Therefore, if R is a CC-ring then $R\times A$ is also a CC-ring. Hence, the result follows from Theorem 2.1. □

Theorem 2.1 shows that the commuting graph of a finite CC-ring is CN-integral. Further, if R is a finite CC-ring and A is any finite commutative ring then, by Corollary 2.2, the commuting graph of $R\times A$ is also CN-integral. In the next result we consider a particular class of CC-rings and compute the CN-spectrum and CN-energy of its commuting graph.

Theorem 2.3. *Let R be a finite ring such that the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p\times\mathbb{Z}_p$, where p is a prime. Then* CN-spec$(\Gamma_c(R))$ *is given by*

$$\{(-((p-1)|Z(R)|-2))^{(p+1)((p-1)|Z(R)|-1)}, (((p-1)|Z(R)|-1)((p-1)|Z(R)|-2))^{p+1}\}$$

and $E_{cn}(\Gamma_c(R)) = 2(p+1)((p-1)|Z(R)|-1)((p-1)|Z(R)|-2)$.

Proof. By Theorem 1.3, we have $\Gamma_c(R) = (p+1)K_{(p-1)|Z(R)|}$. Hence, the result follows from Theorem 1.1 considering $k=1$, $l_1 = p+1$ and $m_1 = (p-1)|Z(R)|$. □

If R is a non-commutative ring of order p^2 or p^3 for any prime p then $|Z(R)| = 1$ or p respectively. Therefore, $\frac{R}{Z(R)} \cong \mathbb{Z}_p\times\mathbb{Z}_p$ and hence we have the following corollary.

Corollary 2.4. *Let R be a non-commutative ring and p be any prime. Then the following statements hold.*

(a) *If* $|R| = p^2$ *then* CN-spec$(\Gamma_c(R)) = \{(-(p-3))^{(p+1)(p-2)}, ((p-2)(p-3))^{p+1}\}$ *and* $E_{cn}(\Gamma_c(R)) = 2(p+1)(p-2)(p-3)$.

(b) *If* $|R| = p^3$ *then* CN-spec$(\Gamma_c(R)) = \{(-(p^2-p-2))^{(p+1)(p^2-p-1)}, ((p^2-p-1)(p^2-p-2))^{p+1}\}$ *and* $E_{cn}(\Gamma_c(R)) = 2(p+1)(p^2-p-1)(p^2-p-2)$.

Now we consider non-commutative rings of order p^4 and p^5 for any prime p.

Theorem 2.5. *Let* $|R| = p^4$ *and R has unity.*

(a) *If* $|Z(R)| = p$ *then* CN-spec$(\Gamma_c(R)) = \{(-(p^2-p-2))^{(p^2+p+1)(p^2-p-1)}, ((p^2-p-1)(p^2-p-2))^{p^2+p+1}\}$ *or* $\{(-(p^2-p-2))^{l_1(p^2-p-1)}, ((p^2-p-1)(p^2-p-2))^{l_1}, (-(p^3-p-2))^{l_2(p^3-p-1)}, ((p^3-p-1)(p^3-p-2))^{l_2}\}$ *and* $E_{cn}(\Gamma_c(R)) = 2(p^2+p+1)(p^2-p-1)(p^2-p-2)$ *or* $2l_1(p^2-p-1)(p^2-p-2) + 2l_2(p^3-p-1)(p^3-p-2)$, *where* $l_1 + l_2(p+1) = p^2+p+1$.

(b) *If* $|Z(R)| = p^2$ *then* CN-spec$(\Gamma_c(R)) = \{(-(p^3-p^2-2))^{(p+1)(p^3-p^2-1)}, ((p^3-p^2-1)(p^3-p^2-2))^{p+1}\}$ *and* $E_{cn}(\Gamma_c(R)) = 2(p+1)(p^3-p^2-1)(p^3-p^2-2)$.

Proof. The result follows from Theorem 1.1 and [28, Theorem 2.5] recalling that $\Gamma_c(R) = (p^2+p+1)K_{(p^2-p)}$ or $l_1K_{(p^2-p)} \sqcup l_2K_{(p^3-p)}$, where $l_1 + l_2(p+1) = p^2+p+1$, if $|Z(R)| = p$; and $\Gamma_c(R) = (p+1)K_{(p^3-p^2)}$ if $|Z(R)| = p^2$. □

Theorem 2.6. *Let $|R| = p^5$, R has unity and $Z(R)$ is not a field.*

(a) *If $|Z(R)| = p^2$ then* $\text{CN-spec}(\Gamma_c(R)) = \{(-(p^3-p^2-2))^{(p^2+p+1)(p^3-p^2-1)}, ((p^3-p^2-1)(p^3-p^2-2))^{p^2+p+1}\}$ *or* $\{(-(p^3-p^2-2))^{l_1(p^3-p^2-1)}, ((p^3-p^2-1)(p^3-p^2-2))^{l_1}, (-(p^4-p^2-2))^{l_2(p^4-p^2-1)}, ((p^4-p^2-1)(p^4-p^2-2))^{l_2}\}$ *and* $E_{cn}(\Gamma_c(R)) = 2(p^2+p+1)(p^3-p^2-1)(p^3-p^2-2)$ *or* $2l_1(p^3-p^2-1)(p^3-p^2-2)+2l_2(p^4-p^2-1)(p^4-p^2-2)$*, where* $l_1+l_2(p+1) = p^2+p+1$.

(b) *If $|Z(R)| = p^3$ then* $\text{CN-spec}(\Gamma_c(R)) = \{(-(p^4-p^3-2))^{(p+1)(p^4-p^3-1)}, ((p^4-p^3-1)(p^4-p^3-2))^{p+1}\}$ *and* $E_{cn}(\Gamma_c(R)) = 2(p+1)(p^4-p^3-1)(p^4-p^3-2)$.

Proof. The result follows from Theorem 1.1 and [28, Theorem 2.7] recalling that $\Gamma_c(R) = (p^2+p+1)K_{(p^3-p^2)}$ or $l_1K_{(p^3-p^2)} \sqcup l_2K_{(p^4-p^2)}$, where $l_1+l_2(p+1) = p^2+p+1$, if $|Z(R)| = p^2$; and $\Gamma_c(R) = (p+1)K_{(p^4-p^3)}$ if $|Z(R)| = p^3$. □

In the next three theorems p, q denote distinct primes.

Theorem 2.7. *Let R be a non-commutative ring of order p^2q such that $Z(R) = \{0\}$.*

(a) *If $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$ then*

$$\text{CN-spec}(\Gamma_c(R)) = \left\{(-(t-3))^{\frac{(p^2q-1)(t-2)}{(t-1)}}, ((t-2)(t-3))^{\frac{p^2q-1}{(t-1)}}\right\} \text{ and}$$

$$E_{cn}(\Gamma_c(R)) = \frac{2(p^2q-1)(t-2)(t-3)}{(t-1)}.$$

(b) *If $l_1(p-1)+l_2(q-1)+l_3(p^2-1)+l_4(pq-1) = p^2q-1$ then*

$$\begin{aligned}\text{CN-spec}(\Gamma_c(R)) = \Big\{&(-(p-3))^{l_1(p-2)}, ((p-2)(p-3))^{l_1}, (-(q-3))^{l_2(q-2)},\\ &((q-2)(q-3))^{l_2}, (-(p^2-3))^{l_3(p^2-2)}, ((p^2-2)(p^2-3))^{l_3},\\ &(-(pq-3))^{l_4(pq-2)}, ((pq-2)(pq-3))^{l_4}\Big\}\end{aligned}$$

and $E_{cn}(\Gamma_c(R)) = 2l_1(p-2)(p-3)+2l_2(q-2)(q-3)+2l_3(p^2-2)(p^2-3)+2l_4(pq-2)(pq-3)$.

Proof. Parts (a) and (b) follow from Theorem 1.1, recalling the facts (proved in [27, Theorem 2.9]) that $\Gamma_c(R) = \frac{p^2q-1}{t-1}K_{t-1}$ or $l_1K_{p-1} \sqcup l_2K_{q-1} \sqcup l_3K_{p^2-1} \sqcup l_4K_{pq-1}$ according as $t \in \{p, q, p^2, pq\}$ and $(t-1) \mid (p^2q-1)$; or $l_1(p-1)+l_2(q-1)+l_3(p^2-1)+l_4(pq-1) = p^2q-1$. □

We would like to remark that the conditions in [27, Theorem 2.9] were stated incorrectly. We conclude this section with the following two results.

Theorem 2.8. *Let R be a non-commutative ring with unity having order p^3q. If $|Z(R)| = pq$ then* $\text{CN-spec}(\Gamma_c(R))$ *is given by*

$$\left\{(-(p^2q-pq-2))^{(p+1)(p^2q-pq-1)}, ((p^2q-pq-1)(p^2q-pq-2))^{p+1}\right\}$$

and $E_{cn}(\Gamma_c(R)) = 2(p+1)(p^2q-pq-1)(p^2q-pq-2)$.

Proof. The result follows from Theorem 1.1, recalling the fact (proved in [27, Theorem 2.12]) that $\Gamma_c(R) = (p+1)K_{p^2q-pq}$ if R is a non-commutative ring with unity having order p^3q and $|Z(R)| = pq$. □

Theorem 2.9. *Let R be a non-commutative ring with unity having order p^3q and $|Z(R)| = p^2$.*

(a) *If* $(q-1) \mid (pq-1)$ *then* CN-spec$(\Gamma_c(R))$ *is given by*

$$\left\{(-(p^2q-p^2-2))^{\frac{(pq-1)(p^2q-p^2-1)}{q-1}}, ((p^2q-p^2-1)(p^2q-p^2-2))^{\frac{pq-1}{q-1}}\right\}$$

and $E_{cn}(\Gamma_c(R)) = \frac{2(pq-1)(p^2q-p^2-1)(p^2q-p^2-2)}{q-1}$.

(b) *If* $(p-1) \mid (pq-1)$ *then* CN-spec$(\Gamma_c(R))$ *is given by*

$$\left\{(-(p^3-p^2-2))^{\frac{(pq-1)(p^3-p^2-1)}{p-1}}, ((p^3-p^2-1)(p^3-p^2-2))^{\frac{pq-1}{p-1}}\right\}$$

and $E_{cn}(\Gamma_c(R)) = \frac{2(pq-1)(p^3-p^2-1)(p^3-p^2-2)}{p-1}$.

(c) *If* $l_1(p-1)+l_2(q-1) = pq-1$ *then* CN-spec$(\Gamma_c(R))$ *is given by*

$$\Big\{(-(p^3-p^2-2))^{l_1(p^3-p^2-1)}, ((p^3-p^2-1)(p^3-p^2-2))^{l_1},$$
$$(-(p^2q-p^2-2))^{l_2(p^2q-p^2-1)}, ((p^2q-p^2-1)(p^2q-p^2-2))^{l_2}\Big\}$$

and $E_{cn}(\Gamma_c(R)) = 2l_1(p^3-p^2-1)(p^3-p^2-2) + 2l_2(p^2q-p^2-1)(p^2q-p^2-2)$.

Proof. Parts (a), (b) and (c) follow from Theorem 1.1, recalling the facts (proved in [27, Theorem 2.12]) that $\Gamma_c(R) = \frac{pq-1}{q-1}K_{p^2q-p^2}$, $\frac{pq-1}{p-1}K_{p^3-p^2}$ and $l_1K_{p^3-p^2} \sqcup l_2K_{p^2q-p^2}$ if $(q-1) \mid (pq-1)$, $(p-1) \mid (pq-1)$ and $l_1(p-1)+l_2(q-1) = pq-1$ respectively, where R is a non-commutative ring with unity having order p^3q and $|Z(R)| = p^2$. □

3 Some consequences

In this section, we derive some consequences of the results obtained in Section 2.

Proposition 3.1. *Let R be a finite ring such that the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then, by Theorem 2.3, it follows that $\Gamma_c(R)$ is CN-integral.

We also have

$$E_{cn}(\Gamma_c(R)) = 2(p+1)((p-1)|Z(R)|-1)((p-1)|Z(R)|-2).$$

Since $|V(\Gamma_c(R))| = (p^2-1)|Z(R)|$, by (1.1) we have

$$E_{cn}(K_{(p^2-1)|Z(R)|}) = 2((p^2-1)|Z(R)|-1)((p^2-1)|Z(R)|-2).$$

Clearly

$$\begin{aligned}((p^2-1)|Z(R)|-1)&((p^2-1)|Z(R)|-2)\\ &> ((p^2-1)|Z(R)|-(p+1))((p^2-1)|Z(R)|-2(p+1))\\ &= (p+1)((p-1)|Z(R)|-1)((p-1)|Z(R)|-2).\end{aligned}$$

Thus $E_{cn}(K_{(p^2-1)|Z(R)|}) > E_{cn}(\Gamma_c(R))$. □

As an immediate consequence of Proposition 3.1 we have the following corollary.

Corollary 3.2. *If R is a non-commutative ring of order p^2 or p^3 then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proposition 3.3. *Let $|R| = p^4$ and R has unity. Then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. By Theorem 2.5, it follows that $\Gamma_c(R)$ is CN-integral.

If $|Z(R)| = p$ then $V(\Gamma_c(R))$ has order $p^4 - p$ and so, by (1.1), we have

$$E_{cn}(K_{p^4-p}) = 2(p^4 - p - 1)(p^4 - p - 2).$$

We also have

$$\begin{aligned}(p^4 - p - 1)(p^4 - p - 2) &= (p^4 - p)(p^4 - p - 3) + 2\\ &> p(p^3 - 1)(p^4 - p - 3)\\ &= (p^2 + p + 1)(p^2 - p)(p^4 - p - 3)\\ &> (p^2 + p + 1)(p^2 - p - 1)(p^2 - p - 2)\end{aligned}$$

and

$$\begin{aligned}(p^4 - p - 1)(p^4 - p - 2) &> (p^2 + p + 1)(p^2 - p)(p^4 - p - 3)\\ &= l_1(p^2 - p)(p^4 - p - 3) + l_2(p + 1)(p^2 - p)(p^4 - p - 3)\\ &= l_1(p^2 - p)(p^4 - p - 3) + l_2(p^3 - p)(p^4 - p - 3)\\ &> l_1(p^2 - p - 1)(p^2 - p - 2) + l_2(p^3 - p - 1)(p^3 - p - 2),\end{aligned}$$

where $l_1 + l_2(p + 1) = p^2 + p + 1$. Therefore, by Theorem 2.5, it follows that

$$E_{cn}(K_{p^4-p}) > E_{cn}(\Gamma_c(R)).$$

If $|Z(R)| = p^2$ then $V(\Gamma_c(R))$ has order $p^4 - p^2$ and so, by (1.1), we have

$$E_{cn}(K_{p^4-p^2}) = 2(p^4 - p^2 - 1)(p^4 - p^2 - 2).$$

Since

$$\begin{aligned}(p^4 - p^2 - 1)(p^4 - p^2 - 2) &= (p^4 - p^2)(p^4 - p^2 - 3) + 2\\ &> (p + 1)(p^3 - p^2)(p^4 - p^2 - 3)\\ &> (p + 1)(p^3 - p^2 - 1)(p^3 - p^2 - 2),\end{aligned}$$

by Theorem 2.5, it follows that

$$E_{cn}(K_{p^4-p^2}) > E_{cn}(\Gamma_c(R)).$$

Hence, $\Gamma_c(R)$ is not CN-hyperenergetic. □

Proposition 3.4. *Let $|R| = p^5$, R has unity and $Z(R)$ is not a field. Then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. By Theorem 2.6, it follows that $\Gamma_c(R)$ is CN-integral.

If $|Z(R)| = p^2$ then $V(\Gamma_c(R))$ has order $p^5 - p^2$ and so, by (1.1), we have

$$E_{cn}(K_{p^5-p^2}) = 2(p^5 - p^2 - 1)(p^5 - p^2 - 2).$$

We also have

$$\begin{aligned}(p^5 - p^2 - 1)(p^5 - p^2 - 2) &= (p^5 - p^2)(p^5 - p^2 - 3) + 2\\ &> (p^2 + p + 1)(p^3 - p^2)(p^5 - p^2 - 3)\end{aligned}$$

and

$$\begin{aligned}(p^5 - p^2 - 1)(p^5 - p^2 - 2) &> (p^2 + p + 1)(p^3 - p^2)(p^5 - p^2 - 3)\\ &> l_1(p^3 - p^2)(p^5 - p^2 - 3) + l_2(p + 1)(p^3 - p^2)(p^5 - p^2 - 3)\\ &= l_1(p^3 - p^2)(p^5 - p^2 - 3) + l_2(p^4 - p^2)(p^5 - p^2 - 3)\\ &> l_1(p^3 - p^2 - 1)(p^3 - p^2 - 2) + l_2(p^4 - p^2 - 1)(p^4 - p^2 - 2),\end{aligned}$$

where $l_1 + l_2(p+1) = p^2 + p + 1$. Therefore, by Theorem 2.6, it follows that

$$E_{cn}(K_{p^5-p^2}) > E_{cn}(\Gamma_c(R)).$$

If $|Z(R)| = p^3$ then $V(\Gamma_c(R))$ has order $p^5 - p^3$ and so, by (1.1), we have

$$E_{cn}(K_{p^5-p^3}) = 2(p^5 - p^3 - 1)(p^5 - p^3 - 2).$$

Since

$$\begin{aligned}(p^5 - p^3 - 1)(p^5 - p^3 - 2) &= (p^5 - p^3)(p^5 - p^3 - 3) + 2 \\ &> (p+1)(p^4 - p^3)(p^5 - p^3 - 3) \\ &> (p+1)(p^4 - p^3 - 1)(p^4 - p^3 - 2),\end{aligned}$$

by Theorem 2.5, it follows that

$$E_{cn}(K_{p^5-p^3}) > E_{cn}(\Gamma_c(R)).$$

Hence, $\Gamma_c(R)$ is not CN-hyperenergetic. □

Proposition 3.5. *Let R be a non-commutative ring of order p^2q such that $Z(R) = \{0\}$. Then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. By Theorem 2.7, it follows that $\Gamma_c(R)$ is CN-integral. Note that $V(\Gamma_c(R))$ has order $p^2q - 1$ and so, by (1.1), we have

$$E_{cn}(K_{p^2q-1}) = 2(p^2q - 2)(p^2q - 3).$$

We shall complete the proof considering the following cases.

Case 1: $(t-1) \mid (p^2q - 1)$ where $t \in \{p, q, p^2, pq\}$.

Let $p^2q - 1 = n(t-1)$ for some positive integer $n > 2$. We have

$$\begin{aligned}(p^2q - 2)(p^2q - 3) &= (n(t-1) - 1)(n(t-1) - 2) \\ &= n^2\left(t - \frac{n+1}{n}\right)\left(t - \frac{n+2}{n}\right) > n(t-2)(t-3),\end{aligned}$$

since $\frac{n+1}{n}, \frac{n+2}{n} < 2$. Therefore, by Theorem 2.7(a), we have

$$E_{cn}(K_{p^2q-1}) > E_{cn}(\Gamma_c(R)).$$

Case 2: $l_1(p-1) + l_2(q-1) + l_3(p^2-1) + l_4(pq-1) = p^2q - 1$.

We have

$$\begin{aligned}(p^2q - 2)(p^2q - 3) =&(p^2q - 1 - 1)(p^2q - 1 - 2) \\ =&(p^2q - 1)^2 - 3(p^2q - 1) + 2 \\ >&(p^2q - 1)^2 - 3(p^2q - 1) \\ =&(p^2q - 1)(p^2q - 4) \\ =&l_1(p-1)(p^2q - 4) + l_2(q-1)(p^2q - 4) + l_3(p^2-1)(p^2q - 4) \\ &+ l_4(pq-1)(p^2q - 4) \\ >&l_1(p-2)(p-3) + l_2(q-2)(q-3) + l_3(p^2-2)(p^2-3) \\ &+ l_4(pq-2)(pq-3),\end{aligned}$$

since $p - 1 > p - 2$, $q - 1 > q - 2$, $p^2 - 1 > p^2 - 2$, $pq - 1 > pq - 2$ and $p^2q - 4 > p - 3, q - 3, p^2 - 3, pq - 3$. Therefore, by Theorem 2.7(b), we have

$$E_{cn}(K_{p^2q-1}) > E_{cn}(\Gamma_c(R)).$$

This completes the proof. □

Proposition 3.6. *Let R be a non-commutative ring with unity having order p^3q such that $|Z(R)|$ is not a prime. Then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. By Theorem 2.8 and Theorem 2.9, it follows that $\Gamma_c(R)$ is CN-integral. If $|Z(R)| = pq$ then $V(\Gamma_c(R))$ has order $p^3q - pq$ and so, by (1.1), we have

$$E_{cn}(K_{p^3q-pq}) = 2(p^3q - pq - 1)(p^3q - pq - 2).$$

Note that

$$p^3q - pq - 1 > p^3q - pq - 1 - p = (p+1)(p^2q - pq - 1)$$

and

$$p^3q - pq - 2 > p^3q - pq - 2 - 2p = (p+1)(p^2q - pq - 2).$$

Therefore,

$$\begin{aligned}(p^3q - pq - 1)(p^3q - pq - 2) >&(p+1)^2(p^2q - pq - 1)(p^2q - pq - 2)\\ >&(p+1)(p^2q - pq - 1)(p^2q - pq - 2).\end{aligned}$$

Hence, by Theorem 2.8, we have

$$E_{cn}(K_{p^3q-pq}) > E_{cn}(\Gamma_c(R)).$$

If $|Z(R)| = p^2$ then $V(\Gamma_c(R))$ has order $p^3q - p^2$ and so, by (1.1), we have

$$E_{cn}(K_{p^3q-p^2}) = 2(p^3q - p^2 - 1)(p^3q - p^2 - 2).$$

We shall complete the proof considering the following cases.
Case 1: $(p-1) \mid (pq-1)$.
Let $pq - 1 = n(p-1)$ for some positive integer $n > 2$. We have

$$p^3q - p^2 - 1 = p^2(pq-1) - 1 = n(p^3 - p^2) - 1 > n(p^3 - p^2 - 1)$$

and

$$p^3q - p^2 - 2 = p^2(pq-1) - 2 = n(p^3 - p^2) - 2 > n(p^3 - p^2 - 2).$$

Therefore,

$$\begin{aligned}(p^3q - p^2 - 1)(p^3q - p^2 - 2) >&n^2(p^3 - p^2 - 1)(p^3 - p^2 - 2)\\ >&n(p^3 - p^2 - 1)(p^3 - p^2 - 2).\end{aligned}$$

Hence, by Theorem 2.9(a), we have

$$E_{cn}(K_{p^3q-p^2}) > E_{cn}(\Gamma_c(R)).$$

Case 2: $(q-1) \mid (pq-1)$.
Let $pq - 1 = n(q-1)$ for some positive integer $n > 2$. We have

$$p^3q - p^2 - 1 = p^2(pq-1) - 1 = n(p^2q - p^2) - 1 > n(p^2q - p^2 - 1)$$

and

$$p^3q - p^2 - 2 = p^2(pq-1) - 2 = n(p^2q - p^2) - 2 > n(p^2q - p^2 - 2).$$

Therefore,

$$\begin{aligned}(p^3q - p^2 - 1)(p^3q - p^2 - 2) >&n^2(p^2q - p^2 - 1)(p^2q - p^2 - 2)\\ >&n(p^2q - p^2 - 1)(p^2q - p^2 - 2).\end{aligned}$$

Hence, by Theorem 2.9(b), we have

$$E_{cn}(K_{p^3q-p^2}) > E_{cn}(\Gamma_c(R)).$$

Case 3: $l_1(p-1)+l_2(q-1)=pq-1$.

We have $l_1(p^3-p^2)+l_2(p^2q-p^2)=p^3q-p^2$ and

$$\begin{aligned}(p^3q-p^2-1)(p^3q-p^2-2) &= (p^3q-p^2)^2-3(p^3q-p^2)+2\\ &>(p^3q-p^2)^2-3(p^3q-p^2)\\ &=(p^3q-p^2)(p^3q-p^2-3)\\ &=l_1(p^3-p^2)(p^3q-p^2-3)+l_2(p^2q-p^2)(p^3q-p^2-3)\\ &>l_1(p^3-p^2-1)(p^3-p^2-2)+l_2(p^2q-p^2-1)(p^2q-p^2-2),\end{aligned}$$

since $p^3-p^2>p^3-p^2-1$, $p^3q-p^2-3>p^3-p^2-2$, $p^2q-p^2>p^2q-p^2-1$ and $p^3q-p^2-3>p^2q-p^2-2$. Therefore, by Theorem 2.9(c), we have

$$E_{cn}(K_{p^3q-p^2})>E_{cn}(\Gamma_c(R)).$$

This completes the proof. □

Proposition 3.7. *If R is a finite 4-centralizer ring then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. If R is a finite 4-centralizer ring then, by [11, Theorem 3.2], we have that the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$. Hence, the result follows from Proposition 3.1 considering $p=2$. □

Proposition 3.8. *If R is a finite 5-centralizer ring then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. If R is a finite 5-centralizer ring then, by [11, Theorem 4.3], we have that the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_3\times\mathbb{Z}_3$. Hence, the result follows from Proposition 3.1 considering $p=3$. □

Proposition 3.9. *If R is a finite $(p+2)$-centralizer ring of order p^k, for any prime p, then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. If R is a finite $(p+2)$-centralizer ring of order p^k then, by [11, Theorem 2.12], the additive quotient group $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p\times\mathbb{Z}_p$. Hence, the result follows from Proposition 3.1. □

The commuting probability of a finite ring R is the probability that a randomly chosen pair of elements of R commute. Let $\Pr(R)$ be the commuting probability of R. Then

$$\Pr(R):=\frac{|\{(r,s)\in R\times R: rs=sr\}|}{|R|^2}.$$

The study of $\Pr(R)$ was initiated by MacHale [22] in 1976. Recent results on $\Pr(R)$ can be found in [8, 9, 13, 17]. The following result shows that $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic if $\Pr(R)=\frac{5}{8}$.

Proposition 3.10. *Let R be a finite ring with $\Pr(R)=\frac{5}{8}$ then*

$$\text{CN-spec}(\Gamma_c(R))=\left\{(-(|Z(R)|-2))^{3|Z(R)|-1},((|Z(R)|-1)(|Z(R)|-2))^3\right\}$$

and $E_{cn}(\Gamma_c(R))=6(|Z(R)|-1)(|Z(R)|-2)<E_{cn}(K_{3|Z(R)|})$.

Proof. If $\Pr(R)=\frac{5}{8}$ then, by [22, Theorem 1], we have $\frac{R}{Z(R)}\cong\mathbb{Z}_2\times\mathbb{Z}_2$. Therefore, putting $p=2$ in Theorem 2.3 we get the required result. □

Proposition 3.11. *Let R be a finite ring and p the smallest prime divisor of $|R|$. If $\Pr(R)=\frac{p^2+p-1}{p^3}$ then $\Gamma_c(R)$ is CN-integral but not CN-hyperenergetic.*

Proof. If p the smallest prime divisor of $|R|$ and $\Pr(R) = \frac{p^2+p-1}{p^3}$, by [22, Theorem 3], we have $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from Proposition 3.1. □

Notice that all the rings considered in this paper are finite CC-rings. By Theorem 2.1, it follows that the commuting graph of a finite CC-ring is CN-integral. We conclude this paper with the following conjecture.

Conjecture 3.12. If R is a finite CC-ring then $\Gamma_c(R)$ is not CN-hyperenergetic.

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