Palestine Journal of Mathematics

Vol 13(1)(2024) , 77–90

THE IDENTITY-FILTER GRAPH OF LATTICES

Shahabaddin Ebrahimi Atani

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 97H50, 06A11; Secondary 16G30, 06D25.

Keywords and phrases: lattice, essential filter; identity-filter graph.

The author thanks the referee and editor for their useful suggestions.
Corresponding Author: S. Ebrahimi Atani

Abstract *Let L be a lattice with 1 and 0. A filter F of a lattice L is called an identity-filter if there exists $a \in L \setminus \{1\}$ such that $a \vee F = \{1\}$. The identity-filter graph $\mathbb{IG}(L)$ of L is a graph whose vertices are all identity-filters $F \neq \{1\}$ of L and two distinct filters F and G are adjacent if and only if $F \cap G = \{1\}$. The basic properties and possible structures of the graph $\mathbb{IG}(L)$ and its subgraph $\mathbb{IG}_{fg}(L)$ induces by vertices which are finitely generated as filters of L are investigated.*

1 Introduction

Associating a graph to an algebraic structure is a research subject and has attracted considerable attention. In fact, the research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. There has been a lot of activity over the past several years in associating a graph to an algebraic system such as a ring, module or lattice, for instance see [1,3, 4, 7, 10, 12, 14]. The main aim of this article is that of extending some results obtained for ring theory to in lattice theory. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [5, 6, 7, 8, 9, 10, 11, 13]).

Let L be a distributive lattice with 0 and 1 with $\mathbb{I}(L)$ its set of identity-filters. In the present paper, we are interested in investigating the identity-filter graphs of lattices to use other notions of the annihilating-ideal graphs, and associate which exist in the literature as laid forth in [4, 14]. The purpose of this paper is to investigate a graph associated to a lattice L called the identity-filter graph of L. This will result in characterization of lattices in terms of some specific properties of those graphs. The identity-filter graph of L is a simple graph $\mathbb{IG}(L)$ with vertices $\mathbb{I}(L)^* = \mathbb{I}(L) \setminus \{1\}$, and two distinct vertices are adjacent if and only if the intersection of the corresponding filters is $\{1\}$. We also study the graph of $\mathbb{IG}_{fg}(L)$ which is a subgraph of $\mathbb{IG}(L)$ generated by vertices which, as filters of L are finitely generated. The concept of the annihilating-ideal graph of a commutative ring R was introduced and studied by Behboodi and Rakeei in [4]. The annihilating-ideal graph of R, denoted by $\mathbb{AG}(R)$, is a graph whose vertex set is the set of all nonzero annihilator ideals of R and two vertices I and J are adjacent whenever $IJ = (0)$. Finitely generated annihilating-ideal graph of a commutative ring R was investigated by Taheri and Tehranian in [14].

Here is a brief outline of the article. Among many results in this paper, the first, introduction section contains elementary observations needed later on. In Section 2, we investigate some finiteness conditions and connectivity of identity-filter graphs. For instance, it is proved that if L is not a L-domain and an element $x \neq 1$ of L has a complement $y \neq 1$ in L, then $\mathbb{IG}(L)$ has ACC *(resp.* DCC*) on vertices if and only if L is a Noetherian (resp. an Artinian) lattice (Theorem 2.6). Also, it is shown that If L is a complemented lattice, then $\mathbb{IG}(L)$ has n ($n \geq 1$) vertices if and only if L has only n nontrivial filters (Corollary 2.10). It is shown in Theorem 2.17 that $\mathbb{IG}(L)$ is connected with $\mathrm{diam}(\mathbb{IG}(L)) \leq 3$ and if $\mathbb{IG}(L)$ contains a cycle, then $\mathrm{gr}(\mathbb{IG}(L)) \leq 4$.*

An element x of L is called identity join of lattice $L \neq \{1\}$, if there exists $1 \neq y \in L$ such that $x \vee y = 1$. The set of all identity join of a lattice L is denoted $\mathcal{I}(L)$. It is proved in Theorem 2.18 that if L is a complemented lattice which is not a L-domain, then $\mathbb{IG}(L)$ is a complete graph if and only if $\mathcal{I}(L)$ is a simple filter of L and $L = \mathcal{I}(L) \odot (1 :_L \mathcal{I}(L))$, where $(1 :_L \mathcal{I}(L))$ is a simple filter.

In Section 3, we study some finiteness conditions and connectivity of finitely generated identity-filter graphs and establish some basic connections between $\mathbb{IG}(L)$ and $\mathbb{IG}_{fg}(L)$. It is shown in Theorem 3.2 that if L is not a L-domain and an element $x \neq 1$ of L has a complement $y \neq 1$ in L, then $\mathbb{IG}_{fg}(L)$ has ACC on vertices if and only if L is a Noetherian lattice. Also, we show that if L is a complemented lattice which is not a L-domain, then $\mathbb{IG}_{fg}(L)$ contains a universal vertex if and only if $\mathbb{IG}(L)$ contains a universal vertex (Theorem 3.5) and $\mathbb{IG}(L)$ is a complete (star) graph if and only if $\mathbb{IG}_{fg}(L)$ is a complete (star) graph (Theorem 3.8 and Theorem 3.14). In this section, diameter and girth of the graph $\mathbb{IG}_{fg}(L)$ are studied (Theorem 3.1, Proposition 3.11, Theorem 3.12 and Corollary 3.13). It is show that if L is a complemented lattice which is not a L-domain such that $\mathrm{gr}(\mathbb{IG}_{fg}(L)) = 4$, then $\mathbb{IG}(L)$ is a complete bipartite graph if and only if $\mathbb{IG}_{fg}(L)$ is a complete bipartite graph (Theorem 3.17). Consequently, if L is a complemented lattice which is not a L-domain, then $\mathbb{IG}(L)$ is a bipartite graph if and only if $\mathbb{IG}_{fg}(L)$ is a bipartite graph (Theorem 3.19).

Let G be a simple graph with vertex set $\mathcal{V}$ (G) and edge set $\mathcal{E}$ (G). For every vertex $v \in \mathcal{V}$ (G), the degree of v, denoted by $\deg_G(v)$, is defined the cardinality of the set of all vertices which are adjacent to v. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G are adjacent and K_n will stand for a complete graph with n vertices. Let $u, v \in \mathcal{V}$ (G). We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. The distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$. The diameter of G is $\mathrm{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. The girth of a graph G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\mathrm{gr}(G) = \infty$. A subset $S \subseteq \mathcal{V}(G)$ is independent if no two vertices of S are adjacent. For a positive integer k, a k-partite graph is a graph whose vertices can be partitioned into k nonempty independent sets. For terminology and notation not defined here, the reader is referred to [15].

By a lattice we mean a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter P of L is called prime if $x \vee y \in P$, then $x \in P$ or $y \in P$. The set of all prime filters of L is denoted $\mathrm{Spec}(L)$. A proper filter $\mathfrak{m}$ of L is said to be maximal if E is a filter in L with $\mathfrak{m} \subsetneqq E$, then $E = L$. The set of all maximal filters of L is denoted $\mathrm{Max}(L)$. If L is a lattice, then L is Noetherian (resp. Artinian) if any non-empty set of filters of L has a maximal member (resp. minimal member) with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending chain condition) on filters of L. L is called L-domain if $a \vee b = 1$, then either $a = 1$ or $b = 1$. For terminology and notation not defined here, the reader is referred to [5].

Lemma 1.1. *Let L be a lattice [5, 6].*

(1) A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.

(2) If F_1, F_2 are filters of L and $a \in L$, then $F_1 \vee F_2 = \{a_1 \vee a_2 : a_1 \in F_1, a_2 \in F_2\}$ and $a \vee F_1 = \{a \vee a_1 : a_1 \in F_1\}$ are filters of L and $F_1 \cap F_2 = F_1 \vee F_2 \subseteq F_1, F_2$.

(3) If L is distributive, F is a filter of L and $a \in L$, then $(1 :_L F) = \{x \in L : x \vee F = \{1\}\}$ and $(1 :_L T(\{a\}) = (1 :_L a) = \{x \in L : a \vee x = 1\}$ are filters of L.

(4) If L is distributive and F_1, F_2 are filters of L, then $F_1 \wedge F_2 = \{a \wedge b : a \in F_1, b \in F_2\}$ is a filter of L, $F_1, F_2 \subseteq F_1 \wedge F_2$ (for if $x \in F_1$, then $x = x \wedge 1 \in F_1 \wedge F_2$) and if $F_1 \subseteq F_2$, then $F_1 \wedge F_2 = F_2$.

Let H be subset of a lattice L. Then the filter generated by H, denoted by $T(H)$, is the intersection of all filters that is containing H. A filter F is called finitely generated (resp cyclic) if there is a finite subset H (resp. $a \in F$) of F such that $F = T(H)$ (resp. $T(\{a\})$).

Lemma 1.2. *Let L be a lattice [8, 9].*

(1) Let A be an arbitrary non-empty subset of L. Then $T(A) = \{x \in L : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in A$ $(1 \leq i \leq n)\}$. Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$ and $T(F) = F$.

(2) If F and G are filters of L, then $T(G \cup F) = F \wedge G$;

(3) (modular law) If F, G and H are filters of L with $F \subseteq G$, then $G \cap (F \wedge H) = F \wedge (G \cap H)$.

A lattice L is called semisimple, *if for each proper filter F of L, there exists a filter G of L such that $L = F \wedge G$ and $F \cap G = \{1\}$). In this case, we say that F is a direct meet of L, and we write $L = F \odot G$. A filter F of L is called a semisimple filter, if every subfilter of F is a direct meet. A simple filter is a filter that has no filters besides the $\{1\}$ and itself.*

Let $\Lambda = \{F_i : i \in I\}$ be a set of filters of L. Then it is easy to see that $\bigwedge_{i \in I} F_i = \{\bigwedge_{i \in I'} f_i : f_i \in F_i, I' \subset I, I'$ is finite$\}$ is a filter of L (if $\Lambda = \emptyset$, then we set $\bigwedge_{i \in I} F_i = \{1\}$). $L = \bigodot_{i \in I} F_i$ is said to be a direct decomposition of L into the meet of the filters $\{F_i : i \in I\}$ if (1) $L = \bigwedge_{i \in I} F_i$ and (2) $\{F_i : i \in I\}$ is independent i.e for each $j \in I$, $F_j \cap \bigwedge_{j \neq i \in I} F_i = \{1\}$. For each filter F of L, $\mathrm{Soc}(F) = \bigwedge_{i \in \Lambda} F_i$, where $\{F_i\}_{i \in \Lambda}$ is the set of all simple filters of L contained in F.

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(L, \leq)$, we define a relation on L, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on L, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leq_Q$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. It is straightforward to check that $(\frac{L}{F}, \leq_Q)$ is a poset. The following notation below will be kept in this paper: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X = \{a \wedge F, b \wedge F\}$. By definition of $\leq_Q$, $(a \vee b) \wedge F$ is an upper bound for the set X. If $c \wedge F$ is any upper bound of X, then we can easily show that $(a \vee b) \wedge F \leq_Q c \wedge F$. Thus $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$. Thus $(\frac{L}{F}, \leq_Q)$ is a lattice. We need the following Lemma proved in [9, Lemma 4.3].

Lemma 1.3. *Let G be a a subfilter of a filter F of L.*

(1) If $a \in F$, then $a \wedge F = F$. By the definition of $\leq_Q$, it is easy to see that $1 \wedge F = F$ is the greatest element of $\frac{L}{F}$.

(2) If $a \in F$, then $a \wedge F = b \wedge F$ (for every $b \in L$) if and only if $b \in F$. In particular, $c \wedge F = F$ if and only if $c \in F$. Moreover, if $a \in F$, then $a \wedge F = F = 1 \wedge F$.

(3) By the definition $\leq_Q$, we can easily show that if L is distributive, then $\frac{L}{F}$ is distributive.

(4) $\frac{F}{G} = \{a \wedge G : a \in F\}$ is a filter of $\frac{L}{G}$.

(5) If K is a filter of $\frac{L}{G}$, then $K = \frac{F}{G}$ for some filter F of L.

(6) If H is a filter of L such that $G \subseteq H$ and $\frac{F}{G} = \frac{H}{G}$, then $F = H$.

(7) If H and V are filters of L containing G, then $\frac{F}{G} \cap \frac{H}{G} = \frac{V}{G}$ if and only if $V = H \cap F$.

(8) If H is a filter of L containing G, then $\frac{F \bigwedge H}{G} = \frac{H}{G} \bigwedge \frac{F}{G}$.

2 Finiteness conditions and connectivity of $\mathbb{IG}(L)$

Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1 and 0. In this section, we collect basic properties concerning the graph $\mathbb{IG}(L)$. Our starting point is the following proposition:

Proposition 2.1. *If F is a filter of L, then the following conditions hold:*

(1) L is Noetherian if and only if both F and $\frac{L}{F}$ are Noetherian;

(2) L is Artinian if and only if both F and $\frac{L}{F}$ are Artinian;
(3) L is Noetherian if and only if every filter of L is finitely generated.

Proof. (1) Let L be a Noetherian lattice. Since every subfilter of F is a filter of L, it is clear from the definition that F is Noetherian. By Lemma 1.3, as ascending chain of filters of $\frac{L}{F}$ must have the form $\frac{H_1}{F} \subseteq \frac{H_2}{F} \subseteq \cdots$, where $H_1 \subseteq H_2 \subseteq \cdots$ is an ascending chain of filters of L all of which contain F. Since the latter chain must eventually become stationary, so must the former by Lemma 1.3. Conversely, let $H_1 \subseteq H_2 \subseteq \cdots$ be an ascending chain of filters of L. Then $H_1 \cap F \subseteq H_2 \cap F \subseteq \cdots$ is an ascending chain of subfilters of F, and so there is a positive integer s such that $H_s \cap F = H_{s+i} \cap F$ for all positive integer i. By Lemma 1.3, $\frac{H_1 \bigwedge F}{F} \subseteq \frac{H_2 \bigwedge F}{F} \subseteq \cdots$ is a chain of filters of $\frac{L}{F}$. Then there exists a positive integer t such that $\frac{H_t \bigwedge F}{F} = \frac{H_{t+i} \bigwedge F}{F}$ for all positive integer i, so that $H_t \wedge F = H_{t+i} \wedge F$ for all i. Let $u = \max\{s, t\}$. We show that for each positive integer i, $H_u = H_{u+i}$. It suffices to show that $H_{u+i} \subseteq H_u$. Let $x \in H_{u+i}$. Then $x = a \wedge b \in H_{u+i} \wedge F = H_u \wedge F$ for some $a \in H_u$ and $b \in F$. Now H_{u+i} is a filter gives $b \in H_{u+i}$; so $b \in H_{u+i} \cap F = H_u \cap F$ which implies that $x = a \wedge b \in H_u$, as required.

(2) This can be proved in a very similar manner to the way in which (1) was proved above, and we omit it.

(3) Assume that L is Noetherian and let G be a filter of L. Suppose to the contrary, that G is not finitely generated. Let Ω be the set of all subfilters of G which are finitely generated (so $\Omega \neq \emptyset$ since $T(\{1\}) = \{1\} \in \Omega$). It follows from the maximal condition that Ω has a maximal member with respect to inclusion, $H = T(\{x_1, \cdots, x_n\})$ say; so $H \subsetneqq G$. Let $x \in G \setminus H$. Then $H \wedge T(\{x\}) = T(\{x_1, \cdots x_n, x\})$ is a finitely generated subfilter of G with $H \subsetneqq H \wedge T(\{x\})$, a contradiction. Thus G must be finitely generated. Conversely, let $F_1 \subseteq F_2 \subseteq \cdots$ be an ascending chain of filters of L. Then $G = \bigcup_{i \in \mathbb{N}} F_i$ is a filter of L. By assumption, suppose that it is generated by $f_1, \cdots, f_n$. For each $i = 1, \cdots n$, there exists positive integer m_i such that $f_i \in F_{m_i}$. If $k = \max\{m_1, \cdots m_n\}$, then $f_i \in F_k$ for all $i = 1, \cdots n$. Hence $F_k = F_{k+i}$ for all $i \in \mathbb{N}$, as needed. □

Lemma 2.2. *For the lattice L, the following conditions hold:*
(1) If F_1 and F_2 are nontrivial filters of L such that $L = F_1 \wedge F_2$, then G is a filter of L if and only if $G = G_1 \wedge G_2$ for some subfilter G_1 of F_1 and subfilter G_2 of F_2;
(2) A filter $F \neq \{1\}$ is an identity-filter if and only if there exists a filter $G \neq \{1\}$ of L such that $G \vee F = G \cap F = \{1\}$.

Proof. (1) Let G be a filter of L and set

$$G_1 = \{x \in F_1 : x \wedge y \in G \text{ for some } y \in F_2\}.$$

If $x, z \in G_1$ and $a \in L$, then $x \wedge c, z \wedge d \in G$ for some $c, d \in F_2$ gives $(x \wedge z) \wedge (c \wedge d) \in G$ and $(x \vee a) \wedge c = (x \wedge c) \vee (c \wedge a) \in G$; hence G_1 is a subfilter of F_1. Similarly, $G_2 = \{x \in F_2 : x \wedge y \in G \text{ for some } y \in F_1\}$ is a subfilter of F_2. Since the inclusion $G \subseteq G_1 \wedge G_2$ is clear we will prove the reverse inclusion. If $x \in G_1$, then $x \wedge y \in G$ for some $y \in F_2$. Now G is a filter gives $x \in G$ by Lemma 1.1; so $G_1 \subseteq G$. Similarly, $G_2 \subseteq G$. Thus $G_1 \wedge G_2 \subseteq G$ and so we have equality. Conversely, assume that G_1 and G_2 are subfilters of F_1 and F_2, respectively. We show that $G_1 \wedge G_2$ is a filter of L. Clearly, if $X, Y \in G_1 \wedge G_2$, then $X \wedge Y \in G_1 \wedge G_2$. Let $a \wedge b \in G_1 \wedge G_2$ for some $a \in G_1, b \in G_2$ and $c \in L$. Then $c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b) \in G_1 \wedge G_2$ by Lemma 1.1. This completes the proof.

(2) If $a \vee F = \{1\}$ for some $1 \neq a \in L$, then $T(\{a\}) \cap F = \{1\}$. Conversely, if $G \cap F = \{1\}$ for some filter $G \neq \{1\}$ of L, then there is an element $1 \neq b \in G$ such that $b \vee F = \{1\}$, as needed. □

If $x \in L$, then a complement of x in L is an element $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice L is complemented if every element of L has a complement in L.

Lemma 2.3. *If L is not a L-domain and an element $x \neq 1$ of L has a complement $y \neq 1$ in L, then $\frac{L}{(1:_L T(\{x\})} \cong T(\{x\})$.*

Proof. Set $F = (1 :_L x)$ (so $(1 :_L x) \neq \{1\}$). We show that the mapping $\varphi : L \to T(\{x\})$ for which $\varphi(a) = a \vee x$ for all $a \in L$ is a lattice morphism. Let $a, b \in L$. Then $\varphi(a \wedge b) =$

$(a\wedge b)\vee x = (a\vee x)\wedge(b\vee x) = \varphi(a)\wedge\varphi(b)$. Similarly, $\varphi(a\vee b) = \varphi(a)\vee\varphi(b)$. If $a\wedge F = b\wedge F$, then there are elements $f, f' \in F$ such that $f'\wedge a = f\wedge b$. Hence $\varphi(b) = b\vee x = (b\vee x)\wedge 1 = (b\vee x)\wedge(f\vee x) = x\vee(f\wedge b) = x\vee(f'\wedge a) = x\vee a = \varphi(a)$. It follows that there is indeed a mapping $\psi : \frac{L}{F} \to T(\{x\})$ by the formula $\psi(a\wedge F) = \varphi(a)$, and it is clear that ψ is serjective. Next note that, for all $a\wedge F, b\wedge F \in \frac{L}{F}$, we have $\psi((a\wedge F)\wedge_Q(b\wedge F)) = \psi((a\wedge b)\wedge F) = \varphi(a\wedge b) = \varphi(a)\wedge\varphi(b) = \psi(a\wedge F)\wedge_Q\psi(b\wedge F)$. Similarly, $\psi((a\wedge F)\vee_Q(b\wedge F)) = \psi(a\wedge F)\vee_Q\psi(b\wedge F)$. It remains to show that ψ is injective. If $\psi(a\wedge F) = \psi(b\wedge F)$, then $a\vee x = b\vee x$. It follows that $(a\vee x)\wedge y = (b\vee x)\wedge y$, and so $(a\wedge y)\vee 0 = (b\wedge y)\vee 0$; hence $a\wedge y = b\wedge y$. Thus $a\wedge F = b\wedge F$. □

Proposition 2.4. *For the lattice L the following conditions hold:*
(1) The graph $\mathbb{IG}(L)$ is a null graph if and only if L is a simple lattice;
(2) If F is a filter of L, then $F\cap(1 :_L F) = \{1\}$;
(3) If $\mathcal{V}(\mathbb{IG}(L)) \neq \emptyset$, then $\mathbb{IG}(L)$ is not an empty graph.

Proof. (1) The proof is clear.

(2) If $x \in F\cap(1 :_L F)$, then $x\vee F = \{1\}$ gives $x = x\vee x \in x\vee F = \{1\}$. Thus $F\cap(1 :_L F) = \{1\}$.

(3) If $G \in \mathcal{V}(\mathbb{IG}(L))$, then $H = (1 :_L G) \neq \{1\}$. Since by (2), $G\cap H = G\vee H = \{1\}$, we get $\{1\} \neq G \subseteq (1 :_L H)$ which implies that $H \in \mathcal{V}(\mathbb{IG}(L))$. Now the assertion follows from (2). □

Henceforth we will assume that all considered lattices L are not simple since all definitions of graph theory are for non-null graph [15].

Proposition 2.5. *Let S be a simple filter of a complemented lattice L which is not a L-domain. Then the following conditions hold:*
(1) $(1 :_L S)$ is a maximal filter of L.
(2) $L = S\odot(1 :_L S)$.

Proof. (1) Let $x \in S\setminus\{1\}$. Since S is simple, we conclude that $S = T(\{x\})$. It follows from Lemma 2.3 that $\frac{L}{(1:_L T(\{x\})} \cong T(\{x\})$; hence $\frac{L}{(1:_L T(\{x\})}$ is a simple lattice. Thus $(1 :_L S)$ is a maximal filter of L.

(2) By Proposition 2.4 (2), $S\cap(1 :_L S) = \{1\}$. As

$$(1 :_L S) \subsetneqq S\wedge(1 :_L S) \subseteq L,$$

we get $L = S\wedge(1 :_L S)$; hence $L = S\odot(1 :_L S)$. □

Let L be a lattice. We say that the identity-filter graph $\mathbb{IG}(L)$ has ACC *(resp.* DCC*) on vertices if L has* ACC *(resp.* DCC*) on $\mathbb{I}(L)^*$. Let $\mathbb{F}(L)$ be the set of all filters of L and set $\mathbb{F}(L)^* = \mathbb{F}(L)\setminus\{1\}$.*

Compare the next theorem with Theorem 1.1 in [4].

Theorem 2.6. *If L is not a L-domain and an element $x \neq 1$ of L has a complement $y \neq 1$ in L, then $\mathbb{IG}(L)$ has* ACC *(resp.* DCC*) on vertices if and only if L is a Noetherian (resp. an Artinian) lattice.*

Proof. Let $\mathbb{IG}(L)$ has ACC (resp. DCC) on vertices and set $F = (1 :_L x)$,

$$A = \{G \in \mathbb{F}(L) : G \subseteq T(\{x\})\}$$

and $B = \{G \in \mathbb{F}(L) : G \subseteq F\}$. If $z \in G \in A$, then there exists $a \in L$ such that $z = x\vee a$; so $y\vee z = y\vee(x\vee a) = 1$ which implies that $y \in (1 :_L G) \neq \{1\}$. Thus $A \subseteq \mathcal{V}(\mathbb{IG}(L))$. If $G \in B$, then $x\vee G = \{1\}$ gives $x \in (1 :_L G) \neq \{1\}$; hence $B \subseteq \mathcal{V}(\mathbb{IG}(L))$. It follows that the filters $T(\{x\})$ and F have ACC (resp. DCC) on subfilters i.e. $T(\{x\})$ and F are Noetherian (resp Artinian) filters. Now Propositions 2.1 and Lemma 2.3 gives R is a Noetherian (resp. an Artinian) lattice. The other implication is clear. □

A filter $F \neq \{1\}$ of L is called L-second if for each $a \in L$, either $a \vee F = \{1\}$ or $a \vee F = F$. By [6, Proposition 2.1], F is L-second if and only if the only subfilters of F are $\{1\}$ and F itself (i.e. F is simple) and in this case, $|F| = 2$. Moreover, if L is a Artinian lattice and F is a filter of L with $F \neq \{1\}$, then F contains only a finite number of simple filters by [6, Theorem 2.2 (i)].

Proposition 2.7. *(1) If L is an Artinian complemented lattice, then every proper filter F of L with $F \neq \{1\}$ is a vertex of $\mathbb{IG}(L)$.*

(2) If L is an Artinian complemented L-domain, then L is a simple lattice.

Proof. (1) By [6, Theorem 2.2 (iii)], $L = S_1 \wedge \cdots \wedge S_n$, where $S_1, \cdots, S_n$ are distinct simple filters of L (i.e. L is semisimple). As $S_i \cap S_j = S_i \vee S_j = \{1\}$, we get $\{1\} \subsetneqq S_j \subseteq (1 :_L S_i)$ for all $1 \leq i \neq j \leq n$. Let G be a nontrivial filter of L (i.e. different from $\{1\}$ and L). There is a simple filter S of L such hat $S \nsubseteq G$ (otherwise, $L = G$ which is impossible). Then S is a simple filter gives $S \cap G = S \vee G = \{1\}$; hence $\{1\} \subsetneqq S \subseteq (1 :_L G)$. This completes the proof.

(2) Assume to the contrary, that L is not simple. So we can write $L = S_1 \odot S_2$, where $S_1 = T(\{x\})$ and $S_2 = T(\{y\})$ are distinct simple filters of L by [6, Theorem 2.2 (iii)]. Since $S_1 \cap S_2 = S_1 \vee S_2 = \{1\}$, $x \vee y = 1$; so either $x = 1$ or $y = 1$ which is impossible. Thus L is simple. □

*Let R be a non-domain. In [**4**], the following questions were investigated: (1) When $|\mathbb{A}(R)^*| < |\mathbb{I}(R)^*|$ and (2) $|\mathbb{A}(R)^*| = |\mathbb{I}(L)^*|$, as a conjecture? In that paper, the conjecture above is true for all Artinian rings as well as all decomposable rings (see [4, Propositions 1.3 and 1.6]). In the following example, it is shown that the conjecture above is not true for all Artinian lattices and the condition " L is a complemented lattice" in Proposition 2.7 (1) cannot be omitted.*

Example 2.8. (1) Let $L = \{0, a, b, c, d, 1\}$ be a lattice with $0 \leq d \leq c \leq a \leq 1$, $0 \leq d \leq c \leq b \leq 1$, $a \vee b = 1$ and $a \wedge b = c$. An inspection will show that the nontrivial filters of L are $S_1 = \{1, a\}$, $S_2 = \{1, b\}$, $S_3 = \{1, a, b, c\}$ and $S_4 = \{1, a, b, c, d\}$ with $(1 :_L S_1) \neq \{1\}$, $(1 :_L S_2) \neq \{1\}$, $(1 :_L S_3) = \{1\}$ and $(1 :_L S_4) = \{1\}$. Moreover, L is Artinian which is not a complemented L-domain, but $2 = |\mathbb{I}(L)^*| < |\mathbb{F}(L)^*| = 4$. This provides an answer to a question (1) which is investigated in [**4**]. Also, this show that the condition " L is a complemented lattice" is not superficial in Proposition 2.7 (1).

(2) Let $L = \{0, a, b, c, 1\}$ be a lattice with $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. An inspection will show that the nontrivial filters of L are $S_1 = \{1, a, c\}$, $S_2 = \{1, b, c\}$ and $S_3 = \{1, c\}$ with $(1 :_L S_1) = \{1\}$, $(1 :_L S_2) = \{1\}$ and $(1 :_L S_3) = \{1\}$. Moreover, L is Artinian which is not a complemented L-domain, but $0 = |\mathbb{I}(L)^*| < |\mathbb{F}(L)^*| = 3$.

(3)Let $D = \{a, b, c\}$. Then $L(D) = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion greatest element D and least element $\emptyset$ (note that if $x, y \in L(D)$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). It can be easily seen that the set of all nontrivial filters $L(D)$ is $\{\{D\}, F_1, F_2, F_3, F_4, F_5, F_6\}$, where $F_1 = \{D, \{a, b\}\}$, $F_2 = \{D, \{a, c\}\}$, $F_3 = \{D, \{b, c\}\}$,

$$F_4 = \{D, \{a, c\}, \{a, b\}\{a\}\},$$

$F_5 = \{D, \{b, c\}, \{a, b\}\{b\}\}$ and $F_6 = \{D, \{a, c\}, \{c, b\}\{c\}\}$ with $(1 :_L S_i) \neq \{1\}$ for $1 \leq i \leq 6$. Moreover, L is an Artinian complemented lattice which is not a L-domain with $|\mathbb{I}(L)^*| = |\mathbb{F}(L)^*|$.

Compare the next theorem with Theorem 1.4 in [4].

Theorem 2.9. *If L is a complemented lattice which is not a L-domain, then the following conditions are equivalent:*

(1) $\mathbb{IG}(L)$ is a finite graph;

(2) L has only finitely many filters;

(3) Every vertex of $\mathbb{IG}(L)$ has finite degree.

Proof. (1) ⇒ (2) Let $|\mathcal{V}(\mathbb{IG}(L))| = n$ $(n \geq 1)$. Then by Theorem 2.6, L is an Artinian lattice which implies that every nontrivial filter of L is a vertex of $\mathbb{IG}(L)$ by Proposition 2.7. Hence L has only n nontrivial filters.

The implication (2) ⇒ (3) is clear.

(3) $\Rightarrow$ (1) Let every vertex of $\mathbb{IG}(L)$ has finite degree. Assume to the contrary, that $\mathbb{IG}(L)$ is an infinite graph. Let $G = T(\{x\})$ be a vertex of $\mathbb{IG}(L)$ and $F = (1 :_L G)$. Then $G \cap F = \{1\}$ by Proposition 2.4. Let H be an arbitrary subfilter of G. Then $H \cap F \subseteq G \cap F = \{1\}$ gives H and F are adjacent in $\mathbb{IG}(L)$. Thus if the set of subfilters of G (respectively F) is infinite, then F (receptively G) has infinite degree which is impossible. Hence the set of subfilters of G and F are finite and so they are Artinian filters. Since $\frac{L}{F} \cong G$ by Lemma 2.3, we get L is an Artinian lattice by Proposition 2.1. It follows that every nontrivial filter of L is a vertex of $\mathbb{IG}(L)$ by Proposition 2.7. We split the proof into two cases:

Case 1. $|\max(L)| = 1$. Since G has finite number of subfilters, L has a minimal (simple) filter $T(\{y\}) = K$. If $\mathfrak{m} = (1 :_L K)$, then $\mathfrak{m}$ is a maximal filter of L by Proposition 2.5 and so every proper filter of L is contained in $\mathfrak{m}$. So if H is any vertex of $\mathbb{IG}(L)$, then $K \cap H \subseteq \mathfrak{m} \cap K = \{1\}$ gives K is adjacent to H i.e K is adjacent to all other vertices of $\Gamma_M(L)$. Since K has finite degree, $\mathbb{IG}(L)$ is a finite graph.

Case 2. $|\max(L)| \geq 2$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be different maximal filters of L. Without loss of generality, let $y \in \mathfrak{m}_1$ and $y \notin \mathfrak{m}_2$; so $L = T(\{y\}) \wedge \mathfrak{m}_2$ with $\mathfrak{m}_2 \cap T(\{y\}) = \{1\}$. Then for each subfilter G of $T(\{y\})$, $G \cap \mathfrak{m}_2 \subseteq \mathfrak{m}_2 \cap T(\{y\}) = \{1\}$ gives the vertex G of $\mathbb{IG}(L)$ is adjacent to $\mathfrak{m}_2$. Thus the set of subfilters of $T(\{y\})$ is finite. Similarly, the set of subfilters of $\mathfrak{m}_2$ is finite. It follows that the set of filters of L is finite by Lemma 2.2. Hence $\mathbb{IG}(L)$ is a finite graph. □

Corollary 2.10. *If L is a complemented lattice, then $\mathbb{IG}(L)$ has n ($n \geq 1$) vertices if and only if L has only n nontrivial filters.*

Proof. By Proposition 2.7 and Theorem 2.9, we have $\mathbb{IG}(L)$ has n vertices if and only if L has only n nontrivial filters. □

Lemma 2.11. *If $F_1, \cdots, F_n$ are filters of L, then*

$$\bigcap_{i=1}^{n}(1 :_L F_i) = (1 :_L \bigwedge_{i=1}^{n} F_i).$$

Proof. Since $F_i \subseteq \bigwedge_{i=1}^n F_i$ for all $1 \leq i \leq n$, $(1 :_L \bigwedge_{i=1}^n F_i) \subseteq (1 :_L F_i)$, i.e. $(1 :_L \bigwedge_{i=1}^n F_i) \subseteq \bigcap_{i=1}^n (1 :_L F_i)$ holds. For the other inclusion, let $x \in \bigcap_{i=1}^n (1 :_L F_i)$. Then $x \vee F_i = \{1\}$ for all $1 \leq i \leq n$; so $x \vee \bigwedge_{i=1}^n F_i = \bigwedge_{i=1}^n (x \vee F_i) = \{1\}$, and so we have equality. □

Theorem 2.12. *Let L be a Noetherian lattice. If all nontrivial filters of L are vertices of $\mathbb{IG}(L)$, then L has only finitely many maximal filters.*

Proof. Assume to the contrary, that $\{\mathfrak{m}_i : i \in \mathbb{N}\}$ are distinct maximal filters of L. By assumption, $(1 :_L \mathfrak{m}_i) \neq \{1\}$ for all $i \in \mathbb{N}$. Then there exist $1 \neq a_i \in L$ such that $a_i \vee \mathfrak{m}_i = \{1\}$ which implies that $\mathfrak{m}_i = (1 :_L a_i)$ by maximality of $\mathfrak{m}_i$ for all $i \in \mathbb{N}$. If $i \neq t$, then $\mathfrak{m}_i \wedge \mathfrak{m}_t = L$ gives $0 = a \wedge b$ for some $a \in \mathfrak{m}_i$ and $b \in \mathfrak{m}_t$. It follows that $a_i = a_i \vee 0 = (a_i \vee a) \wedge (a_i \vee b) = a_i \vee b \in \mathfrak{m}_t$ which gives for each $i \in \mathbb{N}$, a_i is in every maximal filter $\mathfrak{m}_t$ for $i \neq t$. Since L is Noetherian, the chain $T(\{a_1\}) \subsetneqq T(\{a_1\}) \wedge T(\{a_2\}) \subsetneqq \cdots \subsetneqq \bigwedge_{i=1}^n T(\{a_i\}) \cdots$ must stabilize, and each step is proper since $\bigcap_{i=1}^s \mathfrak{m}_i = (1 :_L \bigwedge_{i=1}^s T(\{a_i\}))$ for each s by Lemma 2.10. Thus $|\max(L)| < \infty$. □

Lemma 2.13. *Let F be a filter of L that is maximal among all $(1 :_L x)$ of elements $x \neq 1$ of L. Then F is a prime filter.*

Proof. Let $a \neq 1$ be an element of L such that $F = (1 :_L a)$. If $x \vee y \in F$ for some $x, y \in L$ with $x \notin F$, then $a \vee x \neq 1$ gives $F \subseteq (1 :_L a \vee x)$; so $y \in (1 :_L a \vee x) = F$ by maximality of F. Thus F is prime. □

Lemma 2.14. *For the lattice L, the following conditions hold:*
(1) $\mathbb{I}(L)^* \neq \emptyset$ if and only if L is not a L-domain;
(2) $\mathbb{I}(L)^* \neq \emptyset$ if and only if $|\mathcal{I}(L)| \geq 2$.

Proof. (1) Let $\mathbb{I}(L)^* \neq \emptyset$. Then there exists a nontrivial filter F of L such that $(1 :_L F) \neq \{1\}$. Take $1 \neq y \in F$, as $F \neq \{1\}$. By assumption, there is an element $1 \neq x \in L$ such that $x \vee F = \{1\}$ which implies that $x \vee y = 1$, i.e. the result holds. Conversely, suppose that L is not a L-domain. Then there are elements $x \neq 1$ and $y \neq 1$ such that $x \vee y = 1$. Set $G = T(\{x\})$. Then $G \in \mathbb{I}(L)^*$ since $1 \neq y \in (1 :_L G)$.

(2) If G is a vertex of $\mathbb{IG}(L)$, then there exists $1 \neq x \in L$ such that $x \vee G = \{1\}$; so $x \vee y = 1$ for some $1 \neq y \in G$ which implies that $x, y \in \mathcal{I}(L)$. Conversely, if $1 \neq a \in \mathcal{I}(L)$, then $a \vee b = 1$ for some $1 \neq b \in L$ which implies that $T(\{a\}) \in \mathbb{I}(L)^*$, as required. □

Theorem 2.15. *If L is a Noetherian lattice, then either* $\mathbb{I}(L)^* = \emptyset$ *or at least one of the vertices of* $\mathbb{IG}(L)$ *is a prime filter.*

Proof. Suppose $\mathbb{I}(L)^* \neq \emptyset$. Then L is not a L-domain by Lemma 2.14. It is easy to see that the set of all $(1 :_L a)$ of elements $a \neq 1$ of L is a subset of $\mathbb{I}(L)^*$. Since L is Noetherian, there is a filter $F = (1 :_L z)$ of L which is maximal among all $(1 :_L a)$ of elements $a \neq 1$ of L. It follows from Lemma 2.13 that F is prime. Moreover, since L is not a L-domain, $F \neq \{1\}$ and so $F \in \mathbb{I}(L)^*$. □

Theorem 2.16. *Let* $\mathfrak{m}$ *be a maximal filter of a complemented lattice L which is not a L-domain. Then* $\mathfrak{m} \in \mathbb{I}(L)^*$ *if and only if* $\mathrm{Soc}(L) \neq \{1\}$.

Proof. Let $\mathfrak{m} \in \mathbb{I}(L)^*$ be a maximal filter and set $G = (1 :_L \mathfrak{m})$. Then $G \vee \mathfrak{m} = \{1\}$. Take $1 \neq a \in G$, as $G \neq \{1\}$. Now $a \vee \mathfrak{m} = \{1\}$ gives $\mathfrak{m} = (1 :_L a)$ by maximality of $\mathfrak{m}$. By Lemma 2.3, $\frac{L}{\mathfrak{m}} \cong T(\{a\})$ which gives $T(\{a\})$ is a simple filter of L, i.e. $\mathrm{Soc}(L) \neq \{1\}$. Conversely, assume that $\mathrm{Soc}(L) \neq \{1\}$ and S is a simple filter of L. Then $S = T(\{s\})$ for some $s \in S$. Set $H = (1 :_L S)$. Since $\frac{L}{H} \cong S$ and S is a simple filter of L, $H \neq \{1\}$ is a maximal filter of L which is not a L-domain and so $S \vee H = G \cap H = \{1\}$ gives $H \in \mathbb{I}(L)^*$. □

Theorem 2.17. *For the lattice L, the following conditions hold:*

(1) $\mathbb{IG}(L)$ *is connected graph of diameter not bigger than* 3*;*

(2) If $\mathbb{IG}(L)$ *contains a cycle, then* $\mathrm{gr}(\mathbb{IG}(L)) \leq 4$.

Proof. (1) Let F and G be vertices in $\mathbb{IG}(L)$ with $F \neq G$. Then $Z = (1 :_L F) \neq \{1\}$, $W = (1 :_L G) \neq \{1\}$, $F \cap Z = \{1\}$ and $W \cap G = \{1\}$. If $F \cap G = \{1\}$, then $\mathrm{d}(F, G) = 1$. If $F \cap G \neq \{1\}$ and $Z \cap W = \{1\}$, then F, G connected by a path $F \backsim Z \backsim W \backsim G$ of length ≤ 3. If $G \cap F \neq \{1\}$ and $W \cap Z \neq \{1\}$, then F, G connected by a path $F \backsim Z \cap W \backsim G$ of length $= 2$.

(2) Suppose that $\mathbb{IG}(L)$ contains a cycle and let $F_1 \backsim \cdots \backsim F_n \backsim F_1$ be a cycle with the minimum length. If $n \leq 4$, we are done. If $n > 4$, then $F_1 \cap F_4 \neq \{1\}$. We split the proof into three cases.

Case 1. $F_1 \cap F_4 = F_1$. Then $F_1 \cap F_3 \subseteq F_3 \cap F_4 = \{1\}$ gives $F_1 \backsim F_2 \backsim F_3 \backsim F_1$ is a cycle which is impossible. The case $F_1 \cap F_4 = F_4$ is similar.

Case 2. $F_1 \cap F_4 = F_2$. Then $F_2 \subseteq F_1$ gives $F_2 \cap F_n \subseteq F_1 \cap F_n = \{1\}$; so $F_2 \backsim \cdots \backsim F_n \backsim F_2$ is a cycle with length $n - 1$, a contradiction. The case $F_1 \cap F_4 = F_3$ is similar.

Case 3. $F_1 \cap F_4 \neq F_1, F_2, F_3, F_4$. Since $F_2 \cap (F_1 \cap F_4) = \{1\} = F_3 \cap (F_1 \cap F_4)$, we have $F_2 \backsim F_1 \cap F_4 \backsim F_3 \backsim F_2$ is a path which is a contradiction. Hence $n \leq 4$ and $\mathrm{gr}(\mathbb{IG}(L)) \leq 4$. □

The next theorem gives a more explicit description of lattices with a complete identity-filter graph. Compare the next theorem with Theorem 2.7 in [4].

Theorem 2.18. *If L is a complemented lattice which is not a L-domain, then* $\mathbb{IG}(L)$ *is a complete graph if and only if* $\mathcal{I}(L)$ *is a simple filter of L and* $L = \mathcal{I}(L) \odot (1 :_L \mathcal{I}(L))$*, where* $(1 :_L \mathcal{I}(L))$ *is a simple filter.*

Proof. One side is clear. To prove the other side, Assume that $\mathbb{IG}(L)$ is a complete graph and let $x \in \mathcal{I}(L) \setminus \{1\}$. Then $x \vee y = 1$ for some $1 \neq y \in L$. We show that $\mathcal{I}(L) = T(\{x\})$. If $x \vee a \in T(\{x\})$ for some $a \in L$, then $(x \vee a) \vee y = 1$ gives $T(\{x\}) \subseteq \mathcal{I}(L)$. For the reverse inclusion, suppose to the contrary, that $\mathcal{I}(L) \nsubseteq T(\{x\})$. So there exists $1 \neq z \in \mathcal{I}(L)$ such that $T(\{x\}) \neq T(\{z\})$. Then $x \vee z = 1$ and either $T(\{x, z\}) \neq T(\{z\})$ or $T(\{x, z\}) \neq T(\{x\})$. If the latter, then $T(\{x, z\}) \cap T(\{x\}) = T(\{x\}) = \{1\}$, a contradiction which implies that $z \in T(\{x, z\}) = T(\{x\})$, a contradiction. If $T(\{x, z\}) = T(\{z\})$, then $x \in T(\{z\})$ gives $x = z \vee b$ for some $b \in L$; hence $x = x \vee x = x \vee (z \vee b) = 1$, a contradiction. Thus $\mathcal{I}(L) \subseteq T(\{x\})$ and so we have equality. It is clear that $\mathcal{I}(L)$ (resp. $(1 :_L \mathcal{I}(L))$ does not have any nontrivial subfilter and so it is a simple filter (resp. $(1 :_L \mathcal{I}(L))$ is a simple filter). Now the assertion follows from Proposition 2.5. □

Compare the next theorem with Theorem 2.10 in [4].

Theorem 2.19. *Let L be a complemented lattice which is not a L-domain. If $\mathbb{I}(L)^* \subseteq \text{Spec}(L)$, then $L = S_1 \odot S_2$ for a pair of simple filters S_1 and S_2.*

Proof. Let $1 \neq x \in L$ be such that $x \vee y = 1$ and $x \wedge y = 0$ for some $1 \neq y \in L$. Then $y \in (1 :_L T(\{x\})) = S_1$ and $x \in (1 :_L T(\{y\})) = S_2$ gives S_1 and S_2 are prime filters of L. If $z \in L$, the $z = z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) \in S_1 \wedge S_2$; so $L = S_1 \wedge S_2$. If $a \in S_1 \cap S_2$, then $a = x \vee c = y \vee d$ for some $c, d \in L$. Thus $a = a \vee a = (x \vee c) \vee (y \vee d) = 1$ gives $L = S_1 \odot S_2$. If $\{1\} \neq S \subseteq S_1$, then $(1 :_L S_1) \subseteq (1 :_L S)$ gives S is a prime filter of L. As $x \vee y = 1 \in S$ and $y \notin S$, we get $x \in S$; hence $S = S_1$. Thus S_1 is simple. Similarly, S_2 is simple, as required. □

Corollary 2.20. *If L is a complemented lattice which is not a L-domain, then the following conditions are equivalent:*

(1) $\mathbb{I}(L)^* \subseteq \text{Max}(L)$*;*
(2) $\mathbb{I}(L)^* = \text{Max}(L)$*;*
(3) $\mathbb{I}(L)^* = \text{Spec}(L)$*;*
(4) $\mathbb{I}(L)^* \subseteq \text{Spec}(L)$*;*
(5) $L = S_1 \odot S_2$ *for a pair of simple filters S_1 and S_2.*

Proof. By Theorem 2.19, it suffices to show that if $\mathfrak{m}$ is a maximal filter of L, then it is prime. Let $x \vee y \in \mathfrak{m}$ with $x, y \notin \mathfrak{m}$. Then $T(\{x\}) \wedge \mathfrak{m} = L = T(\{y\}) \wedge \mathfrak{m}$ gives $m \wedge (x \vee a) = 0 = m' \wedge (y \vee b)$ for some $a, b \in L$ and $m, m' \in \mathfrak{m}$. It follows that $y = y \vee 0 = y \vee (m \wedge (x \vee a)) = (y \vee m) \wedge (x \vee y \vee a) \in \mathfrak{m}$ which is impossible. Thus $\mathfrak{m}$ is prime. □

We close this section with the following proposition, that gives us a characterization for lattices L for which every nontrivial cyclic filter F of L is a vertex of $\mathbb{IG}(L)$.

Proposition 2.21. *Every nontrivial cyclic filter of L is a vertex of $\mathbb{IG}(L)$ if and only if every element in L is an identity join.*

Proof. Let every nontrivial cyclic filter of L is a vertex of $\mathbb{IG}(L)$ and $x \in L$. Then $T(\{x\})$ is a vertex of $\mathbb{IG}(L)$ which implies that $(1 :_L T(\{x\}) \neq \{1\}$; hence there is an element $y \neq 1$ of L such that $y \vee T(\{x\}) = \{1\}$. So $x \vee y = 1$. Thus x is an identity join. The proof of the other implication is similar. □

3 Finiteness conditions and connectivity of $\mathbb{IG}_{fg}(L)$

Let us begin the following theorem.

Theorem 3.1. *If L is not a L-domain, then $\mathcal{V}(\mathbb{IG}_{fg}(L)) \neq \emptyset$ and $\mathbb{IG}_{fg}(L)$ is a connected graph with* $\dim(\mathbb{IG}_{fg}) \leq 3$.

Proof. Since L is not a L-domain, there are elements $x \neq 1$ and $y \neq 1$ of L such that $x \vee y = 1$. Then $y \in (1 :_L x)$ gives $T(\{x\}) \in \mathcal{V}(\mathbb{IG}_{fg}(L))$; so $\mathbb{IG}_{fg}(L)$ is not empty. Suppose that G and F are two distinct vertices of $\mathbb{IG}_{fg}(L)$. If $F \cap G = \{1\}$, we are done. So we may assume that $F \cap G \neq \{1\}$. There exist $K, H \in \mathcal{V}(\mathbb{IG}(L))$ such that $K \cap F = \{1\}$ and $H \cap G = \{1\}$, as $F, G \in \mathcal{V}(\mathbb{IG}(L))$ and $\mathbb{IG}(L)$ is a connected graph by Theorem 2.17 (1). If $K = H$, then $1 \neq h \in H = K$ gives $F \backsim T(\{h\}) \backsim G$ is a path in $\mathbb{IG}_{fg}(L)$. So suppose that $K \neq H$. Without loss of generality, assume that $x \in K \setminus H$ and $y \in H$. Then $T(\{x\}) \neq T(\{y\})$. If $T(\{x\}) \cap T(\{y\}) = \{1\}$, then $F \backsim T(\{x\}) \backsim T(\{y\}) \backsim G$ is a path in $\mathbb{IG}_{fg}(L)$. If $T(\{x\}) \cap T(\{y\}) \neq \{1\}$, then $F \backsim T(\{x\}) \cap T(\{y\}) \backsim G$ is a path in $\mathbb{IG}_{fg}(L)$. This completes the proof. □

Proposition 3.2. *If L is not a L-domain and an element $x \neq 1$ of L has a complement $y \neq 1$ in L, then $\mathbb{IG}_{fg}(L)$ has* ACC *on vertices if and only if L is a Noetherian lattice.*

Proof. One side is clear. To prove the other side, assume that $\mathbb{IG}_{fg}(L)$ has ACC on vertices. Suppose to the contrary, that L is not Noetherian. It follows from Theorem 2.6 that $\mathbb{IG}(L)$ has not ACC on vertices. So there is a strictly ascending chain of filters $F_1 \subsetneqq F_2 \subsetneqq \cdots$, where

$F_i \in \mathcal{V}(\mathbb{IG}(L))$ for $i \in \mathbb{N}$. If $1 \neq f_1 \in F_1$, then $T(\{f_1\}) \subseteq F_1$ gives $\{1\} \neq (1 :_L F_1) \subseteq (1 :_L f_1)$; so $T(\{f_1\}) \in \mathcal{V}(\mathbb{IG}_{fg}(L))$. There exists $f_2 \in F_2$ such that $f_2 \notin F_1$, as $F_1 \subsetneqq F_2$ which implies that $T(\{f_1\}) \subsetneqq T(\{f_1\}) \wedge T(\{f_2\})$. By continuing this process, we have a strictly chain

$$T(\{f_1\}) \subsetneqq T(\{f_1\}) \wedge T(\{f_2\}) \subsetneqq T(\{f_1\}) \wedge T(\{f_2\}) \wedge T(\{f_3\}) \subsetneqq \cdots$$

which is an infinite chain of elements of $\mathcal{V}(\mathbb{IG}_{fg}(L))$ which is impossible. Thus L is Noetherian. □

Theorem 3.3. *If L is not a L-domain and an element $x \neq 1$ of L has a complement $y \neq 1$ in L, then $\mathbb{IG}_{fg}(L)$ is a finite graph if and only if $\mathbb{IG}(L)$ is a finite graph.*

Proof. One side is clear. Suppose that $\mathbb{IG}_{fg}(L)$ is a finite graph. So $\mathbb{IG}_{fg}(L)$ has ACC on vertices gives L is Noetherian by Proposition 3.2 and hence $\mathcal{V}(\mathbb{IG}_{fg}(L)) = \mathcal{V}(\mathbb{IG}(L))$ by Proposition 2.1 (3). Thus $\mathbb{IG}(L)$ is a finite graph. □

Theorem 3.4. *If L is a complemented lattice which is not a L-domain, then the following conditions are equivalent:*
(1) $\mathbb{IG}_{fg}(L)$ is a finite graph;
(2) L has only finitely many filters;
(3) L has only finitely many finitely generated filters;
(4) Every vertex of $\mathbb{IG}_{fg}(L)$ has finite degree.

Proof. (1) $\Rightarrow$ (2) By Theorem 3.3, $\mathbb{IG}(L)$ is a finite graph. Now Theorem 2.9 shows that (2) holds. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (1) Assume to the contrary, that $\mathbb{IG}_{fg}(L)$ is not a finite graph. Then Theorem 3.3 shows that $\mathbb{IG}(L)$ is not a finite graph; so there exists a vertex G of $\mathbb{IG}(L)$ such that G is not a finite degree. If $1 \neq g \in G$, then $T(\{g\})$ is a vertex of $\mathbb{IG}_{fg}(L)$ with infinite degree which is impossible. Thus $\mathbb{IG}_{fg}(L)$ is a finite graph. □

Compare the next theorem with Proposition 2.3 in [14].

Theorem 3.5. *If L is a complemented lattice which is not a L-domain, then the following conditions are equivalent:*
(1) $\mathbb{IG}_{fg}(L)$ contains a universal vertex;
(2) $\mathbb{IG}(L)$ contains a universal vertex;
(3) $L = S \odot (1 :_L S)$, where S is a simple filter of L and $(1 :_L S)$ is a $(1 :_L S)$-domain.

Proof. (1) $\Rightarrow$ (2) Let G be a universal vertex of $\mathbb{IG}_{fg}(L)$. It suffices to show that for every $G \neq K \in \mathcal{V}(\mathbb{IG}(L))$, $G \cap K = \{1\}$. Let there exists $G \neq H \in \mathcal{V}(\mathbb{IG}(L))$, $G \cap H \neq \{1\}$ and look for a contradiction. So there is an element $x \neq 1$ such that $x \in G \cap H$. Since H is not a finitely generated filter, there exists $1 \neq y \in H$ such that $y \notin T(\{x\})$. It follows that $T(\{x\}) \subsetneqq T(\{x, y\}) \subsetneqq H$; so $x \in T(\{x, y\}) \cap G = \{1\}$, as G is a universal vertex which is impossible. Thus G is a universal vertex of $\mathbb{IG}(L)$.

(2) $\Rightarrow$ (3) Suppose that S is a universal vertex of $\mathbb{IG}(L)$and let $H \subsetneqq S$. Then $H = H \cap S = \{1\}$ gives S is a simple subfilter of L. Now the first part of the statement (3) is given by Proposition 2.5. If $(1 :_L S)$ is not a $(1 :_L S)$-domain, then there are elements $a \neq 1$ and $b \neq 1$ of $(1 :_L S)$ such that $a \vee b = 1$. Then $S \wedge T(\{b\})$ is an identity-filter of L, as $(S \wedge T(\{b\})) \cap T(\{a\}) = \{1\}$, which is not adjacent to S, a contradiction. Thus $(1 :_L S)$ is a $(1 :_L S)$-domain.

(3) $\Rightarrow$ (1) Let $L = S \odot (1 :_L S)$, where S is a simple filter of L. By Lemma 2.2, the nontrivial filters of L are of the form $S \wedge F$, S and F, where F is a nontrivial subfilter of $(1 :_L S)$. By our assumption, and since $\mathbb{IG}(L)$ is connected, we do not have any vertices of the form $S \wedge F$ such that $F \neq \{1\}$ by Lemma 2.2 (2) which implies that simple filter S is adjacent to every other vertex of $\mathbb{IG}_{fg}(L)$; so (1) holds. □

Corollary 3.6. *If L is a complemented lattice which is not a L-domain, then the following conditions are equivalent:*
(1) $\mathbb{IG}(L)$ contains a universal vertex;
(2) $\mathbb{IG}(L)$ is a star graph;
(3) $L = S \odot (1 :_L S)$, where S is a simple filter of L and $(1 :_L S)$ is a $(1 :_L S)$-domain.

Proof. The implication $(2) \Rightarrow (1)$ is clear. Indeed the implication $(1) \Rightarrow (3)$ is a direct consequence of Theorem 3.5.

$(3) \Rightarrow (2)$ By an argument like that in Theorem 3.5, S is adjacent to every other vertex, and since $(1 :_L S)$ is a $(1 :_L S)$-domain, none of the filters of the form F can be adjacent to each other. Thus $\mathbb{IG}(L)$ is a star graph. □

Corollary 3.7. *If L is an Artinian complemented lattice which is not a L-domain, then $\mathbb{IG}(L)$ contains a universal vertex if and only if $L = S_1 \odot S_2$, where S_1, S_2 are simple filters of L.*

Proof. Suppose that there exists a vertex of $\mathbb{IG}(L)$ which is adjacent to every other vertex. Then by Theorem 3.5, $L = S_1 \odot D$, where S_1 is a simple filter of L and D is a D-domain. Since L is Artinian, we get D is an Artinian D-domain and so D is also simple by Proposition 2.7 (2). Conversely, if $L = S_1 \odot S_2$, where S_1, S_2 are simple filters, then the graph $\mathbb{IG}(L)$ is a connected graph with two vertices S_1 and S_2, as required. □

Compare the next theorem with Proposition 2.4 in [14].

Theorem 3.8. *If L is a complemented lattice which is not a L-domain, then the following conditions are equivalent:*

(1) $\mathbb{IG}_{fg}(L)$ is a complete graph;

(2) $\mathbb{IG}(L)$ is a complete graph;

(3) $L = S_1 \odot S_2$, where S_1, S_2 are simple filters of L.

Proof. $(1) \Rightarrow (2)$ Let $\mathbb{IG}_{fg}(L)$ be a complete graph. It suffices to show that for every $F, G \in \mathcal{V}(\mathbb{IG}(L))$, $F \cap G = \{1\}$. Assume to the contrary, that there exist two distinct filters $F, G \in \mathcal{V}(\mathbb{IG}(L))$ such that $G \cap F \neq \{1\}$, where at least one of them is not finitely generated, say G. Thus there is an element $x \neq 1$ of L such that $x \in G \cap F$; so $T(\{x\}) \in \mathcal{V}(\mathbb{IG}_{fg}(L))$. As $T(\{x\}) \subsetneqq G$, there exists $y \in G$ such that $y \notin T(\{x\})$ and hence $T(\{x, y\} \cap T(\{x\}) \neq \{1\}$, a contradiction. Thus $\mathbb{IG}_{fg}(L)$ is a complete graph. The implication $(2) \Rightarrow (1)$ is clear.

$(2) \Rightarrow (3)$ Let $\mathbb{IG}(L)$ be a complete graph (so every vertex is universal). Then by Theorem 3.5, $L = S \odot D$, where S is a simple filter of L and D is a D-domain. If D has a nontrivial subfilter F, then F and D are vertices of $\mathbb{IG}(L)$ which are not adjacent, a contradiction. Thus D does not have any nontrivial subfilter so it is a simple filter. The implication $(3) \Rightarrow (2)$ is clear. □

Corollary 3.9. *If L is a complemented lattice which is not a L-domain, then the following conditions are hold:*

(1) $\mathbb{IG}(L)$ is a graph with one vertex if and only if L has only one filter $F \neq \{1\}$;

(2) $\mathbb{IG}(L)$ is a graph with two vertices if and only if $L = S_1 \odot S_2$, where S_1, S_2 are simple filters.

Proof. (1) By Theorem 2.9 this is clear.

(2) One side is clear. To prove the other side, let $\mathbb{IG}(L)$ be a graph with two vertices. As $\mathbb{IG}(L)$ is connected graph, then $\mathbb{IG}(L)$ is a complete (or star) graph. Then by Corollary 3.7 and Theorem 3.8, $L = S_1 \odot S_2$, where S_1, S_2 are simple filters. □

Lemma 3.10. *If L is a complemented lattice which is not a L-domain, then the following conditions are hold:*

(1) $\mathbb{IG}_{fg}(L) \cong K_1$ if and only if L has only one filter $F \neq \{1\}$;

(2) If $\mathbb{IG}_{fg}(L) \cong K_2$, then $L = S_1 \odot S_2$, where S_1, S_2 are simple filters.

Proof. (1) If $\mathbb{IG}_{fg}(L) \cong K_1$, then L is Noetherian by Proposition 3.2 and hence $\mathbb{IG}(L) \cong K_1$ which implies that $|\mathbb{F}(L) \setminus \{1\}| = 1$. Conversely, let L has only one filter $F \neq \{1\}$. Then L is Artinian gives $|\mathcal{V}(\mathbb{IG}(L))| = 1$. Since $\mathbb{IG}_{fg}(L)$ is a non-empty graph, $|\mathcal{V}(\mathbb{IG}_{fg}(L))| = 1$ and so $\mathbb{IG}_{fg}(L) \cong K_1$.

(2) By Proposition 3.2, L is Noetherian and hence $\mathbb{IG}(L) \cong K_2$. Then Corollary 3.9 shows that (2) holds. □

Proposition 3.11. *If L is a complemented lattice which is not a L-domain, then the following conditions are hold:*

(1) $\text{diam}(\mathbb{IG}(L)) = 0$ *if and only if* $\text{diam}(\mathbb{IG}_{fg}(L)) = 0$*;*
(2) $\text{diam}(\mathbb{IG}(L)) = 1$ *if and only if* $\text{diam}(\mathbb{IG}_{fg}(L)) = 1$*;*
(3) If $\text{diam}(\mathbb{IG}(L)) = 2$*, then* $\text{diam}(\mathbb{IG}_{fg}(L)) = 2$*;*
(4) If $\text{diam}(\mathbb{IG}(L)) = 3$*, then* $\text{diam}(\mathbb{IG}_{fg}(L)) = 2$ *or* 3*;*
(5) $\text{diam}(\mathbb{IG}_{fg}(L)) \leq \text{diam}(\mathbb{IG}(L))$.

Proof. (1) By Lemma 3.10, it is easy to see that $\mathbb{IG}(L) \cong K_1$ if and only if $\mathbb{IG}_{fg}(L) \cong K_1$, as needed.

(2) Indeed this is a direct consequence of Theorem 3.8.

(3) By (1) and (2), $\text{diam}(\mathbb{IG}_{fg}(L)) \neq 0, 1$ and so $2 \leq \text{diam}(\mathbb{IG}_{fg}(L)) \leq 3$. Let F and G be vertices of $\mathbb{IG}_{fg}(L)$ such that $F \cap G \neq \{1\}$. Since $F, G \in \mathcal{V}(\mathbb{IG}(L))$ and $\text{diam}(\mathbb{IG}(L)) = 2$, there exists a vertex K of $\mathbb{IG}(L)$ such that $F \backsim K \backsim G$ is a path in $\mathbb{IG}(L)$. If $1 \neq b \in K$, then $F \backsim T(\{b\}) \backsim G$ is a path in $\mathbb{IG}_{fg}(L)$. Thus $\text{diam}(\mathbb{IG}_{fg}(L)) = 2$.

(4) By assumption, $\text{diam}(\mathbb{IG}_{fg}(L)) \neq 0, 1$. By Theorem 3.1, $\text{diam}(\mathbb{IG}_{fg}(L)) = 2$ or 3.

(5) Parts (1), (2), (3) and (4) shows that (5) holds. □

Theorem 3.12. *Let L be a complemented lattice which is not a L-domain. Then* $\text{gr}(\mathbb{IG}(L)) = \text{gr}(\mathbb{IG}_{fg}(L))$.

Proof. Since $\mathbb{IG}_{fg}(L)$ is a subgraph of $\mathbb{IG}(L)$, we have $\text{gr}(\mathbb{IG}(L)) \leq \text{gr}(\mathbb{IG}_{fg}(L))$. So it suffices to show that $\text{gr}(\mathbb{IG}_{fg}(L)) \leq \text{gr}(\mathbb{IG}(L))$. By Theorem 2.17, $\text{gr}(\mathbb{IG}(L)) = \infty, 3$ or 4. If $\text{gr}(\mathbb{IG}(L)) = \infty$, it is clear that $\text{gr}(\mathbb{IG}_{fg}(L)) = \infty$. Suppose that $\text{gr}(\mathbb{IG}(L)) = 3$ and $F_1 \backsim F_2 \backsim F_3 \backsim F_1$ is a cycle in $\mathbb{IG}(L)$. If F_1, F_2 and F_3 are finitely generated, then $\text{gr}(\mathbb{IG}_{fg}(L)) = 3$. Now we split the proof into three cases.

Case 1. F_1 is not finitely generated and F_2, F_3 are finitely generated. Let $1 \neq f_1 \in F_1$. If $T(\{f_1\}) \neq F_2, F_3$, then $T(\{f_1\}) \backsim F_2 \backsim F_3 \backsim T(\{f_1\})$ is a triangle in $\mathbb{IG}_{fg}(L)$ and hence $\text{gr}(\mathbb{IG}_{fg}(L)) = 3$. If $T(\{f_1\}) = F_2$, then there exists $f_2 \in F_1$ such that $f_2 \notin F_2$, as $F_2 \subsetneqq F_1$ which implies that $F_2 \subsetneqq T(\{f_1, f_2\})$. If $T(\{f_1, f_2\}) \neq F_3$, then $T(\{f_1, f_2\}) \backsim F_2 \backsim F_3 \backsim T(\{f_1, f_2\})$ is a cycle in $\mathbb{IG}_{fg}(L)$. If $T(\{f_1, f_2\}) = F_3$, then there exists $f_3 \in F_1$ such that $F_3 \subsetneqq T(\{f_1, f_2, f_3\})$. Then $T(\{f_1, f_2, f_3\}) \backsim F_2 \backsim F_3 \backsim T(\{f_1, f_2, f_3\})$ is a cycle in $\mathbb{IG}_{fg}(L)$; so $\text{gr}(\mathbb{IG}_{fg}(L)) = 3$.

Case 2. F_1, F_2 are not finitely generated and F_3 is finitely generated. Let $1 \neq f_1 \in F_1$. If $T(\{f_1\}) \neq F_3$, then $T(\{f_1\}) \backsim F_3 \backsim F_2 \backsim T(\{f_1\})$ is a triangle in $\mathbb{IG}(L)$, where $T(\{f_1\}), F_3 \in \mathcal{V}(\mathbb{IG}_{fg}(L))$ and $F_2 \notin \mathcal{V}(\mathbb{IG}_{fg}(L))$. By an argument like that as in Case 1, the proof is complete. If $T(\{f_1\} = F_3$, then there exists $f_2 \in F_1$ such that $f_2 \notin F_3$, as $F_3 \subsetneqq F_1$. It follows that $F_3 \subsetneqq T(\{f_1, f_2\}$; so $T(\{f_1, f_2\} \backsim F_2 \backsim F_3 \backsim T(\{f_1, f_2\}$ is a cycle in $\mathbb{IG}(L)$ such that $T(\{f_1, f_2\}), F_3 \in \mathcal{V}(\mathbb{IG}_{fg}(L))$ and $F_2 \notin \mathcal{V}(\mathbb{IG}_{fg}(L))$. Now by same argument in Case 1, we have $\text{gr}(\mathbb{IG}_{fg}(L)) = 3$.

Case 3. F_1, F_2 and F_3 are not finitely generated. Let $1 \neq f_1 \in F_1$. By using of same argument in Case 1 for triangle $T(\{f_1\}) \backsim F_2 \backsim F_3 \backsim T(\{f_1\})$ where $T(\{f_1\}) \in \mathcal{V}(\mathbb{IG}_{fg}(L))$ and $F_2, F_3 \notin \mathcal{V}(\mathbb{IG}_{fg}(L))$, we have $\text{gr}(\mathbb{IG}_{fg}(L)) = 3$. If $\text{gr}(\mathbb{IG}(L)) = 4$, then by a similar argument, we get $\text{gr}(\mathbb{IG}_{fg}(L)) \leq 4$. Hence in every case, $\text{gr}(\mathbb{IG}(L)) = \text{gr}(\mathbb{IG}_{fg}(L))$. □

Corollary 3.13. *If L is a complemented lattice which is not a L-domain, then* $\text{gr}(\mathbb{IG}_{fg}(L)) \leq 4$.

Proof. This is a consequence of Theorem 2.17 and Theorem 3.12. □

Proposition 3.14. *If L is a complemented lattice which is not a L-domain, then $\mathbb{IG}(L)$ is a star graph if and only if $\mathbb{IG}_{fg}(L)$ is a star graph.*

Proof. If $\mathbb{IG}(L)$ is a star graph, then $\mathbb{IG}_{fg}(L)$ is also a star graph, as $\mathbb{IG}_{fg}(L)$ is an induced subgraph of $\mathbb{IG}(L)$. Conversely, assume that $\mathbb{IG}_{fg}(L)$ is a star graph and let G be a universal vertex of $\mathbb{IG}_{fg}(L)$. Let $G \neq H \in \mathcal{V}(\mathbb{IG}(L))$. We claim that H is only adjacent to G. Assume to the contrary, that $H \cap G \neq\neq \{1\}$. So there is an element $a \neq 1$ such that $a \in G \cap H$. Since H is not a finitely generated filter, there exists $1 \neq b \in H$ such that $b \notin T(\{a\})$. It follows that $T(\{a\}) \subsetneqq T(\{a, b\}) \subsetneqq H$; so $a \in T(\{a, b\}) \cap G = \{1\}$, as G is a universal vertex which is impossible. Thus $G \cap H = \{1\}$. Now suppose that there exists $G \neq F \in \mathcal{V}(\mathbb{IG}(L))$ such that $F \cap H = \{1\}$. Thus $G \backsim H \backsim F \backsim G$ is a triangle in $\mathbb{IG}(L)$ and so $\text{gr}(\mathbb{IG}(L)) = 3$; so $\text{gr}(\mathbb{IG}_{fg}(L)) = 3$ by Theorem 3.12, a contradiction, as $\mathbb{IG}_{fg}(L)$ is star graph. □

Theorem 3.15. *If L is a complemented lattice which is not a L-domain, then the following conditions are equivalent:*
(1) $\mathbb{IG}_{fg}(L)$ contains a universal vertex;
(2) $\mathbb{IG}(L)$ contains a universal vertex;
(3) $L = S \odot (1 :_L S)$, where S is a simple filter of L and $(1 :_L S)$ is a $(1 :_L S)$-domain.
(4) $\mathbb{IG}(L)$ is a star graph;
(5) $\mathbb{IG}_{fg}(L)$ is a star graph.

Proof. This is a direct consequence of Theorem 3.5, Corollary 3.6 and Proposition 3.14. □

We need the following lemma proved in [2, Theorem 3.5].

Lemma 3.16. *A connected graph is bipartite if and only if it contains no cycle of odd length.*

Theorem 3.17. *Let L be a complemented lattice which is not a L-domain such that* $\mathrm{gr}(\mathbb{IG}_{fg}(L)) = 4$*. Then $\mathbb{IG}(L)$ is a complete bipartite graph if and only if $\mathbb{IG}_{fg}(L)$ is a complete bipartite graph.*

Proof. If $\mathbb{IG}(L)$ is a complete bipartite graph, then $\mathbb{IG}_{fg}(L)$ is a complete bipartite graph, as $\mathbb{IG}_{fg}(L)$ is an induced subgraph of $\mathbb{IG}(L)$. Conversely, suppose that $\mathbb{IG}_{fg}(L)$ is a complete bipartite with parts $\mathcal{V}$ and $\mathcal{W}$. We may assume that $\mathbb{IG}_{fg}(L) \neq \mathbb{IG}(L)$. Let $G \in \mathcal{V}(\mathbb{IG}(L))$ with $G \notin \mathcal{V}(\mathbb{IG}_{fg}(L))$. We prove that either for each $H \in \mathcal{V}$, $H \cap G = \{1\}$ or for each $K \in \mathcal{W}$, $K \cap G = \{1\}$. As $\mathbb{IG}(L)$ is a connected graph with $\dim(\mathbb{IG}(L)) \leq 3$ and $\mathrm{gr}(\mathbb{IG}(L)) = 4$ (see Theorem 2.17 and Lemma 3.16), we have only one of the following cases:

Case 1. $G \cap H = \{1\}$ for some $H \in \mathcal{V}$. Now we claim that for each $X \in \mathcal{V}$, $X \cap G = \{1\}$. Suppose to the contrary, that there exists $H_1 \in \mathcal{V}$ such that $H_1 \cap G \neq \{1\}$. Then there exists $1 \neq a \in G$ such that $T(\{a\}) \cap H_1 \neq \{1\}$. As $T(\{a\}) \in \mathcal{V}(\mathbb{IG}_{fg}(L))$, we get $T(\{a\}) \in \mathcal{V}$ and hence $T(\{a\}) \cap H \neq \{1\}$, a contradiction. Thus for each $X \in \mathcal{V}$, $X \cap G = \{1\}$.

Case 2. $G \cap K = \{1\}$ for some $K \in \mathcal{W}$. By a similar argument as in Case 1, for each $Y \in \mathcal{W}$, $Y \cap G = \{1\}$.

Case 3. $K \cap G = \{1\}$ for some $K \in \mathcal{V}(\mathbb{IG}(L))$, where either for each $H \in \mathcal{V}$, $H \cap K = \{1\}$ or for each $F \in \mathcal{W}$, $F \cap K = \{1\}$, and for each $H \in \mathcal{V}$, $F \in \mathcal{W}$, $H \cap G \neq \{1\}$, $F \cap G \neq \{\}$. Without loss of generality, let for every $H \in \mathcal{V}$, $H \cap K = \{1\}$. We show that for each $F \in \mathcal{W}$, $G \cap F = \{1\}$. Suppose to the contrary, that there exists $F_1 \in \mathcal{W}$ such that $F_1 \cap G \neq \{1\}$. Then there exists $1 \neq a \in G$ such that $T(\{a\}) \cap F_1 \neq \{1\}$. Since $T(\{a\}) \in \mathcal{V}(\mathbb{IG}_{fg}(L))$, $T(\{a\}) \in \mathcal{W}$ and $K \backsim H \backsim T(\{a\}) \backsim K$ form a triangle in $\mathbb{IG}(L)$, a contradiction. So for each $F \in \mathcal{W}$, $F \cap G = \{1\}$ which is impossible. Hence this case implies a contradiction in general.

Thus for every $G \in \mathcal{V}(\mathbb{IG}(L))$ with $G \notin \mathcal{V}(\mathbb{IG}_{fg}(L))$, either for each $H \in \mathcal{V}$, $H \cap G = \{1\}$ or for each $K \in \mathcal{W}$, $K \cap G = \{1\}$. Set

$$\mathbb{V} = \mathcal{V} \cup \{H \in \mathcal{V}(\mathbb{IG}(L)) : \text{for each } F \in \mathcal{W}, F \cap H = \{1\}\}$$

and $\mathbb{W} = \mathcal{W} \cup \{F \in \mathcal{V}(\mathbb{IG}(L)) : \text{for each } H \in \mathcal{V}, H \cap F = \{1\}\}$. It suffices to show that if $H \in \mathbb{V} \setminus \mathcal{V}$ and $F \in \mathbb{W} \setminus \mathcal{W}$, then $H \cap F = \{1\}$. Assume to the contrary, that $H \cap F \neq \{1\}$. So there exists $1 \neq a \in H$ such that $T(\{a\}) \cap F \neq \{1\}$. Since $T(\{a\}) \in \mathcal{V}(\mathbb{IG}_{fg}(L))$, $T(\{a\}) \in \mathcal{W}$ and $a \in T(\{a\}) = H \cap T(\{a\}) = \{1\}$), a contradiction. Thus $\mathbb{IG}(L)$ is a complete bipartite graph with parts $\mathbb{V}$ and $\mathbb{W}$. □

Lemma 3.18. *For the lattice L, $\mathbb{IG}(L)$ is a bipartite graph if and only if $\mathbb{IG}(L)$ is a triangle-free graph.*

Proof. One side is clear since bipartite graphs are triangle-free. To prove the other side, let $\mathbb{IG}(L)$ be a triangle-free graph. By contrary assume that $\mathbb{IG}(L)$ is not bipartite. So $\mathbb{IG}(L)$ contains an odd cycle by Lemma 3.16. Let $C = F_1 \backsim F_2 \backsim \cdots \backsim F_{2n+1} \backsim F_1$ be a shortest odd cycle in $\mathbb{IG}(L)$ for some $n \in \mathbb{N}$ (so $n \geq 2$). As C is a shortest odd cycle in $\mathbb{IG}(L)$, we have $F_3 \cap F_{2n+1}$ is a vertex. If $F_1 = F_3 \cap F_{2n+1}$, then $F_4 \cap F_1 = \{1\}$ which implies that $F_1 \backsim F_4 \backsim \cdots \backsim F_{2n+1} \backsim F_1$ is an odd cycle which is impossible. So $F_1 \neq F_3 \cap F_{2n+1}$. If $F_2 = F_3 \cap F_{2n+1}$, then F_2, F_3 and F_4 would form a triangle, a contradiction. Hence $F_2 \neq F_3 \cap F_{2n+1}$. Now F_1, F_2 and $F_3 \cap F_{2n+1}$ form a triangle in $\mathbb{IG}(L)$ which is a contradiction. Thus $\mathbb{IG}(L)$ is a bipartite graph. □

Theorem 3.19. *If L is a complemented lattice which is not a L-domain, then $\mathbb{IG}(L)$ is a bipartite graph if and only if $\mathbb{IG}_{fg}(L)$ is a bipartite graph.*

Proof. One side is clear. Let $\mathbb{IG}_{fg}(L)$ be a bipartite graph. Assume to the contrary, that $\mathbb{IG}(L)$ is not a bipartite graph. Then By Theorem 3.12 and Lemma 3.18, $\mathrm{gr}(\mathbb{IG}(L)) = \mathrm{gr}(\mathbb{IG}_{fg}(L)) = 3$ which implies that $\mathbb{IG}_{fg}(L)$ contains an odd cycle, so $\mathbb{IG}_{fg}(L)$ is not bipartite by Lemma 3.16 which is impossible. Thus $\mathbb{IG}(L)$ is a bipartite graph. □

4 Conclusions and future work

In this work we investigated many fundamental properties of the graph $\mathbb{IG}(L)$ such as connectivity, the diameter, the girth, and obtain some interesting results with finiteness conditions on them. However, in future work, shall search the supplement of this graph and research on deeper properties of them.

References

[1] *D.F. Anderson and P.S. Livingston,* The zero-divisor graph of a commutative rings*, J. Algebra* **217***, 434-447, (1999).*

[2] *L. Anderson,* A first course in Discrete Mathematics*, Springer Undergraduate Mathematics Series, 2000.*

[3] *I. Beck,* Coloring of commutative rings*, J. Algebra,* **116***, 208-226, (1988).*

[4] *M. Behboodi and Z. Rakeei,* The annihilating-ideal graph of commutative rings I*, J. Algebra App.,* **10 (4)***, 727-739, (2011).*

[5] *G. Călugăreanu,* Lattice Concepts of Module Theory*, Kluwer Academic Publishers, 2000.*

[6] *S. Ebrahimi Atani and M.sedghi Shanbeh Bazari,* Decomposable filters of lattices*, Kragujevac J. Math.,* **43 (1)***, 59-73, (2019).*

[7] *S. Ebrahimi Atani, S. Dolati Pish Hesari, M. Khoramdel and M. Sedghi Shanbeh Bazari,* A simiprime filter-based identity-summand graph of a lattice*, Le Matematiche Vol. LXX III - Fasc. II, pp. 297-318 (2018).*

[8] *S. Ebrahimi Atani and M. Chenari,* Supplemented property in the lattices*, Serdica Math. J.* **46 (1)***, 73-88, (2020).*

[9] *S. Ebrahimi Atani,* G-Supplemented property in the lattices*, Mathematica Bohemica,* **147 (4)***, 525-545, (2022).*

[10] *V. Kulal, A. Khairnar and K. Masalkar,* Annihilator ideal graph of a lattice*, Palestine J. Math.* **11 (4)***, 195-204, (2022).*

[11] *T. Kavitha,* Simple Expansions in the Lattice of Čech Closure Operators*, Palestine J. of Math.* **8 (1)***, 1-7, (2019).*

[12] *J. Matczuk and A. Majidinya,* Sum-essential graphs of modules*, J. Algebra Appl.,* **20 (11)** *(2021) 2150211. DOI: 10.1142/S021949882150211X.*

[13] *N. Rafi1, R. K. Bandaru and M. Srujana,* Closure prime spectrum of MS-almost distributive lattices*, Palestine J. Math.* **10 (2)***, 834-844, (2021).*

[14] *R. Taheri and A. Tehranian,* Finitely generated annihilating-ideal graph of commutative rings*, Int. J. Industrial Mathematics,* **10 (4)***, 375-383, (2018).*

[15] *D. B. West,* Introduction to Graph Theory*, 2nd edn. Prentice Hall, USA, 2001.*

Author information

Shahabaddin Ebrahimi Atani, Department of Mathematics, Faculty of Mathematical Sciences, Rasht, Iran.
E-mail: `ebrahimiatani@gmail.com`

Received: 2022-05-25
Accepted: 2024-02-14