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CANONICALLY CONSISTENT CAYLEY SIGNED GRAPHS

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The authors dedicate this paper on the occasion of the birthday of Dr. T. Tamizh Chelvam (Manonmaniam Sundaranar University Tirunelveli, Tamilnadu, India) who always encouraged to work for the growth of the subject.

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Abstract *Let S be a subset of a finite group Γ. The* Cayley signed graph*, denoted by $\Sigma = Cay_{\Sigma}(S, \sigma)$ has vertex set Γ and two distinct vertices $x, y \in \Gamma$ are joined by an edge from x to y if and only if there exists $s \in S$ such that $x = sy$, where Σ is a signed graph whose underlying graph is Γ and $\sigma : E(\Gamma) \to \{+, -\}$ is a function defined as*

$$\sigma(x\,y) = \begin{cases} +, & \textit{if } x \in S \textit{ or } y \in S; \\ -, & \textit{otherwise.} \end{cases}$$

In this manuscript, we have characterized the Cayley set and generating sets for which Cayley signed graphs are canonically consistent.

1 Introduction

It is well-known that the structure of Cayley graphs depends on a specific set of generators, i.e., the same algebraic structure can have a different Cayley graph. In 1878, the notion of Cayley graph was introduced by Cayley [1] to illustrate the concept of 'group' and a 'generating' subsets. The formal definition is as follows: The Cayley graph *of group Γ, denote by $Cay(\Gamma, S)$ is a simple graph with the vertex set Γ, and two vertices x and y are adjacent if and only if there exists $s \in S$ such that $x = sy$, where S is a subset of Γ. From the survey of the literature it is found that these graphs play an important role in combinatorial and geometric graph theory. For detailed study of Cayley graphs the reader is referred to [6] and [5].*

*On the social psychology front; according to the Harary model [3], 'social networks' can be represented by a graph often called '*Signed graph*' introduced by the Harary in 1950's. According to Harary; The graph Γ equipped with a signature σ is called a* signed graph*, denoted by $\Sigma := (\Gamma, \sigma) = (V, E, \sigma)$, where $\Gamma = (V, E)$ is an* underlying graph *and $\sigma : E \to \{+, -\}$ is the signature that labels each edge of Γ either by '$+$' or '$-$'. The edge which receives a positive(negative) sign is called a positive(negative) edge. A signed graph is an* all-positive*(*all-negative*) if all its edges are positive(negative); further, it is said to be* homogeneous *if it is either all-positive or all-negative and* heterogeneous *otherwise. By $d^-(v)(d^+(v))$, we mean the negative(positive) degree of a vertex v. For a signed graph Σ, the negation $\eta(\Sigma)$ of a signed graph Σ is a signed graph obtained from Σ by negating the sign of every edge of Σ. From the last two decades, signed graphs have been studied a lot due to its applications in social networks, systems biology, and integrated circuit design. Subsequently signed graphs have turned out to be valuable in many other areas of research. For detailed study of signed graphs the reader is referred to the bibliography paper by Zaslavsky [7]. Inspired by the applications of these two graphs, viz., Cayley graph and signed graph, in this paper, we intend to study the Cayley graph in the realm of signed graphs. In this regard, we have characterized the generating sets S for which $Cay_{\Sigma}(\Gamma, S)$ is $\mathbb{C}$-consistent.*

Throughout the paper, we mean by 'Γ' the finite abelian group and by $\mathbb{Z}_n$, the group of integers modulo n, the sets $Z(\mathbb{Z}_n)$ and $U(\mathbb{Z}_n)$ are defined as; $U(\mathbb{Z}_n) = \{x : gcd(x,n) = 1\}$ and $Z(\mathbb{Z}_n) = \{y : gcd(y,n) \neq 1\}$. Also, $U(\mathbb{Z}_m \times \mathbb{Z}_n)$ is defined as; $U(\mathbb{Z}_m \times \mathbb{Z}_n) = \{(x,y) : gcd(x,m) = 1 \text{ \& } gcd(y,n) = 1\}$. Without exception, the notation '$\mathbb{Z}_2^t$' refers to $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{(t-times)}$. All graphs considered here are finite, simple and undirected. For terminology and notations from group theory and graph theory not defined or mentioned in this paper, the reader is referred to [4] and [3], respectively.

2 Cayley Sets and Generating Sets

In this section, we briefly recall the notion of Cayley set and Generating set and derived some observations needed in the sequel of this paper.

A nonempty subset S of Γ is called Cayley set *or* symmetric Cayley set *if $e \notin S$ and for every $a \in S$, $a^{-1} \in S$. If Cayley set S generates a group Γ, then S is called* generating set *or* symmetric generating set. *Consequently, for a given group Γ of order n*

$$1 \leq |S| \leq n - 1. \tag{2.1}$$

However, if S generates Γ, then

$$2 \leq |S| \leq n - 1. \tag{2.2}$$

The following example illustrate the above concepts:

Example 2.1. Let $\Gamma \cong \mathbb{Z}_4$. Then possible Cayley sets are $S_1 = \{2\}$, $S_2 = \{1,3\}$, $S_3 = \{1,2,3\}$ and out of them only $S_2 = \{1,3\}$ is a generating set.

Observation 2.1. The following observations are straightforward and can directly be obtained from the definition of Cayley set and generating set:

i) Let S be Cayley set and $\{a, a^{-1}\} \subseteq S$, where $a \in U(\Gamma)$. Then S becomes a generating set.

ii) If $\Gamma \cong \mathbb{Z}_p$; p is a prime number, then Cayley sets and generating sets are both equal.

If $|S|$ is either 1 or $(n-1)$, then such S is called an extreme Cayley set and if $|S|$ is either 2 or $(n-1)$, then such S is called an extreme generating set. Notice that if $|\Gamma|$ is odd, then $|S|$ can be even. However, if $|\Gamma|$ is even, then $|S|$ may be even or odd.

3 Canonically Consistent Cayley Signed Graphs

A marked signed graph is a signed graph each vertex of which is designated to be positive or negative and it is consistent if every cycle in the signed graph possesses an even number of negative vertices. Consistent marked graphs were introduced by Beineke and Harary [2], and the concept was motivated by communication networks. A marked signed graph is an ordered pair $\Sigma_\mu = (\Sigma, \mu)$, where $\Sigma = (\Gamma, \sigma)$ is a signed graph and $\mu : V(\Sigma) \to \{+,-\}$ is a function from the vertex set $V(\Sigma)$ into the set $\{+,-\}$, called marking *of Σ. In particular, σ induces a unique marking μ_σ defined by*

$$\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e),$$

where E_v is the set of edges incident at v in Σ, is called a canonical marking *of Σ. If every vertex of a given signed graph Σ is canonically marked, then a cycle Z in Σ is said to be* canonically consistent *($\mathbb{C}$-consistent) if it contains an even number of negative vertices and the given signed graph Σ is said to be $\mathbb{C}$-consistent if every cycle in it is $\mathbb{C}$-consistent.*

Lemma 3.1. *If a signed graph Σ is an all-positive, then it is $\mathbb{C}$-Consistent.*

Proof. If a signed graph Σ is an all-positive, then all the vertices will receive positive signs under canonical marking. Therefore, Σ is trivially $\mathbb{C}$-consistent trivially. $\square$

The examples of $\mathbb{C}$*-consistent Cayley signed graph* $Cay_\Sigma(\Gamma, S)$ *associated with* Γ *are shown in Figure 1 and Figure 2, in which dashed line represents negative edges and solid line represents the positive edges, respectively.*

Example 3.2. Let $\Gamma \cong \mathbb{Z}_6$. Then there are total seven Cayley sets, namely, $S_1 = \{3\}$, $S_2 = \{2, 4\}$, $S_3 = \{1, 5\}$, $S_4 = \{2, 3, 4\}$, $S_5 = \{1, 3, 5\}$, $S_6 = \{1, 2, 4, 5\}$, $S_7 = \{1, 2, 3, 4, 5\}$, whose canonically marked Cayley signed graphs associated with Cayley sets S_1, S_2, S_3, S_4, S_5, S_6 and S_7 are shown in Figure 1. From Figure 1, one can notice that $Cay_\Sigma(\mathbb{Z}_6, S_1)$ does not consist of

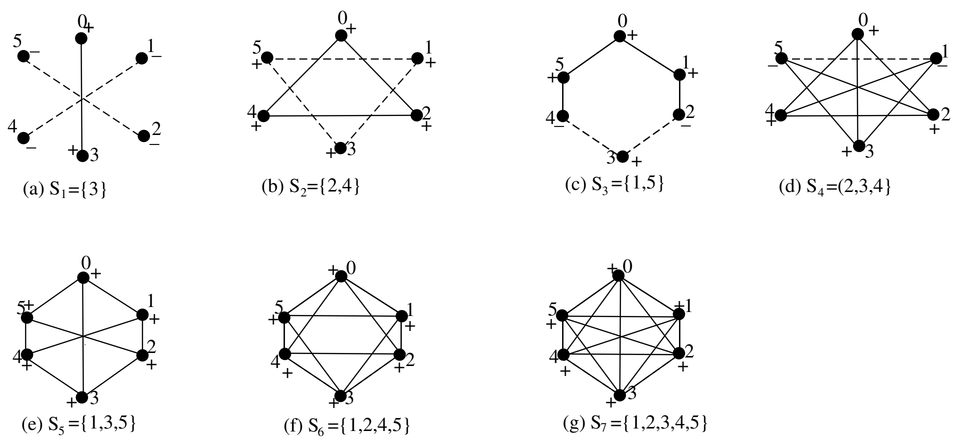

Figure 1. The canonically marked Cayley signed graphs $Cay_\Sigma(\mathbb{Z}_6, S)$

negative cycle as the underlying graph $Cay(\mathbb{Z}_6, S_1)$ is 1-regular graph. Therefore, $Cay_\Sigma(\mathbb{Z}_6, S_1)$ is $\mathbb{C}$-consistent. Clearly, $Cay_\Sigma(\mathbb{Z}_6, S_2)$ has two components of C_3 in which one of them is an all-positive and other is an all-negative and under the canonical marking all vertices will be marked with positive sign, which indicates that $Cay_\Sigma(\mathbb{Z}_6, S_2)$ is $\mathbb{C}$-consistent. Also $Cay_\Sigma(\mathbb{Z}_6, S_3)$ is a cycle graph C_6 having exactly two negative edges. Now under the canonical marking precisely two vertices 2 and 4 are marked with negative sign and remaining four vertices are marked positive sign. This depicts that cycle (graph) consists of even number of negatively marked vertices, and hence $Cay_\Sigma(\mathbb{Z}_6, S_3)$ is $\mathbb{C}$-consistent.

On the other hand, in $Cay_\Sigma(\mathbb{Z}_6, S_4)$ there exist a cycle Z, namely, $Z = (0, 3, 1, 4, 0)$, under the canonical marking among all the vertices only two vertices, namely, 1 and 5 are marked with negative sign and rest four vertices are marked with positive sign in $Cay_\Sigma(\mathbb{Z}_6, S_4)$, this means that there exist a cycle Z in which only one vertex has $(-)ive$ marking which provide the presence of cycle with odd number of negative vertices in $Cay_\Sigma(\mathbb{Z}_6, S_4)$. Thus $Cay_\Sigma(\mathbb{Z}_6, S_4)$ is not $\mathbb{C}$-consistent. For the remaining Cayley signed graph, namely, $Cay_\Sigma(\mathbb{Z}_6, S_5)$, $Cay_\Sigma(\mathbb{Z}_6, S_6)$, and $Cay_\Sigma(\mathbb{Z}_6, S_7)$ it can easily be observed that all are homogeneous all-positive, therefore by Lemma 3.1, all are $\mathbb{C}$-consistent.

Example 3.3. Let $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then there are seven possible Cayley sets, i.e. $S_1 = \{(0, 1)\}$, $S_2 = \{(1, 0)\}$, $S_3 = \{(1, 1)\}$, $S_4 = \{(0, 1), (1, 0)\}$, $S_5 = \{(0, 1), (1, 1)\}$, $S_6 = \{(1, 0), (1, 1)\}$, $S_7 = \{(0, 1), (1, 0), (1, 1)\}$. Whose canonically marked Cayley signed graphs associated with Cayley sets S_1, S_2, S_3, S_4, S_5, S_6 and S_7 are shown in Figure 2.

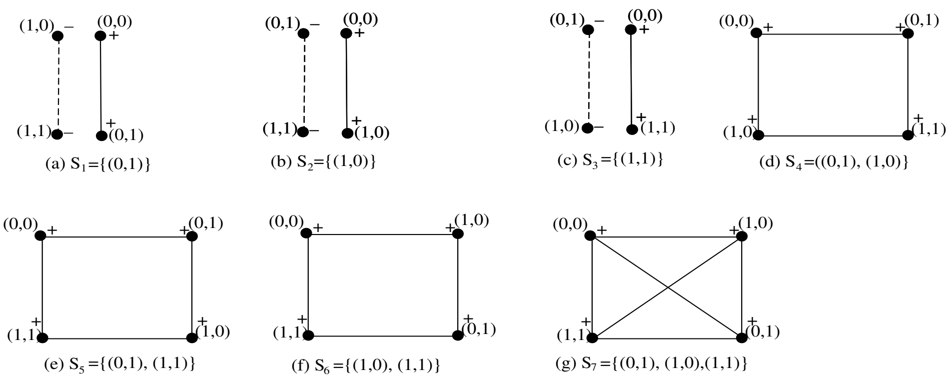

Figure 2. The canonically marked signed Cayley graphs

Notice that the Cayley signed graphs associated with Cayley sets S_1, S_2 and S_3 are 1-regular graph. Therefore, there does not exist any cycle in $Cay_\Sigma(\mathbb{Z}_6, S_1)$, $Cay_\Sigma(\mathbb{Z}_6, S_2)$ and $Cay_\Sigma(\mathbb{Z}_6, S_3)$ and hence, they are $\mathbb{C}$-consistent trivially. On the other hand the Cayley signed graphs associated with Cayley sets S_4, S_5, S_6, and S_7 are homogeneous all-positive, therefore in light of Lemma 3.1, these are all $\mathbb{C}$-consistent.

From the foregoing analysis in examples one can observe that there are some Cayley sets/Generating sets with respect to which Cayley signed graph is $\mathbb{C}$-Consistent. Therefore one can have the following problem:

Problem 3.1. Characterize the Cayley sets/Generating sets S with respect to which Cayley signed graph is canonically consistent.

Towards attempting the Problem 3.1, several results have been established. First we shall establish the result for $\mathbb{C}$-Consistent Cayley signed graphs.

Theorem 3.4. *Let Γ be a finite abelian group of order $n \geqslant 2$ and S be an extreme Cayley set. Then $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.*

Proof. Let S be an extreme Cayley set. Now first suppose that $|S| = 1$, then $Cay(\Gamma, S)$ is 1-regular graph and clearly there is no cycle in $Cay_\Sigma(\Gamma, S)$. Therefore, $Cay_\Sigma(\Gamma, S)$ is trivially $\mathbb{C}$-Consistent. Secondly, if $|S| = n - 1$, then it can easily be seen that all non-zero elements belongs to S, therefore $Cay_\Sigma(\Gamma, S)$ is an all-positive graph. Since all the vertices receive positive sign under the canonical marking, so $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent. □

Theorem 3.5. *Let Γ be a finite abelian group of order $n > 2$ and S be a Cayley set with $|S| = 2$. Then $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.*

Proof. Let Γ be a finite abelian group of order $n > 2$ and S be a Cayley set with $|S| = 2$. This implies that $Cay(\Gamma, S)$ is a 2-regular graph. It means either $Cay(\Gamma, S)$ is a cycle graph or is isomorphic to copies of cycles. Now these two cases arise.
Case-1: If $Cay(\Gamma, S)$ is isomorphic to a cycle graph, then an edge in its corresponding signed graph $Cay_\Sigma(\Gamma, S)$ is negative if and only if none of end vertices belongs to S. Since each negative edge produces two negative vertices under the canonical marking and the underlying graph is cycle, therefore the negative vertices will be even in number. Hence, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-consistent.
Case-2: If $Cay(\Gamma, S)$ is isomorphic to copies of cycle graph, then in its corresponding signed graph $Cay_\Sigma(\Gamma, S)$ an edge is positive if and only if both end vertices belong to S. Since $|S| = 2$, so at most four edges in $Cay_\Sigma(\Gamma, S)$ are positive in one component and the remaining other components are all-negative. Also it is known that each negative edge produces two negative vertices under the canonical marking, therefore in each component(cycle) there are even number(may be zero) of negatively marked vertices through $\mathbb{C}$-marking in $Cay_\Sigma(\Gamma, S)$. Hence, $Cay_\Sigma(\Gamma, S)$ is consistent. □

Theorem 3.6. *Let Γ be a finite cyclic group with $|\Gamma| = 2p$, $p > 2$, p is a prime number and S be a Cayley set such that $S \subseteq Z^*(\Gamma) \setminus \{p\}$ and $|S| = 4$. Then $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.*

Proof. Let $\Gamma \cong \mathbb{Z}_n$ with $|\Gamma| = 2p$, $p > 3$ and S be Cayley set such that $S \subseteq Z^*(\Gamma) \setminus \{p\}$ and $|S| = 4$. This implies that S contains all even positive integers. It is well-known that the difference of two even integers or difference of two odd integers is always even. In this way one can notice that even integer and zero are connected with positive edges and odd integer with odd integer are connected with negative edges in $Cay_\Sigma(\Gamma, S)$ but $|S| = 4$, so the negative degree of odd integer is four and the negative degree of even integer is zero. Therefore, under canonical marking all vertices receive positive signs. Hence, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent. □

Theorem 3.7. *Let Γ be a finite cyclic group and S be a Cayley set. Then $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent if one of following conditions hold:*

(i) $S = U(\Gamma) \cup \{p\}$*, where* $|\Gamma| = 2p$*,* p *is a prime;*

(ii) $S = \{|\Gamma|/4, 3|\Gamma|/4\}$*, where* $|\Gamma|$ *is multiple of* 4*;*

(iii) $S = \{x : x \neq x^{-1}; \forall x \in \Gamma\}$*, where* $|\Gamma|$ *is an even;*

(iv) $S = U(\Gamma)$*, where* $|\Gamma| = p^k, k \geq 1$*;*

(v) $S = \{|\Gamma|/4, 3|\Gamma|/4\} \cup U(\Gamma)$*, where* $|\Gamma|$ *is an odd multiple of* 4*.*

Proof. (i) Let Γ be a finite cyclic group with $|\Gamma| = 2p$. If $S = U(\Gamma) \cup \{p\}$, then S contains all odd positive integer upto $2p$, and hence $|S| = p$. This indicates that only even positive integers remain outside S. We know in $Cay_\Sigma(\Gamma, S)$ an edge is negative if and only if none of the end vertices belongs to S. Since the difference of two even positive integers is always an even integer, there does not exist a negative edge in $Cay_\Sigma(\Gamma, S)$. Therefore, $Cay_\Sigma(\Gamma, S)$ is homogenous an all-positive, and hence in view of Lemma 3.1, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

(ii) If $S = \{|\Gamma|/4, 3|\Gamma|/4\}$, where $|\Gamma|$ is multiple of 4(say $4k$), then $Cay_\Sigma(\Gamma, S)$ is isomorphic to $\underbrace{C_4 \cup C_4 \cup \cdots \cup C_4}_{k-times}$ in which all edges are positive in one component of $Cay_\Sigma(\Gamma, S)$ formed by vertices, namely, $0, |\Gamma|/4, 3|\Gamma|/4$ and $|\Gamma|/2$. In $Cay_\Sigma(\Gamma, S)$ the remaining components are an all-negative. Therefore, every vertex will receive positive sign under the canonical marking. Hence, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

(iii) Let $|\Gamma|$ is even. Then there is only one non-trivial self inverse element (say x) in Γ. If S contains all non-zero elements except x, then each edge has an end vertex in S. This shows that $Cay_\Sigma(\Gamma, S)$ is an all-positive, and hence by Lemma 3.1, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

(iv) If $S = U(\Gamma)$, where $|\Gamma| = p^k, k \geq 1$, then the elements which do not belong to S are some multiples of p. Note that in $Cay_\Sigma(\Gamma, S)$ an edge is negative if and only if none of the end vertices belongs to S. Also, the difference of multiples of p is again a multiple of p, which ensures that no two elements outside S are adjacent in $Cay_\Sigma(\Gamma, S)$ with negative edge. This implies that there is no negative edge in $Cay_\Sigma(\Gamma, S)$. Therefore in view of Lemma 3.1, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

(v) If $S = \{|\Gamma|/4, 3|\Gamma|/4\} \cup U(\Gamma)$, where $|\Gamma|$ is an odd multiple of 4, then S contains all odd integers upto $|\Gamma|$. This implies only even numbers are outside S and the difference of two even numbers is always even number, so there does not exist a negative edge in $Cay_\Sigma(\Gamma, S)$. This shows that $Cay_\Sigma(\Gamma, S)$ is an all-positive. Thus, in light of Lemma 3.1, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent. □

Theorem 3.8. *Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, where $p_i's$ are primes, $k_i's$, i and t are positive integers. Assume that $S = U(\Gamma)$ be a generating set of Γ. If at least one of $\mathbb{Z}'_{p_i}s (\ncong \mathbb{Z}_2)$ has $\mathbb{Z}_2$ as a quotient, then $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.*

Proof. Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, $p_i's$ are prime numbers, $k_i's$, i and t are positive integers. Let $S = U(\Gamma)$ be a generating set of Γ and let $u = (u_1, u_2, \ldots, u_t)$ and $v = (v_1, v_2, \ldots, v_t)$ be two vertices of $Cay_\Sigma(\Gamma, S)$. If u and v are adjacent, then $u - v \in S$. Now the following two cases to be tackled:

Case:1 If at least one of vertices u or v belongs to S, then a negative edge does not occur between them. This shows that $Cay_\Sigma(\Gamma, S)$ is an all-positive. Thus, in light of Lemma 3.1, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

Case:2 If none of u and v belong to S, then only negative edge occur between them. Since u does not belong to S, so $u_i \in U(\mathbb{Z}_{{p_i}^{k_i}})$ for some i, and $u_j \in H_j$, where H_j denotes the maximal subgroup in $\mathbb{Z}_{{p_j}^{k_j}}$. Now the positive degree $d^+(u)$ of u is given by $d^+(u) = \prod_i [{p_i}^{k_i} - 2{p_i}^{k_i - 1}] \prod_j [{p_j}^{k_j} - {p_j}^{k_j - 1}]$, which is even. Also $|S|$ is even, therefore $d^-(u) = |S| - d^+(u)$ is also even. Now all the vertices of $Cay_\Sigma(\Gamma, S)$ will receive positive signs under canonical marking. Therefore, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent. $\square$

Theorem 3.9. *Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, where $p_i's$ are primes, $k_i's$, i and t are positive integers. Assume that $S = U(\Gamma)$ be a generating set of Γ. Then Cayley signed graph $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent if and only if one of following conditions hold:*

(i) $t = 1$.

(ii) for $t > 1$, *none of* $\mathbb{Z}_{{p_i}^{k_i}}$ *is isomorphic to* $\mathbb{Z}_2$.

(iii) for $t > 1$, Γ *is isomorphic to* $\mathbb{Z}_2^{t-1} \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$.

Proof. Necessity: Let $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent and each of the above listed conditions is false. In order to violate the condition (i) we must have $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$; $t > 1$ and to violate (ii) at least one of $\mathbb{Z}_{p_i}^{k_t}$ in Γ is isomorphic to $\mathbb{Z}_2$, and to violate the condition (iii) neither Γ is isomorphic to $\mathbb{Z}_2^{t-1} \times \mathbb{Z}_2$; $t > 1$ nor isomorphic to $\mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$; $t > 1$. In view of the above the order of Γ has to be even and the precise form of Γ is $\Gamma \cong \mathbb{Z}_2 \times \Gamma'$. Since $| \mathbb{Z}_{{p_i}^{k_i}} / \mathbb{Z}_{{p_i}^{k'_i}} | \geq 3$, so there are at least three vertices which are in the form of t-tuples, where $t > 1$ and it is easy to see the presence of a cycle in $Cay_\Sigma(\Gamma, S)$ in the following cases:

Case:1 Let $\Gamma \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{(t-1)-times} \times \mathbb{Z}_{p^k}$, where prime p is either $p > 3$ with $k \geq 1$ or $p = 3$ with $k > 1$. Then there exist a cycle in $Cay_\Sigma(\Gamma, S)$, viz., $v_1 - v_2 - v_3 - v_4 - v_1$, where $v_1 = (\underbrace{0, 0, 0, \ldots, 0}_{(t-1)-times}, 1)$, $v_2 = (\underbrace{1, 1, 1, \ldots, 1}_{(t-1)-times}, u_1)$, $v_3 = (\underbrace{0, 0, 0, \ldots, 0}_{t-times})$ and $v_4 = (\underbrace{1, 1, 1, \ldots, 1}_{(t-1)-times}, u_2)$ and $u_i \in U(\mathbb{Z}_{p^k})$. Since vertices v_2 and v_4 belongs to $U(\Gamma)$, so through canonical marking both vertices receive positive sign. Note that v_3 is adjacent to all the vertices which are elements of S, and hence v_3 also receive positive sign. Now, to determine $\mathbb{C}$-Consistency we only have to find the marking received by vertex v_1. To do this we have to calculate the negative degree $d^-(v_1)$ of vertex v_1. It can be noticed that v_1 is adjacent to vertex of form $(\underbrace{1, 1, 1, \ldots, 1}_{(t-1)-times}, b)$, where $b \in Z(\mathbb{Z}_{p^k})$, and hence $d^-(v_1) = p^{k-1}$, which is odd. Therefore, there exists a cycle containing exactly one negative vertex, viz., v_1 and the remaining three vertices are positive. This indicates that the above mentioned cycle is not consistent, and hence $Cay_\Sigma(\Gamma, S)$ is not $\mathbb{C}$-Consistent.

Case:2 Let $\Gamma \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{j-times} \times \underbrace{\mathbb{Z}_{{p_{j+1}}^{k_{j+1}}} \times \mathbb{Z}_{{p_{j+2}}^{k_{j+2}}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}}_{(t-j)-times}$, where primes $p_i > 3$ and $(t - j) > 1$. That is $\Gamma \cong \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_t$, $(t - j) > 1$, where either $p_i > 3$ or $p = 3$ with $k > 1$. Then there exist a cycle in $Cay_\Sigma(\Gamma, S)$, viz., $v_1 - v_2 - v_3 - v_4 - v_1$, where $v_1 = (\underbrace{0, 0, 0, \ldots, 0}_{j-times}, \underbrace{1, 1, 1, \ldots, 1}_{(t-j)-times})$, $v_2 = (\underbrace{1, 1, 1, \ldots, 1}_{j-times}, \underbrace{u_1, u_1, u_1, \ldots, u_1}_{(t-j)-times})$, $v_3 = (\underbrace{0, 0, 0, \ldots, 0}_{t-times})$ and $v_4 = (\underbrace{1, 1, 1, \ldots, 1}_{j-times}, \underbrace{u_2, u_2, u_2, \ldots, u_2}_{(t-j)-times})$. Now we are tempted to show the above cycle is not consistent. To do this we will determine the negative degree of each vertex. Since $v_2, v_4 \in S$, so $d^-(v_2) = d^-(v_4) = 0$. Clearly the vertex v_3 is adjacent to all the vertices which are elements of S, and hence $d^-(v_3) = 0$. Now, we shall calculate negative degree $d^-(v_1)$ of v_1. Note that v_1 is adjacent to the vertices of form $(\underbrace{1, 1, 1, \cdots, 1}_{j-times}, \underbrace{b_1, b_2, b_3, \cdots, b_t}_{(t-j)-times})$, where at least one $b_i \in Z(\Gamma_i)$. Now depending upon the choices of b_i the negative degree of v_1 is given by $d^-(v_1) = [|\Gamma_{j+1}| - \phi(|\Gamma_{j+1}|)] \times \phi(|\Gamma_{j+2}|) \times \cdots \times \phi(|\Gamma_t|) + [|\Gamma_{j+2}| - \phi(|\Gamma_{j+2}|)] \times \phi(|\Gamma_{j+1}|) \times \cdots \times \phi(|\Gamma_t|) + \cdots + [|\Gamma_t| - \phi(|\Gamma_t|)] \times \phi(|\Gamma_{j+1}|) \times \cdots \times \phi(|\Gamma_{t-1}|) - [(|\Gamma_{j+1}| - \phi(|\Gamma_{j+1}|)) \times (|\Gamma_{j+2}| - \phi(|\Gamma_{j+2}|)) \times \cdots \times (|\Gamma_t| - \phi(|\Gamma_t|))]$, where ϕ is Euler's phi function. One can easily observe that the negative degree $d^-(v_1)$ is odd. Therefore, the vertex v_1 receives the negative sign under the canonical marking. Thus, there exists a cycle in $Cay_\Sigma(\Gamma, S)$ containing one negative vertex and three positive vertices. This implies that $Cay_\Sigma(\Gamma, S)$ is not $\mathbb{C}$-Consistent.

Case:3 Let $\Gamma \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{j-times} \times \underbrace{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3}_{(t-j)-times}$; $(t-j) > 1$. Then there exist a cycle in $Cay_\Sigma(\Gamma, S)$, viz., $v_1 - v_2 - v_3 - v_4 - v_1$, where $v_1 = (\underbrace{0,0,0,\ldots,0}_{j-times}, \underbrace{2,2,2,\ldots,2}_{(t-j)times})$, $v_2 = (\underbrace{1,1,1,\ldots,1}_{t-times})$, $v_3 = (\underbrace{0,0,0,\ldots,0}_{(t-1)-times}, 2)$ and $v_4 = (\underbrace{1,1,1,\ldots,1}_{(t-1)-times}, 0)$.

Since $v_2 \in S$, so $d^-(v_2) = 0$. Now, we shall calculate the negative degree of each vertices v_1, v_3 and v_4. Note that v_1 is adjacent to vertices of the form $(\underbrace{1,1,1,\cdots,1}_{j-times}, \underbrace{b_{j+1}, b_{j+2}, b_{j+3}, \ldots, b_t}_{(t-j)-times})$, where each $b_j's$ is either 0 or 2. Therefore, $d^-(v_1) = 2^{t-j} - 1$, which is odd. This indicates that v_1 receives the negative sign through a canonical marking. Note that v_3 is adjacent to the vertices of form $(\underbrace{1,1,1,\ldots,1}_{j-times}, \underbrace{b_{j+1}, b_{j+2}, b_{j+3}, \ldots, c_t}_{(t-j)-times})$, where each $b_j's$ is either 1 or 2 and c_t is either 0 or 1, this gives us $d^-(v_3) = 2^{t-j} - 2$, which is even. Thus, through a canonical marking the vertex v_3 receive the positive sign. The vertex v_4 is adjacent to the vertices of form $(\underbrace{0,0,0,\ldots,0}_{j-times}, \underbrace{b_{j+1}, b_{j+2}, b_{j+3}, \ldots, c_t}_{(t-j)-times})$, where each $b_j's$ is either 0 or 2 and c_t is either 1 or 2, this gives that $d^-(v_4) = 2^{t-j} - 2$, which is even.

Clearly the vertices v_2, v_3 and v_4 in $Cay_\Sigma(\Gamma, S)$ receive the positive sign through canonical marking. Thus the existence of a four cycle can be seen in $Cay_\Sigma(\Gamma, S)$ consisting of exactly one negative vertex which ensures that it is not consistent. Hence $Cay_\Sigma(\Gamma, S)$ is not $\mathbb{C}$-Consistent.

Therefore from the foregoing analysis we found the existence of a cycle in all cases which is not consistent, so our assumption that $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent is wrong. Thus, for $Cay_\Sigma(\Gamma, S)$ to be $\mathbb{C}$-Consistent at least one of the above listed conditions holds.

Sufficiency: Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, $p_i's$ are prime numbers, $k_i's$, i and t are positive integers. Let $S = U(\Gamma)$ be a generating set of Γ. Here our aim is to show that $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent in each of the above listed conditions. If $|S| = 1$ or $|S| = 2$, then $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent by Theorem 3.4 and Theorem 3.5. Now, for $|S| > 2$ we shall tackle each case separately as follows:

For (i), $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}}$ (as $t = 1$) and $S = U(\Gamma)$. Consequently in view of Theorem 3.7(iv), $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

For (ii), let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ $(t > 1)$ among which none of $\mathbb{Z}'_{{p_i}^{k_i}}s$ is isomorphic to $\mathbb{Z}_2$. Let $u = (u_1, u_2, \ldots, u_t)$ and $v = (v_1, v_2, \ldots, v_t)$ be two vertices of $Cay_\Sigma(\Gamma, S)$. If u and v are adjacent in $Cay(\Gamma, S)$, then $u - v \in S$. Now in order to calculate the negative degree of u we apply the similar procedure as done in Theorem 3.8, which gives

$$d^+(u) = \prod_i [{p_i}^{k_i} - 2{p_i}^{k_i - 1}] \prod_j [{p_j}^{k_j} - {p_j}^{k_j - 1}].$$

Note that $d^+(u$ is even, and hence $d^-(u)$ is also even as $|S|$ and $d^+(u)$ are both even. Since u is arbitrary, each vertex of $Cay_\Sigma(\Gamma, S)$ receive a positive sign under the canonical marking. Therefore $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

For (iii), first let $\Gamma \cong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_2$. Then $S = U(\Gamma) = \{\underbrace{(1,1,1,\ldots,1)}_{t-times}\}$, that is $|S| = 1$ which indicates that $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent by the Theorem 3.4.

Next, let $\Gamma \cong \mathbb{Z}_2^{t-1} \times \mathbb{Z}_3$. Then $S = U(\Gamma) = \{\underbrace{(1,1,1,\ldots,1)}_{t-times}, \underbrace{(1,1,1,\ldots,2)}_{t-times}\}$, that is $|S| = 2$. Now in view of Theorem 3.5, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent. Hence the result. □

Theorem 3.10. *Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, $p_i's$ are prime, $k_i's$, i and t are positive integers. Assume that $S = Z^0(\Gamma)$ be a Cayley set of Γ. If at least one $\mathbb{Z}_{{p_i}^{k_i}}$ is isomorphic to $\mathbb{Z}_2$ and $|U(\Gamma)| \geq 3$, then $Cay_\Sigma(\Gamma, S)$ is not $\mathbb{C}$-Consistent.*

Proof. Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group and $S = Z^0(\Gamma)$ be a Cayley set of Γ. In order to prove the desire result it suffices to show the existence of a cycle in $Cay_\Sigma(\Gamma, S)$, which is not consistent. To do this, we tackle the following cases:

Case 1: If some of $\mathbb{Z}_{{p_i}^{k_i}}$ is isomorphic to $\mathbb{Z}_2$, but not all, then we can choose $u = (1, 1, \ldots, 1)$, $v = (1, 1, \ldots, a)$ and $w = (1, 1, \ldots, b)$ be three arbitrary distinct elements from $U(\Gamma)$ as $|U(\Gamma)| \geq 3$, where $a, b \in U(\mathbb{Z}_{{p_t}^{k_t}})$. Note that in $Cay_\Sigma(\Gamma, S)$ an edge is negative if and only if none of end vertices belongs to S. Here all three vertices are adjacent with each other through negative edge and the negative degree of each of u, v and w is equal and is given by

$$d^-(u) = d^-(v) = d^-(w) = \prod_i ({p_i}^{k_i} - {p_i}^{k_i - 1}) - 1,\ 1 \leq i \leq t.$$

Consequently, the negative degree of each of u, v and w is odd and therefore all the vertices receive the negative sign through canonical marking. Since the vertices u, v and w are adjacent to each other in $Cay_\Sigma(\Gamma, S)$, so there exist an all-negative triangle in which all three vertices are marked with negative sign, this would ensure the existence of negative cycle which is not consistent. Hence $Cay_\Sigma(\Gamma, S)$ is not $\mathbb{C}$-Consistent.

Case 2: If exactly one of $\mathbb{Z}_{{p_i}^{k_i}}$ is isomorphic to $\mathbb{Z}_2$, then we can choose the four elements $u_1 = (1, 1, \ldots, 1)$, $u_2 = (0, 0, \ldots, 1)$, $u_3 = (1, 0, \ldots, 0)$, and $u_4 = (0, 0, \ldots, 0)$. Then there exist a cycle $(u_1, u_2, u_3, u_4, u_1)$ in $Cay_\Sigma(\Gamma, S)$. The negative degree of u_1 is odd as it is adjacent to all other elements of $U(\Gamma)$ and the negative degree of remaining other vertices u_2, u_3, and u_4 is even, therefore under the canonical marking u_1 receive the negative sign and other vertices, namely, u_2, u_3, and u_4 receive the positive sign, this would ensure the existence of positive cycle which is not consistent. Hence $Cay_\Sigma(\Gamma, S)$ is not $\mathbb{C}$-Consistent. □

Theorem 3.11. *Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, $p_i's$ are prime, $k_i's$, i and t are positive integers. Assume that $S = Z^0(\Gamma)$ be a Cayley set of Γ. Then Cayley signed graph $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-consistent if and only if one of following conditions hold:*

(i) for $t = 1$, Γ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_{p^k}$, $p > 2$;

(ii) for $t > 1$, none of $\mathbb{Z}_{{p_i}^{k_i}}$ has $\mathbb{Z}_2$ as a quotient;

(iii) for $t > 1$, $\Gamma \cong {\mathbb{Z}_2}^t$.

Proof. Necessity: Let us suppose that $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent and each of listed conditions is false. Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group and $S = Z^0(\Gamma)$ be a Cayley set of Γ. In order to violate (i) and (iii), Γ is neither isomorphic to $\mathbb{Z}_4$ nor $\mathbb{Z}_{p^k}$ nor ${\mathbb{Z}_2}^t$, $t > 1$ and to violate (ii) atleast one of $\mathbb{Z}_{{p_i}^{k_i}}$ has $\mathbb{Z}_2$ as a quotient in Γ. This indicate that $|\Gamma|$ must be even and the number of elements a, $a \in U(\Gamma)$ is also even. Note that in $Cay_\Sigma(\Gamma, S)$ an edge is negative if and only if none of end vertices belongs to S. Since each element of $U(\Gamma)$ is adjacent to every element of $U(\Gamma)$ by negative edge, so the negative degree of a is odd $\forall\ a$, $a \in U(\Gamma)$. Consider three distinct elements u_1, u_2 and u_3 from the set $\Gamma \setminus S$, i.e., $U(\Gamma)$. Since u_1, u_2 and u_3 are mutually adjacent in $Cay_\Sigma(\Gamma, S)$ with negative edge, so there exists an all-negative triangle, namely, (u_1, u_2, u_3, u_1). Now it is easy to see that under the canonical marking each element of $U(\Gamma)$ receive the negative sign and one can find the presence of an all-negative triangle in $Cay_\Sigma(\Gamma, S)$, namely, (u_1, u_2, u_3, u_1) in which all the three vertices are negatively marked. Therefore, there exist a cycle in $Cay_\Sigma(\Gamma, S)$ which is not consistent and hence $Cay_\Sigma(\Gamma, S)$ is not $\mathbb{C}$-Consistent, a contradiction to our assumption. Hence, if $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent, then one of the conditions must hold.

Sufficiency: Let $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ be a finite abelian group, $p_i's$ are prime numbers, $k_i's$, i and t are positive integers. Let $S = Z^0(\Gamma)$ be a Cayley set of Γ. Here our aim is to show that $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent in each of the above listed conditions. We shall tackle each case separately as follows:

(i) If $t = 1$ and $\Gamma \cong \mathbb{Z}_4$, then $|S| = 1$. Therefore, in view of Theorem 3.4, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent. Next if $t = 1$ and $\Gamma \cong \mathbb{Z}_{p^k}$, then $|S|$ is even and $Cay(\Gamma, S)$ is an even regular graph. Note that in $Cay_\Sigma(\Gamma, S)$ an edge is negative if and only if none of the end vertices belongs to S. Since the positive edges lie only between multiple of p and 0, so the number of negative edges incident at a vertex u, $u \notin S$ are also even. This implies that the negative degree of u is even $\forall u$, $u \in \Gamma$. Therefore, all the vertices receive positive sign under the canonical marking. Thus, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

(ii) For $t > 1$ and $\Gamma \cong \mathbb{Z}_{{p_1}^{k_1}} \times \mathbb{Z}_{{p_2}^{k_2}} \times \cdots \times \mathbb{Z}_{{p_t}^{k_t}}$ in which none of $\mathbb{Z}_{{p_i}^{k_i}}$ has $\mathbb{Z}_2$ as a quotient. Note that in $Cay_\Sigma(\Gamma, S)$ an edge is negative if and only if none of end vertices belongs to S. Let

u be vertex which does not belong to S. Then negative degree of vertex u in $Cay_\Sigma(\Gamma, S)$ is given by

$$d^-(u) = \prod_i [{p_i}^{k_i} - {p_i}^{k_i - 1}] - \prod_i [{p_i}^{k_i} - 2{p_i}^{k_i - 1}] - 1,\ 1 \le i \le t.$$

This implies that $d^-(u)$ is even for all u, $u \notin S$. Since the negative degree of the vertices which belong to S is zero, so all the vertices in $Cay_\Sigma(\Gamma, S)$ are marked with positive signs through canonical marking. Hence, $Cay_\Sigma(\Gamma, S)$ is $\mathbb{C}$-Consistent.

(iii) For $t > 1$ and $\Gamma \cong {\mathbb{Z}_2}^t$, then S contains all non-zero elements except the one element, precisely $\underbrace{(1, 1, 1, \ldots, 1)}_{t-times}$. Since there is only one element which does not belong to S, so there is no negative edge in $Cay_\Sigma(\Gamma, S)$, and hence it is an all-positive graph. Therefore in light of Lemma 3.1, $Cay_\Sigma(\Gamma, S)$ is balanced. □

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