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Uncertain Measurable S-acts

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Abstract In this paper, we define Uncertain Measurable S-acts and morphisms between two Uncertain Measurable S-acts on monoids. Next, we construct the new category, namely Uncertain Meas Act-S, of these objects and morphisms. We prove that the category of these new objects is closed under product, coproduct, pushout and pullback. Also, we show that the category of Uncertain Measurable S-acts is closed under equalizer and coequalizer.

1 Introduction

Real decisions are usually made in the state of indeterminacy. There exist two mathematical systems for modelling indeterminacy, one is probability theory and the other is uncertainty theory. From the mathematical viewpoint, uncertainty theory is essentially an alternative theory of measure. Thus, uncertainty theory begins with a measurable space [11]. Uncertainty theory, founded by Liu, is a branch of mathematics for dealing with human uncertainty. There are three fundamental concepts in uncertainty theory. The first concept, called uncertain measure, is introduced based on three Axioms: normality Axiom, duality Axiom , and subadditivity Axiom for presenting the degree that an uncertain event may occur. The second one, called the uncertain variable, is brought in to show quantities in uncertainty. The third one, called uncertainty distribution (a real function), is put forward for describing uncertain variables. For more, see [1], [2], [3], [4], [5], [6], [12] and [13]. In this paper, we construct the category of uncertain S-acts and study some properties of this category , such as product, coproduct, pullback, pushout, equalizer, and coequalizer of uncertain S-acts. First, we give some preliminaries needed in the sequel. Let S be a monoid. By a *(right) S-act* or *act over S*, we mean a non-empty set A together with a map $A \times S \to A, (a, s) \mapsto as$, such that for all $a \in A, s, t \in S$, $(as)t = a(st)$ and $a1 = a$. A non-empty subset $B \subseteq A$ is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. Clearly, S is an S-act with the operation as the action. Let A and B be two S-acts. A mapping $f : A \to B$ is called an S-*homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. The category of all S-acts as well as all S-homomorphisms between them is denoted by **Act-**S. In this category, monomorphisms are exactly one-to-one S-homomorphisms, and epimorphisms are surjective S-homomorphisms. A *congruence* on an S-act A is an equivalence relation ρ on A for which $a\rho a'$ implies that $(as)\rho(a's)$ for $a, a' \in A$ and $s \in S$. Fore more, see [8], [9] and [10] .

Let X be a non-empty set. The pair (X, σ_X) is called measurable space if σ_X be a σ-algebra on X. A sub-sigma-algebra of σ_X is a subset of σ_X that is also a sigma-algebra. Consider two measurable spaces (A, σ_A) and (B, σ_B). A function $f : A \longrightarrow B$ is called (σ_A, σ_B)-measurable, briefly measurable if $f^{-1}(B_1) \in \sigma_A$ for any $B_1 \in \sigma_B$.

We recall from [7], Let A be an S-act with σ-algebra σ_A on A. The pair (A, σ_A) is said to be a measurable S-act on monoid S if for any $s \in S$, $\lambda_s : A \longrightarrow A$, $\lambda_s(a) = as, a \in A$ is measurable. Also, consider two measurable S-acts (A, σ_A) and (B, σ_B). An S-homomorphism $f : A \longrightarrow B$ is called *Measurable S-morphism*, briefly, MS-morphism if $f^{-1}(B') \in \sigma_A$ for any $B' \in \sigma_B$. The category of measurable S-acts and MS- morphisms between two measurable S-acts is denoted by Meas Act-S.

2 Uncertain Measurable S-acts In Category Act-S

In this section, we define the category of uncertain measurable S-acts, which namely Uncer Act-S and shows this category is closed under products, coproducts, pullbacks, pushouts, equalizers, and coequalizers.

We recall from [11], for any non-empty set X and σ be a σ-algebra over X, any element $Y \in \sigma$ is renamed to an event in uncertainty theory. Let $\mathcal{M}$ be an uncertain measure, i.e. a function $\mathcal{M} : \sigma \longrightarrow [0,1]$ such that it satisfies in the following conditions:
(i) (Normality Axiom) $\mathcal{M}(X) = 1$ for the universal set X.
(ii) (Duality Axiom) $\mathcal{M}(Y) + \mathcal{M}(Y^c) = 1$ for any event $Y \in \sigma$.
(iii) (Sub additivity Axiom) For every countable sequence of events $Y_1, Y_2, \cdots$, we have $\mathcal{M}(\bigcup_{i=1}^{\infty}(Y_i)) \leq \sum_{i=1}^{\infty} \mathcal{M}(Y_i)$
Then, the triple $(X, \sigma, \mathcal{M})$ is called an uncertainty space.
Moreover, for $(X_k, \sigma_k, \mathcal{M}_k)$, $k = 1, 2, \cdots$ of uncertainty spaces, The product uncertain measure $\mathcal{M}$ is an uncertain measure if satisfying in the below Axiom.
(iv) (Product Axiom) $\mathcal{M}\{\prod_{k=1}^{\infty} Y_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}(Y_k)$, where Y_k is arbitrarily chosen events from σ_k for $k = 1, 2, \cdots$, respectively.
Although probability measure satisfies the above three Axioms, probability theory is not a special case of uncertainty theory because the product probability measure does not satisfy the fourth Axiom. We recall from [11], Uncertain measure $\mathcal{M}$ is a monotone.

Definition 2.1. Let A be an S-act. A triple $(A, \sigma_A, \mathcal{M}_A)$, is called an *Uncertain Measurable S-act* if $(A, \sigma_A, \mathcal{M}_A)$ is an uncertain space. Let $(A, \sigma_A, \mathcal{M}_A)$ and $(B, \sigma_B, \mathcal{M}_{\mathcal{B}})$ be two Uncertain Measurable S-acts. A MS-morphism $f : A \longrightarrow B$ is called *Uncertain Measurable S-morphism*, briefly UMS- morphism if $\mathcal{M}_{\mathcal{A}}(f^{-1}(B')) \leq \mathcal{M}_{\mathcal{B}}(B')$ for any $B' \in \sigma_B$.

The identity morphism of Uncertain Measurable S-act $(A, \sigma_A, \mathcal{M})$ is the identity S-homomorphism of the S-act A. Clearly, all Uncertain Measurable S-acts and UMS-morphisms between two Uncertain Measurable S-acts construct a category, which we denote by *Uncer Meas Act-S*. In the category Uncer Act-S, any UMS-morphism $f : (A, \sigma_A, \mathcal{M}) \longrightarrow (B, \sigma_B, \mathcal{M}')$ is an isomorphism if it is an isomorphism in the category of measurable S-acts and $\mathcal{M}'(f(A')) \leq \mathcal{M}(A')$ for any $A' \in \sigma_A$.

Example 2.2. (i) Let B be a subact of S-act A on monoid S. Consider σ-algebra $\sigma_A = \{\emptyset, B, B - A, A\}$ on S-act A. Define $\mathcal{M} : \sigma \longrightarrow [0,1]$ such that

$$\mathcal{M}(X) = \begin{cases} 0 & X = \emptyset \\ 1 & X = A \\ 0.5 & o.w \end{cases}$$

which is an uncertain measure. So $(A, \sigma_A, \mathcal{M})$ is an Uncertain Measurable S-act.

(ii) Consider monoid $S = (\mathbb{N}, .)$ with usual multiplication, measurable S-acts $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and $(2\mathbb{N}, \mathcal{P}(2\mathbb{N}))$ and uncertain measures $\mathcal{M} : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1]$ and $\mathcal{M}' : \mathcal{P}(2\mathbb{N}) \longrightarrow [0,1]$ such that $\mathcal{M}(\mathbb{N}) = \mathcal{M}'(2\mathbb{N}) = 1$, $\mathcal{M}(\emptyset) = \mathcal{M}'(\emptyset) = 0$, $\mathcal{M}(X) = 0.5$ for any $X \neq \emptyset, \mathbb{N}$ and $\mathcal{M}'(Y) = 0.5$ for any $Y \neq \emptyset, 2\mathbb{N}$.
Now define S-homomorphism $f : \mathbb{N} \longrightarrow 2\mathbb{N}$ such that $f(n) = 2n$. Obviously, $f : (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathcal{M}') \longrightarrow (2\mathbb{N}, \mathcal{P}(2\mathbb{N}), \mathcal{M}')$ is an UMS-morphism.
(iii) Consider monoid $S = \{0, 1\}$ and S-act $A = \{a, b\}$ with action $a.0 = b, a.1 = a, b.0 = b$ and $b.1 = b$. Consider σ-algebras $\sigma_A = \{\emptyset, \{a\}, \{b\}, A\}$ and $\sigma'_A = \{\emptyset, A\}$. Obviously, $f : (A, \sigma'_A) \longrightarrow (A, \sigma_A)$, which $f(x) = b$ for any $x \in A$ is a MS-morphism. Consider real number $0 < c < 1$ and define $\mathcal{M}_A : \sigma_A \longrightarrow [0,1]$ such that $\mathcal{M}_A(\emptyset) = 0, \mathcal{M}_A(\{a\}) = c, \mathcal{M}_A(\{b\}) = 1 - c$ and $\mathcal{M}_A(A) = 1$. Also, define $\mathcal{M}'_A : \sigma'_A \longrightarrow [0,1]$ such that $\mathcal{M}'_A(\emptyset) = 0$ and $\mathcal{M}'_A(A) = 1$. It is not difficult to see $(A, \sigma_A, \mathcal{M}_A)$ and $(A, \sigma'_A, \mathcal{M}'_A)$ are Uncertain Measurable S-acts and f is not an UMS-morphism.

Theorem 2.3. *The category Uncertain Meas Act-S is closed under products.*

Proof. Consider family $\{(A_i, \sigma_{A_i}, \mathcal{M}_i)\}_{i\in I}$ of Uncertain Measurable S-acts. We recall from [7], the product of family $\{(A_i, \sigma_{A_i})\}_{i\in I}$ is $(\prod_{i\in I} A_i, \sigma_{\prod_{i\in I} A_i})$, which $\prod_{i\in I} A_i$ is cartesian product of A_i, $\sigma_{\prod_{i\in I} A_i} = \{B \subseteq \prod_{i\in I} A_i | \exists D_i \in \sigma_{A_i}$ for any $i \in I$ such that $B = \pi_i^{-1}(D_i)\}$ and $\pi_i : \prod_{i\in I} A_i \longrightarrow A_i$, $i \in I$ is the projection map. Define $\mathcal{M} : \sigma_{\prod_{i\in I} A_i} \longrightarrow [0, 1]$ such that for any $B \in \sigma_{\prod_{i\in I} A_i}$,

$$\mathcal{M}(B) = \begin{cases} \bigwedge_{i\in I} \mathcal{M}_i(D_i) & \bigwedge_{i\in I} \mathcal{M}_i(D_i) > 0.5 \\ 1 - \bigwedge_{i\in I} \mathcal{M}_i(D_i^c) & \bigwedge_{i\in I} \mathcal{M}_i(D_i^c) > 0.5 \\ 0.5 & o.w \end{cases}$$

We show that $\mathcal{M}$ is an uncertain measure. Clearly, $\mathcal{M}$ is satisfying in the normality Axiom. We check $\mathcal{M}$ for duality Axiom. For any $B = \pi^{-1}(D_i) \in \sigma_{\prod_{i\in I} A_i}$, if $\bigwedge_{i\in I} \mathcal{M}_i(D_i) > 0.5$, we have $\mathcal{M}(B) = \bigwedge_{i\in I} \mathcal{M}_i(D_i)$. By the definition of $\mathcal{M}(B)$, we can conclude that $\mathcal{M}(B)+\mathcal{M}(B^c) = 1$. If $\bigwedge_{i\in I} \mathcal{M}_i(D_i^c) > 0.5$, similarly we can prove it. Now, suppose that $\bigwedge_{i\in I} \mathcal{M}_i(D_i) \leq 0.5$ and $\bigwedge_{i\in I} \mathcal{M}_i(D_i^c) \leq 0.5$, we have $\mathcal{M}(B) = \mathcal{M}(B^c) = 0.5$. Now, we check $\mathcal{M}$ for the monotonicity Axiom. Suppose that $B \subseteq B', B = \pi_i^{-1}(D_i), B' = \pi_i^{-1}(D'_i) \in \sigma_{\prod_{i\in I} A_i}$. First, suppose that $\bigwedge_{i\in I} \mathcal{M}_i(D_i) > 0.5$. Since $B \subseteq B'$, we have $D_i \subseteq D'_i, i \in I$. Hence, $0.5 < \bigwedge_{i\in I} \mathcal{M}_i(D_i) \leq \bigwedge_{i\in I} \mathcal{M}_i(D'_i)$ and $\mathcal{M}(B) \leq \mathcal{M}(B')$. Suppose that $\bigwedge_{i\in I} \mathcal{M}_i(D_i'^c) > 0.5$. Since $B \subseteq B'$, we have $B'^c \subseteq B^c$ and so $D_i'^c \subseteq D_i^c$. Therefore $0.5 < \bigwedge_{i\in I} \mathcal{M}_i(D_i'^c) \leq \bigwedge_{i\in I} \mathcal{M}_i(D_i^c)$ and $\mathcal{M}(B) = 1 - \bigwedge_{i\in I} \mathcal{M}_i(D_i^c) \leq 1 - \bigwedge_{i\in I} \mathcal{M}_i(D_i'^c) = \mathcal{M}(B')$. The other cases are easy to check. Before checking the sub additivity Axiom, we claim $\mathcal{M}_i(D_i \cap D'_i) \geq \mathcal{M}_i(D_i)+\mathcal{M}_i(D'_i)-1, i \in I$. Since, for any $i \in I, \mathcal{M}_i$ is satisfied in sub additivity Axiom, we have $\mathcal{M}_i(D_i \cap D'_i) = 1-\mathcal{M}_i((D_i \cap D'_i)^c) = 1-\mathcal{M}_i(D_i^c \cup D_i'^c) \geq 1-(\mathcal{M}_i((D_i^c)+\mathcal{M}_i(D_i'^c))) = 1-(1-\mathcal{M}_i(D_i))-(1-\mathcal{M}_i(D'_i)) = \mathcal{M}_i(D_i) + \mathcal{M}_i(D'_i) - 1$. Now, consider $B = \pi_i^{-1}(D_i), B' = \pi_i^{-1}(D'_i) \in \sigma_{\prod_{i\in I} A_i}$. First, suppose that $\mathcal{M}(B), \mathcal{M}(B') < 0.5$. We have $\mathcal{M}(B \cup B') = 1-\mathcal{M}(\pi_i^{-1}(D_i^c) \cap \pi_i^{-1}(D_i'^c)) = 1 - \mathcal{M}(\pi_i^{-1}(D_i^c \cap D_i'^c))$. Clearly, there exists $j \in I$ such that $1 - \mathcal{M}(\pi_i^{-1}(D_i^c \cap D_i'^c) = 1 - \mathcal{M}_j(D_j^c \cap D_j'^c)$. Hence, $\mathcal{M}(B \cup B') = 1 - \mathcal{M}_j(D_j^c \cap D_j'^c) \leq 1 - \mathcal{M}_j(D_j^c) + 1 - \mathcal{M}_j(D_j'^c) \leq 1 - \wedge\mathcal{M}_i(D_i^c) + 1 - \wedge\mathcal{M}_i(D_i'^c) = \mathcal{M}(B) + \mathcal{M}(B')$. Let $\mathcal{M}(B) \geq 0.5$ and $\mathcal{M}(B') < 0.5$. If $\mathcal{M}(B \cup B') = 0.5$, then $\mathcal{M}(B \cup B') \leq \mathcal{M}(B) + \mathcal{M}(B')$. If $\mathcal{M}(B \cup B') > 0.5$, we have $\mathcal{M}(B^c \cap B'^c) < 0.5$. So, $\mathcal{M}(B \cup B') = 1 - \mathcal{M}(B^c \cap B'^c) \leq 1 - \mathcal{M}(B^c) + \mathcal{M}(B') = \mathcal{M}(B) + \mathcal{M}(B')$. If $\mathcal{M}(B), \mathcal{M}(B') \geq 0.5$, we have $\mathcal{M}(B) + \mathcal{M}(B') \geq 1 \geq \mathcal{M}(B \cup B')$. Now, we show that $\pi_i, i \in I$ are UMS-morphism. Let $D_j \in \sigma_{A_j}, j \in I$. We show that $\mathcal{M}(\pi_j^{-1}(D_j)) \leq \mathcal{M}_j(D_j)$. We have $\mathcal{M}(\pi_j^{-1}(D_j)) = \mathcal{M}(A_1 \times A_2 \times \cdots \times D_j \times \cdots)$. So $\bigwedge_{i\in I} \mathcal{M}_i(D_i) = \mathcal{M}_j(D_j)$. If $\bigwedge_{i\in I} \mathcal{M}_i(D_i) > 0.5$. Then $\mathcal{M}(\pi_j^{-1}(D_j)) = \mathcal{M}_j(D_j)$. If $\bigwedge_{i\in I} \mathcal{M}_i(D_i^c) > 0.5$, then $\mathcal{M}(\pi_j^{-1}(D_j)) = 1 - \mathcal{M}_j(D_j^c) = \mathcal{M}_j(D_j)$. The other case is obtained easily. Therefore $\pi_j, j \in I$ is an UMS-morphism. For any Uncertain Measurable S-act $(B, \sigma_B, \mathcal{M}_B)$ and every family of UMS-morphisms $g_i : (Y, \sigma_Y, \mathcal{M}_Y) \longrightarrow (A_i, \sigma_{A_i}, \mathcal{M}_i)$, by [7], there exists a unique MS-morphism $h : (Y, \sigma_Y) \longrightarrow (\prod_{i\in I} A_i, \sigma_{\prod_{i\in I} A_i})$ such that $\pi h = g_i$. We show that h is an UMS-morphism. Let $B \in \sigma_{\prod_{i\in I} A_i}$. We have $B = \pi_i^{-1}(D_i), D_i \in \sigma_{A_i}$. We proved that $\mathcal{M}_Y(h^{-1}(B)) \leq \mathcal{M}(B)$. We have $\mathcal{M}_Y(h^{-1}(B)) = \mathcal{M}_Y(g_i^{-1}\pi_i(B)) = \mathcal{M}_Y(g_i^{-1}(D_i)) \leq \mathcal{M}_i(D_i)$. First, suppose that $\bigwedge_{i\in I} \mathcal{M}_i(D_i) > 0.5$.
Since $\mathcal{M}_Y(h^{-1}(B)) \leq \mathcal{M}_i(D_i), i \in I$, we have $\mathcal{M}_Y(h^{-1}(B)) \leq \bigwedge_{i\in I} \mathcal{M}_i(D_i) = \mathcal{M}(B)$. Now suppose that $\bigwedge_{i\in I} \mathcal{M}_i(D_i^c) > 0.5$. Since $\bigwedge_{i\in I} \mathcal{M}_i(D_i^c) \leq \mathcal{M}_i(D_i^c), i \in I$, we can conclude that $\mathcal{M}_Y(h^{-1}(B)) \leq \mathcal{M}_i(D_i) = 1 - \mathcal{M}_i(D_i^c) \leq 1 - \bigwedge_{i\in I} \mathcal{M}_i(D_i^c) = \mathcal{M}(B)$. In the other case, we easily observe $\mathcal{M}_Y(h^{-1}(B)) \leq \mathcal{M}(B)$. Thus h is an UMS-morphism, and the category Uncer Act-S is closed under products.

□

Theorem 2.4. *The category Uncertain Meas Act-S is closed under coproducts.*

Proof. Let $\{(A_i, \sigma_{A_i}, \mathcal{M}_i)\}_{i\in I}$ be a family of Uncertain Measurable S-acts. We recall from [7], the pair $(\coprod_{i\in I} A_i, \sigma_{\coprod_{i\in I} A_i})$ is the coproduct of the $\{(A_i, \sigma_{A_i})\}_{i\in I}$, in which $(\coprod_{i\in I} A_i, \iota_i)$ is the disjoint union of A_i, $i \in I$ with the canonical injection $\iota_i : A_i \longrightarrow \coprod_{i\in I} A_i$ and $\sigma_{\coprod_{i\in I} A_i} =$

$\{\coprod_{i\in I} D_i | D_i \in \sigma_{A_i}$, for any $i \in I\}$. Define $\mathcal{M} : \sigma_{\coprod_{i\in I} A_i} \longrightarrow [0,1]$ such that for any $D = \coprod_{i\in I} D_i \in \sigma_{\coprod_{i\in I} A_i}$

$$\mathcal{M}(D) = \begin{cases} \bigvee_{i\in I} \mathcal{M}_i(D_i) & \bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5 \\ 1 - \bigvee_{i\in I} \mathcal{M}_i(D_i^c) & \bigvee_{i\in I} \mathcal{M}_i(D_i^c) < 0.5 \\ 0.5 & o.w \end{cases}$$

We show that $\mathcal{M}$ is an uncertain measure. It follows easily that $\mathcal{M}(\coprod_{i\in I} A_i) = 1$. Now, we show that $\mathcal{M}$ satisfy in the duality Axiom. Let $D \in \sigma_{\coprod_{i\in I} A_i}$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5$. By the definition of $\mathcal{M}(D)$, we have $\mathcal{M}(D^c) = 1 - \bigvee_{i\in I} \mathcal{M}_i(D_i)$ and so $\mathcal{M}(D) + \mathcal{M}(D^c) = 1$. If $\bigvee_{i\in I} \mathcal{M}_i(D_i^c) < 0.5$, we can prove similarly. Now suppose that $\bigvee_{i\in I} \mathcal{M}_i(D_i) > 0.5$. There are two cases, $\bigvee_{i\in I} \mathcal{M}_i(D_i^c) > 0.5$ or $\bigvee_{i\in I} \mathcal{M}_i(D_i^c) < 0.5$. In the first case we have $\mathcal{M}(D) = \mathcal{M}(D^c) = 0.5$ and in the second case we have $\mathcal{M}(D^c) = \bigvee_{i\in I} \mathcal{M}_i(D_i^c)$ and $\mathcal{M}(D) = 1 - \bigvee_{i\in I} \mathcal{M}_i(D_i^c)$. The case $\bigvee_{i\in I} \mathcal{M}_i(D_i) = \bigvee_{i\in I} \mathcal{M}_i(D_i^c) = 0.5$ makes it obvious that $\mathcal{M}(D) + \mathcal{M}(D^c) = 1$. Now, we show that $\mathcal{M}$ is monotone. Let $D \subseteq D'$, $D = \coprod_{i\in I} D_i$, $D' = \coprod_{i\in I} D_i' \in \sigma_{\coprod_{i\in I} A_i}$, we show that $\mathcal{M}(D) \leq \mathcal{M}(D')$. Since $\iota_i^{-1}(D) \subseteq \iota_i^{-1}(D')$, $i \in I$, we conclude that $\mathcal{M}_i(D_i) \leq \mathcal{M}_i(D_i')$, $i \in I$. We show that for four cases. First suppose that $\bigvee_{i\in I} \mathcal{M}_i(D_i') < 0.5$, then $\bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5$ and so $\mathcal{M}(D) = \bigvee_{i\in I} \mathcal{M}_i(D_i) \leq \bigvee_{i\in I} \mathcal{M}_i(D_i') = \mathcal{M}(D')$. If $\bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i') > 0.5$, cleary it follows $\mathcal{M}(D) \leq \mathcal{M}(D')$. If $\bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i'^c) < 0.5$, we have $\mathcal{M}(D) = \bigvee_{i\in I} \mathcal{M}_i(D_i) \leq 1 - \bigvee_{i\in I} \mathcal{M}_i(D_i'^c) = \mathcal{M}(D')$. Now, let $\bigvee_{i\in I} \mathcal{M}_i(D_i) > 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i'^c) < 0.5$. Then $\mathcal{M}(D) = 0.5 \leq 1 - \bigvee_{i\in I} \mathcal{M}_i(D_i'^c) = \mathcal{M}(D')$. We claim that $\mathcal{M}(D)$ is satisfies in sub additivity Axiom. We show that $\mathcal{M}(D \cup D') \leq \mathcal{M}(D) + \mathcal{M}(D')$, $D = \coprod_{i\in I} D_i$, $D' = \coprod_{i\in I} D_i' \in \sigma_{\coprod_{i\in I} A_i}$. If $\bigvee_{i\in I} \mathcal{M}_i(D_i) > 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i') > 0.5$, then $\bigvee_{i\in I} \mathcal{M}_i(D_i \cup D_i') > 0.5$ and so $\mathcal{M}(D \cup D') = 0.5 \leq \mathcal{M}(D) + \mathcal{M}(D')$. If $\bigvee_{i\in I} \mathcal{M}_i(D_i) > 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i'^c) > 0.5$, then $\bigvee_{i\in I} \mathcal{M}_i(D_i \cup D_i') > 0.5$. Hence $\mathcal{M}(D \cup D') = 0.5 \leq \mathcal{M}(D) + \mathcal{M}(D')$. Now let $\bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i') < 0.5$.
If $\bigvee_{i\in I} \mathcal{M}_i(D_i \cup D_i') < 0.5$, we have $\mathcal{M}(D \cup D') = \bigvee_{i\in I} \mathcal{M}_i(D_i \cup D_i') = \mathcal{M}_j(D_j \cup D_j') \leq \mathcal{M}_j(D_j) + \mathcal{M}_j(D_j') \leq \bigvee_{i\in I} \mathcal{M}_i(D_i') + \bigvee_{i\in I} \mathcal{M}_i(D_i') = \mathcal{M}(D) + \mathcal{M}(D')$. If $\bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5$, $\bigvee_{i\in I} \mathcal{M}_i(D_i') < 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i((D_i \cup D_i')^c) < 0.5$, we have $\mathcal{M}(D \cup D') = 1 - \bigvee_{i\in I} \mathcal{M}_i((D_i \cup D_i')^c) = 1 - \mathcal{M}_i(D_i^c \cap D_i^c) \leq 1 - (\mathcal{M}_i(D_i) + \mathcal{M}_i(D_i') - 1) \leq \mathcal{M}(D) + \mathcal{M}(D')$. Now, suppose that $\bigvee_{i\in I} \mathcal{M}_i(D_i) \geq 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i(D_i') < 0.5$. Thus, $\bigvee_{i\in I} \mathcal{M}_i(D_i \cup D_i') > 0.5$ and so $\mathcal{M}(D \cup D') = 0.5 \leq \mathcal{M}(D) + \mathcal{M}(D')$.
If $\bigvee_{i\in I} \mathcal{M}_i(D_i) < 0.5$, $\bigvee_{i\in I} \mathcal{M}_i(D_i') < 0.5$ and $\bigvee_{i\in I} \mathcal{M}_i((D_i \cup D_i')) \geq 0.5$. We have $\mathcal{M}(D \cup D') = 1 - \mathcal{M}(D^c \cup D'^c) \leq 1 - \mathcal{M}(D^c \cup D'^c) + \mathcal{M}(D') \leq 1 - \mathcal{M}(D^c) + \mathcal{M}(D') = \mathcal{M}(D) + \mathcal{M}(D')$. The other cases can be proved clearly. So $\mathcal{M}$ is an uncertain measure.

For any $D = \coprod_{i\in I} D_i \in \sigma_{\coprod_{i\in I} A_i}$, we have $\mathcal{M}_i(\iota_i^{-1}(D)) = \mathcal{M}_i(D_i) = \mathcal{M}(D_i) \leq \mathcal{M}(D)$. Thus $\iota_i, i \in I$ are UMS-morphism. For any Uncertain Measurable S-act $(B, \sigma_B, \mathcal{M}_B)$ and every family of UMS-morphisms $g_i : (A_i, \sigma_{A_i}, \mathcal{M}_i) \longrightarrow (B, \sigma_B, \mathcal{M}_B)$, by [7], there exists a unique MS-morphism $h : (\coprod_{i\in I} A_i, \sigma_i) \longrightarrow (B, \sigma_B)$ such that $h\iota_i = g_i$. We show that h is an UMS-morphism. Let $Y \in \sigma_B$. We have $\mathcal{M}(h^{-1}(Y)) = \mathcal{M}(\iota_i g_i^{-1}(Y))$.
If $\bigvee_{i\in I} \mathcal{M}_i(\iota_i^{-1}\iota_i g_i^{-1}(Y)) < 0.5$ then $\mathcal{M}(h^{-1}(Y)) = \bigvee_{i\in I} \mathcal{M}_i(g_i^{-1}(Y)) \leq \mathcal{M}_B(Y)$.
If $\bigvee_{i\in I} \mathcal{M}_i(\iota_i^{-1}\iota_i g_i^{-1}(Y^c)) < 0.5$, we have $\mathcal{M}(h^{-1}(Y)) = 1 - \bigvee_{i\in I} \mathcal{M}_i(g_i^{-1}(Y^c))$. On the other hand, for any $i \in I$, we have $\mathcal{M}_{\mathcal{B}}(Y^c) \leq \mathcal{M}_i(g_i^{-1}(Y^c))$ and so $\mathcal{M}_{\mathcal{B}}(Y^c) \leq \bigvee_{i\in I} \mathcal{M}_i(g_i^{-1}(Y^c))$. Hence, we have $\mathcal{M}(h^{-1}(Y)) \leq 1 - \mathcal{M}_{\mathcal{B}}(Y^c) = \mathcal{M}_{\mathcal{B}}(Y)$, and the proof is complete.

□

We recall, in the category of Act-S, the pullback of S-homomorphism $f_1 : X_1 \longrightarrow Y$ and $f_2 : X_2 \longrightarrow Y$ is the pair $(P, (\rho_1, \rho_2))$ such that $P = \{(x_1, x_2) \in X_1 \times X_2 | f_1(x_1) = f_2(x_2)\}$ and ρ_i is the restriction to P of the i-th projection π_i from P onto X_i, $i = 1, 2$.

Theorem 2.5. *The category Uncertain Meas Act-S is closed under pullbacks.*

Proof. Consider UMS-morphisms $f_1 : (X_1, \sigma_{X_1}, \mathcal{M}_1) \longrightarrow (Y, \sigma_Y, \mathcal{M}_Y)$ and $f_2 : (X_2, \sigma_{X_2}, \mathcal{M}_2) \longrightarrow (Y, \sigma_Y, \mathcal{M}_Y)$. Since [7], the pullback of MS-morphisms $f_1 : (X_1, \sigma_{X_1}) \longrightarrow (Y, \sigma_Y)$ and $f_2 : (X_2, \sigma_{X_2}) \longrightarrow (Y, \sigma_Y)$ is the measurable S-act (P, σ_P) with MS-morphisms $\rho_i : (P, \sigma_P) \longrightarrow$

(X_i, σ_{X_i}), $i = 1, 2$, which $\sigma_P = \{P \cap D | D \in \sigma_{X_1 \times X_2}\}$, $\sigma_{X_1 \times X_2}$ is σ- algebra on $X_1 \times X_2$ and $\rho_i = \pi_i|_P$, $i = 1, 2$. Define $\mathcal{M}_P : \sigma_P \longrightarrow [0, 1]$ such that $\mathcal{M}'_P = \mathcal{M}_{|P}$, which $\mathcal{M} : \sigma_{\prod_{i \in I} A_i} \longrightarrow [0, 1]$ is defined in Theorem 2.3. By this definition, it is easy to see that $(P, \sigma_P, \mathcal{M}_P)$ is Uncertain Measurable S-acts and $\rho_i : (P, \sigma_P, \mathcal{M}_P) \longrightarrow (X_i, \sigma_{X_i}, \mathcal{M}_i)$, $i = 1, 2$, are UMS-morphism. For any Uncertain Measurable S-act $(P', \sigma_{P'}, \mathcal{M}_{P'})$ and UMS-morphisms $g_i : (P', \sigma_{P'}, \mathcal{M}_{P'}) \longrightarrow (P, \sigma_P, \mathcal{M}_P)$, by [7], there exists a unique MS-morphism $h : (P', \sigma_{P'}) \longrightarrow (P, \sigma_P)$ such that $\rho_i h = g_i$. We show that h is an UMS-morphism. Let $B = P \cap \pi^{-1}(D_i) = \rho^{-1}(D_i) \in \sigma_P, D_i \in \sigma_{X_1 \times X_2}$. We have $\mathcal{M}_{P'}(h^{-1}(B)) = \mathcal{M}_{P'}(g_i^{-1}(D_i))$. Now, similarly to Theorem 2.3, we can prove that the uniqueness of h, and so the proof is complete. □

We recall from category Act-S, the pushout for S-homomorphisms $f_1 : A \longrightarrow B$ and $f_2 : A \longrightarrow C$ is the S-act $D = \dfrac{B \sqcup C}{\theta}$ which θ is the congruence relation on $B \sqcup C$ generated by all pairs $(f_1(a), f_2(a)), a \in A$ with S-homomorphisms $q_i = \pi u_i$, in which $u_1 : B \longrightarrow B \sqcup C$ and $u_2 : C \longrightarrow B \sqcup C$ are inclusions, π_i is the canonical epimorphism.

Theorem 2.6. *The category Uncertain Meas Act-S is closed under pushouts.*

Proof. Consider UMS-morphisms $f_1 : (A, \sigma_A, \mathcal{M}_\mathcal{A}) \longrightarrow (B, \sigma_B, \mathcal{M}_\mathcal{B})$ and $f_2 : (A, \sigma_A, \mathcal{M}_\mathcal{A}) \longrightarrow (C, \sigma_C, \mathcal{M}_\mathcal{C})$. We recall from [7], the pushout of MS-morphisms $f_1 : (A, \sigma_A) \longrightarrow (B, \sigma_B)$ and $f_2 : (A, \sigma_A) \longrightarrow (C, \sigma_C)$ is MS-morphisms $q_1 : (B, \sigma_B) \longrightarrow (D, \sigma_D)$, $q_2 : (B, \sigma_B) \longrightarrow (D, \sigma_D)$, which $\sigma_D = \{D' \subseteq D | q_1^{-1}(D') \in \sigma_B, q_2^{-1}(D') \in \sigma_C\}$. Define $\mathcal{M} : \sigma_D \longrightarrow [0, 1]$ such that

$$\mathcal{M}(D) = \begin{cases} \mathcal{M}_B(q_1^{-1}(D)) \vee \mathcal{M}_C(q_2^{-1}(D)) & \mathcal{M}_B(q_1^{-1}(D)) \vee \mathcal{M}_C(q_2^{-1}(D)) < 0.5 \\ 1 - \mathcal{M}_B(q_1^{-1}(D)) \vee \mathcal{M}_C(q_2^{-1}(D)) & \mathcal{M}_B(q_1^{-1}(D^c)) \vee \mathcal{M}_C(q_2^{-1}(D^c)) < 0.5 \\ 0.5 & o.w \end{cases}$$

By the similar method of Theorem 2.4, we can prove that the category Uncer Act-S is closed under pushouts. □

Theorem 2.7. *The category Uncertain Meas Act-S is closed under equalizers.*

Proof. Assume $f_1, f_2 : (A, \sigma_A, \mathcal{M}_A) \longrightarrow (B, \sigma_B, \mathcal{M}_B)$, are UM$S$-morphisms. Since [7], the equalizer of morphisms f_1, f_2 in the category Meas-S, is the S-act $E = \{a \in A | f_1(a) = f_2(a)\}$ and $\sigma_E = \{\iota^{-1}(A') | A' \in \sigma_A, \iota : E \longrightarrow A\}$. Consider mapping $\mathcal{M}_E : \sigma_E \longrightarrow [0, 1]$ such that $\mathcal{M}_E(D) = \mathcal{M}_A(D)$ for any $D = \iota^{-1}(A')$, $A' \in \sigma_A$. It is easy to check that $\mathcal{M}_E(D)$ is an Uncertain Measurable S-acts and ι is an UMS-morphism. For any Uncertain Measurable S-act $(E', \sigma_{E'}, \mathcal{M}_{E'})$ and UMS-morphism $g : (E', \sigma_{E'}, \mathcal{M}_{E'}) \longrightarrow (A, \sigma_A, \mathcal{M}_A)$, by [7] there exists a unique MS-morphism $h : (E', \sigma_{E'}, \mathcal{M}_{E'}) \longrightarrow (E, \sigma_E, \mathcal{M}_E)$ such that $\iota h = g$. We show that h is a UMS-morphism. Let $D \in \sigma_E$. We have $\mathcal{M}_{E'}(h^{-1}(D)) = \mathcal{M}_{E'}(h^{-1}\iota^{-1}(A'))$, $A' \in \sigma_A$. So, $\mathcal{M}_{E'}(h^{-1}(D)) = \mathcal{M}_{E'}(g^{-1}(A')) \leq \mathcal{M}_A(D) = \mathcal{M}_E(D)$ and the proof is complete.

□

Theorem 2.8. *The category Uncertain Meas Act-S is closed under coequalizers.*

Proof. Consider UMS-morphisms $f_1, f_2 : (A, \sigma_A, \mathcal{M}_A) \longrightarrow (B, \sigma_B, \mathcal{M}_B)$. Since [7], the co-equalizer of $f_1,, f_2$ in category Meas Act-S is the measurable S-act $(\frac{B}{\nu}, \sigma_{\frac{B}{\nu}})$ with MS-morphism $\pi : (B, \sigma_B) \longrightarrow (\frac{B}{\nu}, \sigma_{\frac{B}{\nu}})$, which $\sigma_{\frac{B}{\nu}} = \{B' \subseteq \frac{B}{\nu} | \pi^{-1}(B') \in \sigma_B\}$, $\pi : B \longrightarrow \frac{B}{\nu}$ is an S-homomorphism such that $\pi(b) = [b]_\nu, b \in B$. Now consider $\mathcal{M}_{\frac{B}{\nu}} : \frac{B}{\nu} \longrightarrow [0, 1]$ such that $\mathcal{M}_{\frac{B}{\nu}}(B') = \mathcal{M}_B(B')$, which is an uncertain measure. Obviously, π is an UMS-morphism. For any Uncertain Measurable S-act $(C, \sigma_C, \mathcal{M}_C)$ and UMS-morphism $g : (B, \sigma_B, \mathcal{M}_B) \longrightarrow (C, \sigma_C, \mathcal{M}_C)$, by [7] there exists a unique MS-morphism $h : (C, \sigma_C) \longrightarrow (\frac{B}{\nu}, \sigma_{\frac{B}{\nu}})$ such that

$hg = \pi$. We show that h is an UMS-morphism. Let $B' \in \sigma_{\frac{B}{\nu}}$. Since $h^{-1}(B') \in \sigma_C$, we have $\mathcal{M}_B(g^{-1}h^{-1}(B')) = \mathcal{M}_B(\pi^{-1}(B')) \leq \mathcal{M}_{\frac{B}{\nu}}(B')$. Thus h is an UMS-morphism and complete our proof. □

3 Conclusion

We show that the category of Uncertain Measurable S-acts is closed under product, coproduct, pushout, and pullback. Also, this category is closed under equalizer and coequalizer.

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