Palestine Journal of Mathematics

Vol 13(2)(2024) , 134–143

FIXED POINT RESULTS FOR Θ-$\mathcal{R}$-WEAK FUZZY CONTRACTION VIA BINARY RELATION

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: 47H10,37C25, 54H25, 55M20, 58C30.

Keywords and phrases: Fixed point theory, contraction principle, fuzzy metric, binary relation, fuzzy contraction

Abstract In this study, we inaugurate a relation-theoretic methodology to explore the outcomes of fuzzy metrical fixed points. We introduce the concept of Θ-$\mathcal{R}$-weak fuzzy contraction and substantiate specific fixed point theorems applicable to this class of mappings within the framework of complete fuzzy metric spaces. This is accomplished through the utilization of the binary relation $\mathcal{R}$. The outcomes presented herein amalgamate, extend, and refine various antecedent findings, thereby contributing to the scholarly discourse in the field.

1 Introduction

L.A. Zadeh [2] introduced the concept of fuzzy sets as a mathematical framework to address ambiguity and vagueness in practical applications. The notion of fuzzy sets has evolved into a pivotal and effective modeling tool. The definition of fuzzy metric, a critical concern in fuzzy topology, has been approached by researchers through various methodologies (see, for instance, [3, 4]). Kramosil and Michalek [5] established the concept of a fuzzy metric space by adapting the probabilistic metric idea to the fuzzy context. George and Veeramani [7] subsequently refined Kramosil and Michalek's fuzzy metric space to establish a Hausdorff topology within this framework.

A cornerstone within nonlinear functional analysis is fixed point theory, representing a pivotal area of study that provides extensive mathematical tools for addressing challenges emerging in various branches of mathematics. The foundational metric fixed point result is intricately linked to the Banach contraction principle [1], extensively investigated and refined through diverse methodologies within the realm of abstract metric spaces

The realm of fixed point theory witnesses a burgeoning area of exploration in relation-theoretic fixed point results, originally conceptualized by Turinici [20] through the introduction of an order-theoretic fixed point framework. Ran and Reurings [21] presented the Banach contraction principle in a natural order-theoretic guise in 2004, offering an application to matrix equations. Alam and Imdad further demonstrated a relation-theoretic adaptation of the Banach contraction principle, synthesizing established order-theoretic theorems with an arbitrary binary relation. Subsequently, various fixed point outcomes were proposed, each providing a unique perspective on binary relations (e.g., see [28, 27]).

In recent years, there has been a burgeoning interest in the exploration of fixed point theory within the realm of fuzzy metric spaces. Gregori and Sapena [8] laid the groundwork by introducing the definition of fuzzy contractive mappings and deriving specific fixed point results. Subsequently, Mihet [13] contributed to this field by proposing the concept of fuzzy ψ-contractive mappings. The study of fuzzy $\mathcal{H}$-contractive mappings was further developed and investigated by Wardowski [14]. A novel contractive condition was introduced by A. Moussaoui et al. [18], employing a set of control functions known as extended $\mathcal{FZ}$-simulation functions. Saleh et al. [24] advanced the field by introducing the notion of fuzzy Θ-contractive mappings, utilizing an auxiliary function $\Theta : (0,1) \to (0,1)$ that satisfies certain appropriate assumptions. For an in-depth exploration of current advancements in metric and fuzzy metric fixed point theory, along with related techniques, refer to (e.g., [8, 14, 15, 16, 17, 18, 19, 22, 23, 25, 31, 32, 33, 34]).

2 Preliminaries

Definition 2.1. [12] An operation $* : [0,1]^2 \to [0,1]$ is a continuous t-norm if $([0,1], *)$ is an Abelien topological monoid with unit 1 such that $\alpha_1 * \alpha_2 \geq \alpha_3 * \alpha_4$ whenever $\alpha_1 \geq \alpha_3$ and $\alpha_2 \geq \alpha_4$, for all $\alpha_i \in [0,1], i = 1,2,3,4$.

Example 2.2.

i) $\alpha_1 *_m \alpha_2 = \min\{\alpha_1, \alpha_2\}$,

ii) $\alpha_1 *_L \alpha_2 = \max[0, \alpha_1 + \alpha_2 - 1]$,

iii) $\alpha_1 *_p \alpha_2 = \alpha_1.\alpha_2$.

Definition 2.3. [7] The 3-tuple $(\Lambda, \zeta, *)$ is said to be a fuzzy metric space if ζ is an arbitrary set, $*$ is a continuous t-norm and ζ is a fuzzy set on $\Lambda^2 \times (0,\infty)$ satisfying:

$(\mathcal{GV}1)$ $\zeta(x,y,t) > 0$,

$(\mathcal{GV}2)$ $\zeta(x,y,t) = 1$ iff $x = y$,

$(\mathcal{GV}3)$ $\zeta(x,y,t) = \zeta(y,x,t)$,

$(\mathcal{GV}4)$ $\zeta(x,z,t+s) \geq \zeta(x,y,t) * \zeta(y,z,s)$,

$(\mathcal{GV}5)$ $\zeta(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

for all $x, y, t \in \Lambda$ and $s, t > 0$.

The function $\zeta(x,y,t)$ can be thought of as the degree of nearness of x and y with respect to the variable t.

Example 2.4. ([7, 10]) Let $j : \Lambda \to \mathbb{R}^+$ be a one-to-one function, $S : \mathbb{R}^+ \to [0,\infty)$ be an increasing continuous function and $\alpha, \beta > 0$. Define

$$\zeta(x,y,t) = \left(\frac{(\min\{j(x), j(y)\})^\alpha + S(t)}{(\max\{j(x), j(y)\})^\alpha + S(t)} \right)^\beta . \tag{2.1}$$

Then $(\zeta, .)$ is a fuzzy metric on Λ.

In particular, by taking $\alpha = \beta = 1$ and j as the identity function, then we obtain the following examples

(i) Let $\Lambda = \mathbb{R}^+$ and take S as the identity function in (2.1), we have

$$\zeta(x,y,t) = \frac{\min\{x,y\} + t}{\max\{x,y\} + t}.$$

Then $(\Lambda, \zeta, *)$ is a fuzzy metric space.

(ii) Let $\Lambda = (0,\infty)$ and take S as the zero function in (2.1), we have

$$\zeta(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}.$$

Then $(\Lambda, \zeta, .)$ is a fuzzy metric space.

Example 2.5. [7] Let $\Lambda = \mathbb{R}$. Define $a * b = ab$ for all $a, b \in [0,1]$ and the function $\zeta : \Lambda \times \Lambda \times (0,\infty) \to [0,1]$ by

$$\zeta(x,y,t) = \left[\exp\left(\frac{|x-y|}{t} \right) \right]^{-1} \text{ for all } x, y \in \Lambda, t > 0.$$

Then $(\Lambda, \zeta, *)$ is a fuzzy metric space.

Example 2.6. [7] Let (Λ, d) be a metric space. Define $a * b = \min(a, b)$ for all $a, b \in [0, 1]$ and

$$\zeta(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}\ , k, m, n \in \mathbb{R}^+.$$

Then $(\Lambda, \zeta, *)$ is a fuzzy metric space.
Letting $k = m = n = 1$, we get

$$\zeta(x, y, t) = \frac{t}{t + d(x, y)},$$

we call this fuzzy metric induced by a metric d the standard fuzzy metric.

Lemma 2.7. *[6] $\zeta(x, y, .)$ is nondecreasing function for all x, y in Λ.*

Definition 2.8. [7] Let $(\Lambda, \zeta, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\} \subseteq \Lambda$ is said to be convergent or converges to $x \in \Lambda$ if $\lim_{n\to\infty} \zeta(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\} \subseteq \Lambda$ is said to be an Cauchy sequence if for all $\varepsilon \in (0, 1)$ and $t > 0$, $\exists n_0 \in \mathbb{N}$ such that $\zeta(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

(iii) A fuzzy metric space in which each Cauchy sequence is convergent is called a complete fuzzy metric space.

Definition 2.9. [8]Let $(\Lambda, \zeta, *)$ be a fuzzy metric space. A mapping $\mathcal{H} : \Lambda \to \Lambda$ is said to be a fuzzy contractive mapping if $\exists k \in (0, 1)$ such that

$$\frac{1}{\zeta(\mathcal{H}x, \mathcal{H}y, t)} - 1 \leq k \left(\frac{1}{\zeta(x, y, t)} - 1 \right),$$

for all $x, y \in \Lambda$ and $t > 0$.

Definition 2.10. [8] A sequence $\{x_n\}$ in a fuzzy metric space $(\Lambda, \zeta, *)$ is said to be fuzzy contractive if $\exists k \in (0, 1)$ such that

$$\frac{1}{\zeta(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left(\frac{1}{\zeta(x_n, x_{n+1}, t)} - 1 \right),$$

for all $n \in \mathbb{N}$ and $t > 0$.

Theorem 2.11. *[8] Let $(\Lambda, \zeta, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. If $\mathcal{H} : \Lambda \to \Lambda$ is a fuzzy contractive mapping then $\mathcal{H}$ has a unique fixed point.*

As a result of his study the following theorem was established by Tirado [11].

Theorem 2.12. *[11] Let $(\Lambda, \zeta, *_L)$ be a complete fuzzy metric space and $\mathcal{H} : \Lambda \to \Lambda$ be a mapping such that*

$$1 - \zeta(\mathcal{H}x, \mathcal{H}y, t) \leq k \left(1 - \zeta(x, y, t)\right).$$

for all $x, y \in \Lambda$,$t > 0$ and for some $k \in (0, 1)$. Then $\mathcal{H}$ has a unique fixed point.

In 2020, Saleh *et al.* [24] brought in the concept of fuzzy Θ-contractive mappings, which was inspired by the results of Jleli *et al.* [19], by employing an auxiliary function $\Theta : (0, 1) \to (0, 1)$ fulfilling the following conditions

(Ω_1) Θ is non-decreasing,

(Ω_2) Θ is continuous,

(Ω_3) $\lim_{n\to\infty} \Theta(\beta_n) = 1$ if and only if $\lim_{n\to\infty} \beta_n = 1$, where $\{\beta_n\}$ is a sequence in $(0, 1)$.

Example 2.13. [24] Let $\Theta : (0,1) \to (0,1)$ be a function defined by

$$\Theta(\beta) = e^{1-\frac{1}{\beta}}, \text{ for all } \beta \in (0,1).$$

Example 2.14. [24] Let $\Theta : (0,1) \to (0,1)$ be a function defined by

$$\Theta(\beta) = 1 - \cos\left(\frac{\pi}{2}\beta\right), \text{ for all } \beta \in (0,1).$$

Definition 2.15. [24] Let $(X, M, *)$ be a fuzzy metric space. A mapping $\mathcal{H} : \Lambda \to \Lambda$ is said to be a fuzzy Θ-contractive mapping w.r.t $\Theta \in \Omega$ if $\exists k \in (0,1)$ such that

$$\zeta(\mathcal{H}x, \mathcal{H}y, t) < 1 \Rightarrow \Theta(\zeta(\mathcal{H}x, \mathcal{H}y, t)) \geq [\Theta(\zeta(x, y, t))]^k,$$

for all $x, y \in \Lambda$ and $t > 0$.

In their work [24], the authors proved that these new class of fuzzy contractions include the classes of Gregori and Sapena [8] and Tirado [11], thereafter they established the following fixed point theorem.

Theorem 2.16. *[24] Let $(\Lambda, \zeta, *)$ be a complete fuzzy metric space and $\mathcal{H} : \Lambda \longrightarrow \Lambda$ be a fuzzy Θ-contractive mapping, then $\mathcal{H}$ has a unique fixed point.*

The following core relation theoretic notions and concepts are necessary in order to establish our results.

Definition 2.17. [26] A subset $\mathcal{R}$ of $\Lambda \times \Lambda$ is called a binary relation on Λ. If $(x, y) \in \mathcal{R}$, then we say that x is related to y under $\mathcal{R}$ (or $x\mathcal{R}y$). If either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$, then we write $[x, y] \in \mathcal{R}$.

Definition 2.18. [26] A binary relation $\mathcal{R}$ on a non-empty set Λ is called:

(i) reflexive if $x\mathcal{R}x$ for all $x \in \Lambda$,

(ii) transitive if $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$ for all $x, y, z \in \Lambda$,

(iii) antisymmetric if $x\mathcal{R}y$ and $y\mathcal{R}x$ imply $x = y$ for all $x, y \in \Lambda$,

Definition 2.19. [30] A binary relation $\mathcal{R}$ on a non-empty set Λ is called $\mathcal{H}$-closed if

$$(x, y) \in \mathcal{R} \Rightarrow (\mathcal{H}x, \mathcal{H}y) \in \mathcal{R},$$

for all $x, y \in \Lambda$, where $\mathcal{H}$ is a self-mapping defined on Λ.

Definition 2.20. [29] Let X be a nonempty set and $\mathcal{R}$ a binary relation on Λ. For $x, y \in \Lambda$, a path of lenght l in $\mathcal{R}$ from x to y is a finite sequence $\{\delta_0, \delta_1, \delta_2, ..., \delta_l\} \subset \vartheta$ satisfying:

$\mathcal{E}1)$ $\delta_0 = x$ and $\delta_l = y$,

$\mathcal{E}2)$ $(\delta_j, \delta_{j+1}) \in \mathcal{R}$ for all j $(0 \leq j \leq l-1)$.

Note that, a path of length l involves $l+1$ elements of Λ.

Let Λ be a nonempty set and $\mathcal{H} : \Lambda \to \Lambda$ be a self-mapping. We will use the following notation:

$$\Lambda(\mathcal{H}, \mathcal{R}) := \{x \in \vartheta : (x, \mathcal{H}x) \in \mathcal{R}\},$$

and

$$\gamma(x, y, \mathcal{R}) := \{\text{the family of all path in } \mathcal{R} \text{ from } x \text{ to } y\}.$$

Definition 2.21. Let $(\Lambda, \zeta, *)$ be a fuzzy metric space equipped with a binary relation $\mathcal{R}$. Define

$$\mathcal{E} = \{(x, y) \in \mathcal{R} : \zeta(x, y, t) < 1\}.$$

3 Main results

First, we introduce the following concept of $\mathcal{R}$-Θ-weak fuzzy contraction.

Definition 3.1. Let $(\Lambda, \zeta, *)$ be a fuzzy metric space equipped with a binary relation $\mathcal{R}$. A self mapping $\mathcal{H} : \Lambda \to \Lambda$ is said to be $\mathcal{R}$-Θ- weak fuzzy contraction if there exist $\Theta \in \Omega$ and $k \in (0,1)$ such that

$$[\Theta(\min\{\zeta(x,y,t), \zeta(x,\mathcal{H}x,t), \zeta(y,\mathcal{H}y,t)\})]^k \leq \Theta(\zeta(\mathcal{H}x,\mathcal{H}y,t)), \tag{3.1}$$

for all $x, y \in \mathcal{E}$ and $t > 0$.

Example 3.2. Let $(\Lambda, \zeta, *)$ represent the complete fuzzy metric space with $\Lambda = [0,1]$ and $\zeta(x,y,t) = \frac{t}{t+d(x,y)}$, where $*$ signifies the product t-norm. In this framework, we turn our attention to the mapping $\mathcal{H} : \Lambda \to \Lambda$, explicitly defined as:

$$\mathcal{H}x = \begin{cases} \frac{1}{4} & \text{if } x = 1, \\ \frac{1}{2} & \text{if } x \in [0,1). \end{cases}$$

Note that, for $x \in [0,1)$ and $y = 1$ with $t > 0$, the expression:

$$\begin{aligned} \zeta(\mathcal{H}x,\mathcal{H}y,t) &= \frac{t}{t + d(\mathcal{H}x,\mathcal{H}y)} \\ &= \frac{t}{t+\frac{1}{4}} < 1. \end{aligned}$$

Consequently, two distinct cases emerge:
Case I: In the scenario where $x \in [\frac{1}{4}, 1)$ and $y = 1$, it follows that

$$\begin{aligned} \min\{\zeta(x,1,t), \zeta(x,\mathcal{H}x,t), \zeta(1,\mathcal{H}1,t)\} &= \zeta(1,\mathcal{H}1,t) \\ &= \frac{t}{t+\frac{3}{4}}. \end{aligned}$$

Case II: Conversely, when $x \in [0, \frac{1}{4})$ and $y = 1$, we obtain that

$$\begin{aligned} \min\{\zeta(x,1,t), \zeta(x,\mathcal{H}x,t), \zeta(1,\mathcal{H}1,t)\} &= \zeta(x,1,t) \\ &= \frac{t}{t+|x-1|}. \end{aligned}$$

In both cases, considering $\Theta(\beta) = e^{1-\frac{1}{\beta}}$ for all $\beta \in (0,1)$ and $k = \frac{1}{3}$, we see that $\mathcal{H}$ constitutes $\mathcal{R}$-Θ-weak fuzzy contraction.

Theorem 3.3. *Let $(\Lambda, \zeta*)$ be a complete fuzzy metric space endowed with a binary relation $\mathcal{R}$ and $\mathcal{H} : \Lambda \to \Lambda$ be a self mapping such that*

(i) $\Lambda(\mathcal{H},\mathcal{R}) \neq \emptyset$,

(iii) *$\mathcal{R}$ is $\mathcal{H}$-closed,*

(iv) *$\mathcal{H}$ is $\mathcal{R} - \Theta$-contraction,*

(v) *$\mathcal{H}$ is continuous.*

Then $\mathcal{H}$ has a fixed point.

Proof. Let x_0 be an arbitrary point in $\Lambda(\mathcal{H},\mathcal{R})$. Define $\{x_n\}$ by

$$\mathcal{H}x_n = x_{n+1},$$

for all $n \geq 0$. If there exists $m \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then it follows that x_{n_0} is a fixed point of $\mathcal{H}$. Hence, assume that $x_n \neq x_{n+1}$ $\forall n \in \mathbb{N}$. Then $\zeta(\mathcal{H}x_{n-1}, \mathcal{H}x_n, t) < 1$ for all $n \in \mathbb{N}$

and $t > 0$. Since $(x_0, \mathcal{H}x_0) \in \mathcal{R}$ and $\mathcal{R}$ is $\mathcal{H}$-closed, then $(x_n, x_{n+1}) \in \mathcal{E}$ for all $n \geq 0$. From (3.1), we get

$$\begin{aligned} 1 > \Theta(\zeta(\mathcal{H}x_{n-1}, \mathcal{H}x_n, t)) &\geq [\Theta(\min\{\zeta(x_{n-1}, x_n, t)), \zeta(x_{n-1}, \mathcal{H}x_{n-1}, t), \zeta(x_n, \mathcal{H}x_n, t)\})]^k \\ &= [\Theta(\min\{\zeta(x_{n-1}, x_n, t)), \zeta(x_{n-1}, x_n, t), \zeta(x_n, x_{n+1}, t)\})]^k \\ &= [\Theta(\min\{\zeta(x_{n-1}, x_n, t)), \zeta(x_n, x_{n+1}, t)\})]^k. \end{aligned} \tag{3.2}$$

If for some $n \in \mathbb{N}$, $\min\{\zeta(x_{n-1}, x_n, t)), M(x_n, x_{n+1}, t)\} = \zeta(x_n, x_{n+1}, t)$, by 3.2, it follows that

$$\begin{aligned} \Theta(\zeta(x_n, x_{n+1}, t)) &\geq ([\Theta(\zeta(x_n, x_{n+1}, t))]^k \\ &> \Theta(\zeta(x_n, x_{n+1}, t), \end{aligned} \tag{3.3}$$

a contradiction. That is, $\min\{\zeta(x_{n-1}, x_n, t)), \zeta(x_n, x_{n+1}, t)\} = \zeta(x_{n-1}, x_n, t)$ for each $n \in \mathbb{N}$. From 3.2, we derive that

$$\begin{aligned} 1 > \Theta(\zeta(x_n, x_{n+1}, t)) &\geq [\Theta(\min\{\zeta(x_{n-1}, x_n, t)), \zeta(x_{n-1}, \mathcal{H}x_{n-1}, t), \zeta(x_n, \mathcal{H}x_n, t)\})]^k \\ &= [\Theta(\zeta(x_{n-1}, x_n, t)))]^k \\ &\geq [\Theta(\zeta(x_{n-2}, x_{n-1}, t))]^{k^2} \\ &\vdots \\ &\geq [\Theta(\zeta(x_0, x_1, t))]^{k^n}. \end{aligned} \tag{3.4}$$

Taking the limit as $n \to \infty$, we deduce

$$\lim_{n\to\infty} \Theta(\zeta(x_n, x_{n+1}, t)) = 1,$$

and by Ω_3, we obtain

$$\lim_{n\to\infty} \zeta(x_n, x_{n+1}, t) = 1, \tag{3.5}$$

Next, we prove the Cauchyness of the sequence $\{x_n\}$. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon \in (0, 1)$, $\ell > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\zeta(x_{m_k}, x_{n_k}, \ell) \leq 1 - \epsilon. \tag{3.6}$$

By Lemma 2.7, we derive that

$$\zeta(x_{m_k}, x_{n_k}, \frac{\ell}{2}) \leq 1 - \epsilon. \tag{3.7}$$

By taking n_k as the lowest value fulfilling (3.7), we get

$$\zeta(x_{m_k-1}, x_{n_k}, \frac{\ell}{2}) > 1 - \epsilon. \tag{3.8}$$

Using (3.1) with $x = x_{m_k-1}$ and $x = x_{n_k-1}$, we obtain

$$\begin{aligned} \Theta(\zeta(x_{m_k}, x_{n_k}, \ell) &\geq [\Theta(\min\{\zeta(x_{m_k-1}, x_{n_k-1}, \ell), \zeta(x_{m_k-1}, \mathcal{H}x_{m_k-1}, \ell), \\ &\qquad \zeta(x_{n_k-1}, \mathcal{H}x_{n_k-1}, \ell)\})]^k, \\ &= [\Theta(\min\{\zeta(x_{m_k-1}, x_{n_k-1}, \ell), \zeta(x_{m_k-1}, x_{m_k}, \ell), \\ &\qquad \zeta(x_{n_k-1}, x_{n_k}, \ell)\}]^k, \\ &> \Theta(\min\{\zeta(x_{m_k-1}, x_{n_k-1}, \ell), \zeta(x_{m_k-1}, x_{m_k}, \ell), \\ &\qquad \zeta(x_{n_k-1}, x_{n_k}, \ell)\}, \end{aligned} \tag{3.9}$$

where,

$$\begin{aligned}\phi_k &= \min\{\zeta(x_{m_k-1}, x_{n_k-1}, \ell), \zeta(x_{m_k-1}, \mathcal{H}x_{m_k-1}, \ell),\\ &\quad \zeta(x_{n_k-1}, \mathcal{H}x_{n_k-1}, \ell)\},\\ &= \min\{\zeta(x_{m_k-1}, x_{n_k-1}, \ell), \zeta(x_{m_k-1}, x_{m_k}, \ell),\\ &\quad \zeta(x_{n_k-1}, x_{n_k}, \ell)\}.\end{aligned}$$

Taking the limit as $k \to \infty$ in the last equality, by 3.5, we obtain

$$\begin{aligned}\lim_{k\to\infty} \phi_k &= \lim_{k\to\infty} \min\{\zeta(x_{m_k-1}, x_{n_k-1}, \ell), \zeta(x_{m_k-1}, \mathcal{H}x_{m_k-1}, \ell),\\ &\quad \zeta(x_{n_k-1}, \mathcal{H}x_{n_k-1}, \ell)\},\\ &= \min\{\zeta(x_{m_k-1}, x_{n_k-1}, \ell), 1, 1\}\\ &= \lim_{k\to\infty} \zeta(x_{m_k-1}, x_{n_k-1}, \ell). \end{aligned} \tag{3.10}$$

On the other hand, by (3.6),(3.8) and $(\mathcal{GV}4)$, we obtain

$$\begin{aligned}1-\epsilon &\geq \zeta(x_{m_k}, x_{n_k}, \ell)\\ &> \zeta(x_{m_k-1}, x_{n_k-1}, \ell)\\ &\geq \zeta(x_{m_k-1}, x_{n_k}, \frac{\ell}{2}) * \zeta(x_{n_k}, x_{n_{k-1}}, \frac{\ell}{2})\\ &> (1-\epsilon) * \zeta(x_{n_k}, x_{n_k-1}, \frac{\ell}{2}).\end{aligned}$$

Taking the limit as $k \to \infty$ in both sides of the last inequality, using (3.5), (3.9) and (3.10), we derive

$$\lim_{k\to\infty} \zeta(x_{m_k}, x_{n_k}, \ell) = \lim_{k\to\infty} \zeta(x_{m_k-1}, x_{n_k-1}, \ell) = 1-\epsilon. \tag{3.11}$$

Letting $k \to \infty$ in (3.9), empolying (3.11) and the continuity of Θ, we get

$$[\Theta(1-\epsilon)]^k \leq \Theta(1-\epsilon),$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. As $(\Lambda, \zeta, *)$ is a complete fuzzy metric space, there exists $\tilde{x} \in \Lambda$ such that $x_n \to \tilde{x}$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \zeta(x_n, \tilde{x}, t) = 1. \tag{3.12}$$

By the continuity of $\mathcal{H}$ and (3.12), we obtain

$$\lim_{n\to\infty} \zeta(x_{n+1}, \mathcal{H}\tilde{x}, t) = \lim_{n\to\infty} \zeta(\mathcal{H}x_n, \mathcal{H}\tilde{x}, t) = 1.$$

The uniqueness of the limit yields $\mathcal{H}\tilde{x} = \tilde{x}$, thus $\tilde{x}$ is a fixed point of $\mathcal{H}$. □

Now, by replacing the continuity of the self mapping $\mathcal{H}$, we have the following result.
Next, we show the uniqueness of the fixed point. Let $\mathcal{F}(\mathcal{H}) := \{x \in \Lambda : x \text{ is a fixed point of } \mathcal{H}\}$.

Theorem 3.4. *In addition to the assumptions of Theorem 3.3, if $\mathcal{R}$ is transitive and $\gamma(x, y, \mathcal{R}) \neq \emptyset$ for all $x, y \in \mathcal{F}(\mathcal{H})$, then $\mathcal{H}$ has a unique fixed point.*

Proof. we argue by contradiction, suppose that $\tilde{x}$ and x^* are two distinct fixed points of $\mathcal{H}$. That is,

$$\mathcal{H}\tilde{x} = \tilde{x} \text{ and } \mathcal{H}x^* = x^* \text{ with } \tilde{x} \neq x^*.$$

Hence $\zeta(\mathcal{H}\tilde{x}, \mathcal{H}x^*, t) < 1$. As $\gamma(\tilde{x}, x^*, \mathcal{R}) \neq \emptyset$, there exists a path $\{\delta_0, \delta_1, \delta_2, ..., \delta_q\}$ of some finite length q in $\mathcal{R}$ from $\tilde{x}$ to x^* such that

$$\delta_0 = \tilde{x}, \delta_k = x^*, (\delta_j, \delta_{j+1}) \in \mathcal{R}, j = 0, 1, 2, ..., q-1.$$

As $\mathcal{R}$ is transitive, we derive that

$$\begin{aligned}&(\tilde{x},\delta_1)\in\mathcal{R}\\&(\delta_1,\delta_2)\in\mathcal{R}\\&\vdots\\&(\delta_{q-1},x^*)\in\mathcal{R}\\\Rightarrow&(\tilde{x},x^*)\in\mathcal{R}.\end{aligned}$$

It follows that

$$\begin{aligned}\Theta(\zeta(\tilde{x},x^*,t))&=\Theta(\zeta(\mathcal{H}\tilde{x},\mathcal{H}x^*,t))\\&\geq[\Theta(\min\{\zeta(\tilde{x},x^*,t),\zeta(\tilde{x},\mathcal{H}\tilde{x},t),\zeta(x^*,\mathcal{H}x^*,t)\})]^k\\&=[\Theta(\min\{\zeta(\tilde{x},x^*,t),\zeta(\tilde{x},\tilde{x},t),\zeta(x^*,x^*,t)\})]^k\\&=[\Theta(\min\{\zeta(\tilde{x},x^*,t),1,1\})]^k\\&\geq([\Theta(\zeta(\tilde{x},x^*,t)\})]^k,\end{aligned}$$

a contradiction. Thus, the fixed point of $\mathcal{H}$ is unique. □

Corollary 3.5. *Let $(\Lambda,\zeta,*)$ be a complete fuzzy metric space endowed with a binary relation $\mathcal{R}$ and $\mathcal{H}:\Lambda\to\Lambda$ be a self mapping such that*

(i) $\Lambda(\mathcal{H},\mathcal{R})\neq\emptyset$,

(iii) *$\mathcal{R}$ is $\mathcal{H}$-closed,*

(iv) $\left[1+\sin\left(\frac{\pi}{2}(\mathcal{L}(x,y,t)-1)\right)\right]^k\leq 1+\sin\left(\frac{\pi}{2}(\zeta(\mathcal{H}x,\mathcal{H}y,t)-1)\right)$, *for all $x,y\in\mathcal{E}$ and $t>0$,*

(v) *$\mathcal{H}$ is continuous.*

$\mathcal{L}(x,y,t)=\min\{\zeta(x,y,t),\zeta(x,\mathcal{H}x,t),\zeta(y,\mathcal{H}y,t)\}$. *Then $\mathcal{H}$ has a fixed point.*

Proof. The proof follows from Theorem 3.3 by taking $\Theta(\beta)=1+\sin\left(\frac{\pi}{2}(\beta-1)\right)$ for all $\beta\in(0,1)$. □

Corollary 3.6. *Let $(\Lambda,\zeta,*)$ be a complete fuzzy metric space endowed with a binary relation $\mathcal{R}$ and $\mathcal{H}:\Lambda\to\Lambda$ be a self mapping such that*

(i) $\Lambda(\mathcal{H},\mathcal{R})\neq\emptyset$,

(iii) *$\mathcal{R}$ is $\mathcal{H}$-closed,*

(iv) $[\mathcal{L}(x,y,t)]^k\leq\zeta(\mathcal{H}x,\mathcal{H}y,t)$, *for all $x,y\in\mathcal{E}$ and $t>0$,*

(v) *$\mathcal{H}$ is continuous.*

Where $\mathcal{L}(x,y,t)=\min\{\zeta(x,y,t),\zeta(x,\mathcal{H}x,t),\zeta(y,\mathcal{H}y,t)\}$. *Then $\mathcal{H}$ has a fixed point.*

Proof. The proof follows from Theorem 3.3 by taking $\Theta(\beta)=\beta$ for all $\beta\in(0,1)$. □

Conclusion

In the current investigation, we introduced the notion of $\mathcal{R}$-Θ-weak fuzzy contraction within the framework of fuzzy metric spaces, incorporating a binary relation. Subsequently, we established results concerning the existence and uniqueness of fixed points for such mappings through the utilization of an auxiliary function. It is noteworthy that by judiciously selecting various instances of the function Θ, we can customize and deduce a diverse range of implications from our fundamental findings. These outcomes may pave the way for further exploration in relation-theoretic and fuzzy fixed point research, particularly its extension to investigating relation-theoretic fuzzy metrical coincidence and common fixed point results.

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Received: 2023-02-15
Accepted: 2024-02-08