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EXISTENCE OF THREE SOLUTIONS FOR A DISCRETE p-LAPLACIAN BOUNDARY VALUE PROBLEM

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Abstract This paper is concerned with boundary value problems for a fourth-order nonlinear difference equation. Sufficient condition are obtained for the existence of at least three solutions, via variational methods and critical point theory. One example is included to illustrate the result.

1 Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of all natural numbers, integers and real numbers, respectively. For any integers a and b with $a \leq b$, $[a,b]_{\mathbb{Z}}$ is defined by the discrete interval $\{a, a+1, ..., b\}$.

Now, we are concerned with the existence of at least three solutions to the fourth-order non linear difference equation

$$\Delta^2(\varphi_p(\Delta^2 u(t-2))) - \Delta(\varphi_p(\Delta u(t-1))) = \alpha f(t,u(t)) + \beta g(t,u(t)),\ t \in [1,N]_{\mathbb{Z}}, \tag{1.1}$$

satisfying the boundary value conditions

$$u(0) = u(N+1) = \Delta u(-1) = \Delta u(N+1) = 0, \tag{1.2}$$

where $N \geq 1$ is an integer, $1 < p < \infty$ is a constant, φ_p is the p-Laplacian operator, that is $\varphi_p(s) = |s|^{p-2}s$, α, β are real parameters positive, f, g $[1,N]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ are two continuous functions, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^0 u(t) = u(t)$, $\Delta^i u(t) = \Delta^{i-1}(\Delta u(t))$ for $i = 1,2,3,4$.

By a solution of (1.1), (1.2), we mean a function $u : [-1, N+2]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ that satisfies both (1.1) and (1.2).

Boundary value problem (BVP, for short) (1.1), (1.2) could be regarded as a discrete analogue of the fourth-order problem

$$\begin{cases} \dfrac{d^2}{dt^2}\left(\varphi_p\left(\dfrac{d^2u(t)}{dt^2}\right)\right) - \dfrac{d}{dt}\left(\varphi_p\left(\dfrac{du(t)}{dt}\right)\right) = \alpha f(t,u(t)) + \beta g(t,u(t)), \quad t \in]0,1[\,, \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{1.3}$$

Difference equations appear in numerous settings and forms, both in mathematics and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields (see, for example [1, 16]). For this reason, in recent years the existence of solutions for difference equations has been studied by many authors, and some results have been obtained by using various methods such as fixed point theorems methods, and the upper and lower solutions methods (see [1, 14, 16] and the references therein). Studying the solvability of difference equations by using variational methods was initiated by Guo and Yu [12]. Since then, by using the critical point theory approaches, such as those based on the mini-max methods and the Morse theory, the existence of solutions for difference equations has been extensively investigated (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 20]).

In this paper, we shall study the existence of at least three solutions of the BVP (1.1), (1.2), via variational methods and critical point theory.

Put

$$F(t,x)=\int_0^x f(t,s)ds \quad \text{and} \quad G(t,x)=\int_0^x g(t,s)ds.$$

To state our main results, we make the following assumptions:

(H_1) There exists δ with $\delta<\lambda_1$ such that

$$\lim_{|x|\to\infty}\sup\max_{t\in[1,N]_{\mathbb{Z}}}\frac{pF(t,x)}{|x|^p}\le\delta,$$

where

$$\lambda_1=\min_{u\in E_N\setminus\{0\}}\frac{\sum_{t=1}^{N+2}|\Delta^2u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}, \tag{1.4}$$

with

$$E_N=\{u:[-1,N+2]_{\mathbb{Z}}\longrightarrow\mathbb{R}\mid u(0)=u(N+1)=\Delta u(-1)=\Delta u(N+1)=0\}. \tag{1.5}$$

We will see in the Section 3 that λ_1 is the first eigenvalue of the nonlinear eigenvalue problem corresponding to the BVP (1.1), (1.2).

(H_2) There exist $c,d\in\,]0,\infty[$ such that $c^p<Nd^p$ and

$$\max_{(t,|x|)\in[1,N]_{\mathbb{Z}}\times[0,c]}F(t,x)<\frac{\lambda_1}{6N}\left(\frac{c}{d}\right)^p\sum_{t=1}^{N}F(t,d).$$

(H_3) There exists $\nu>0$ such that

$$\max\left\{\lim_{|x|\to\infty}\sup\max_{t\in[1,N]_{\mathbb{Z}}}\frac{pG(t,x)}{|x|^p},\ \lim_{x\to0}\sup\max_{t\in[1,N]_{\mathbb{Z}}}\frac{pG(t,x)}{|x|^p}\right\}\leqslant\nu.$$

(H_4) There exists $\rho>0$ such that $\sum_{t=1}^{N}G(t,\rho)>\frac{6\nu\rho^p}{p\lambda_1}$.

The main results in this paper are the following theorems:

Theorem 1.1. *Assume that (H_1) and (H_2) hold, then there exist a nonempty open set $\Lambda\subset[0,1]$ and a positive number γ, for each $\alpha\in\Lambda$ there exist $\xi>0$ such that, for each $\beta\in\,]0,\xi]$, the BVP (1.1), (1.2) has at least three distinct solutions in E_N whose norms are less than γ.*

Theorem 1.2. *Assume that (H_3) and (H_4) hold, then for each compact interval $[a,b]\subset\left]\frac{6\rho^p}{p\sum_{t=1}^{N}G(t,\rho)},\frac{\lambda_1}{\nu}\right[$, there exist $\zeta>0$ such that for each $\beta\in[a,b]$, there exists $\eta>0$ such that for each $\alpha\in\,]0,\eta]$, the BVP (1.1), (1.2) has at least three distinct solutions in E_N whose norms are less than ζ.*

The rest of this paper is organized as follows. In Section 2, contains some preliminary lemmas. Section 3, we introduce the eigenvalue problem (P_0) associated to the BVP (1.1), (1.2). The main results will be proved in Section 4.

2 Variational structure and some lemmas

We consider the vector space defined in (1.5). For $u \in E_N$, define

$$\|u\| = \left(\sum_{t=1}^{N} |u(t)|^p\right)^{1/p}.$$

So $(E_N, \|.\|)$ is an N dimensional reflexive Banach space. In fact, E_N is isomorphic to $\mathbb{R}^N$.
We also put, for every $u \in E_N$

$$\|u\|_\infty = \max_{t\in[1,N]_{\mathbb{Z}}} |u(t)|.$$

Obviously,

$$\|u\|_\infty \leqslant \|u\|.$$

Let the functionals Φ, Ψ_1, and Ψ_2 be defined as follows

$$\Phi(u) = \frac{1}{p}\sum_{t=1}^{N+2} |\Delta^2 u(t-2)|^p + |\Delta u(t-1)|^p,$$

$$\Psi_1(u) = \sum_{t=1}^{N} F(t, u(t)),$$

and

$$\Psi_2(u) = \sum_{t=1}^{N} G(t, u(t)), \tag{2.1}$$

for any $u \in E_N$.
The functional corresponding of BVP (1.1), (1.2) is given by

$$I_{\alpha,\beta} = \Phi - \alpha\Psi_1 - \beta\Psi_2.$$

It is easy to see that Φ, Ψ_1 and Ψ_2 are continuously differentiable and for all $u, v \in E_N$, we obtain

$$\Phi'(u).v = \sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2) + \varphi_p(\Delta u(t-1))\Delta v(t-1),$$

$$\langle \Psi_1'(u), v\rangle = \sum_{t=1}^{N} f(t, u(t))v(t) \quad , \quad \langle \Psi_2'(u), v\rangle = \sum_{t=1}^{N} g(t, u(t))v(t),$$

and

$$I'_{\alpha,\beta}(u).v = \sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2) + \varphi_p(\Delta u(t-1))\Delta v(t-1)$$
$$- \sum_{t=1}^{N} \alpha f(t, u(t))v(t) - \beta g(t, u(t))v(t).$$

Lemma 2.1. *For all $u, v \in E_N$, we have*

1) $\sum_{t=1}^{N+2} \varphi_p(\Delta u(t-1))\Delta v(t-1) = -\sum_{t=1}^{N} \Delta\left(\varphi_p(\Delta u(t-1))\right) v(t).$

2) $\sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2) = \sum_{t=1}^{N} \Delta^2\left(\varphi_p(\Delta^2 u(t-2))\right) v(t).$

Proof. 1) Let $u, v \in E_N$, by the summation by parts formula and the fact that $v(0) = v(N+1) = 0$, it follows that

$$\begin{aligned}\sum_{t=1}^{N} \Delta(\varphi_p(\Delta u(t-1)))v(t) &= [\varphi_p(\Delta u(t-1))v(t)]_1^{N+1} - \sum_{t=1}^{N} \varphi_p(\Delta u(t))\Delta v(t) \\ &= -|\Delta u(0)|^{p-2}\Delta u(0)v(1) - \sum_{t=2}^{N+1} \varphi_p(\Delta u(t-1))\Delta v(t-1) \\ &= -\sum_{t=1}^{N+2} \varphi_p(\Delta u(t-1))\Delta v(t-1).\end{aligned}$$

2) By the summation by parts formula and the fact that $v(0) = v(N+1) = 0$, it follows that

$$\begin{aligned}\sum_{t=1}^{N} \Delta^2 \left(\varphi_p(\Delta^2 u(t-2))\right) v(t) &= \left[\Delta(\varphi_p(\Delta^2 u(t-2)))v(t)\right]_1^{N+1} \\ &\quad - \sum_{t=1}^{N} \Delta(\varphi_p(\Delta^2 u(t-1)))\Delta v(t) \\ &= -\Delta(\varphi_p(\Delta^2 u(-1)))v(1) - \sum_{t=1}^{N} \Delta(\varphi_p(\Delta^2 u(t-1)))\Delta v(t) \\ &= -\sum_{t=0}^{N} \Delta(\varphi_p(\Delta^2 u(t-1)))\Delta v(t) \\ &= -\sum_{t=1}^{N+1} \Delta(\varphi_p(\Delta^2 u(t-2)))\Delta v(t-1).\end{aligned}$$

Similarly, using the summation by parts formula and the fact that $\Delta v(N+1) = \Delta v(-1) = 0$, we get

$$\begin{aligned}\sum_{t=1}^{N} \Delta^2 \left(\varphi_p(\Delta^2 u(t-2))\right) v(t) &= -\left[\varphi_p(\Delta^2 u(t-2))\Delta v(t-1)\right]_1^{N+2} \\ &\quad + \sum_{t=1}^{N+1} \varphi_p(\Delta^2 u(t-1)))\Delta^2 v(t-1) \\ &= \varphi_p(\Delta^2 u(-1))\Delta v(0) + \sum_{t=1}^{N+1} \varphi_p(\Delta^2 u(t-1)))\Delta^2 v(t-1) \\ &= \sum_{t=0}^{N+1} \varphi_p(\Delta^2 u(t-1))\Delta^2 v(t-1) \\ &= \sum_{t=1}^{N+2} \varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2).\end{aligned}$$

This completes the proof of the Lemma 2.1. □

By Lemma 2.1, $I'_{\alpha,\beta}$ can be written as

$$I'_{\alpha,\beta}(u).v = \sum_{t=1}^{N} \left[\Delta^2 \left(\varphi_p(\Delta^2 u(t-2))\right) - \Delta\left(\varphi_p(\Delta u(t-1))\right) - \alpha f(t, u(t)) - \beta g(t, u(t))\right] v(t),$$

for any $v \in E_N$.
Thus, finding solutions of PVP (1.1), (1.2) is equivalent to finding critical point of the functional $I_{\alpha,\beta}$.

Lemma 2.2. *(Minty-Browder see [21]) Let $\theta : E \longrightarrow E^*$ be strictly monotone, coercive and hemicontinuous operator on the real, reflexive Banach space E. Then, the inverse operator $\theta^{-1} : E^* \longrightarrow E$ exists.*

Lemma 2.3. *(see [19]) Let E be a nonempty set and ϕ, ψ two real functions on E. Assume that there are $R > 0$ and $u_0, u_1 \in E$ such that*

$$\phi(u_0) = \psi(u_0) = 0, \quad \phi(u_1) > R, \qquad \sup_{u\in\phi^{-1}(]-\infty,R])} \psi(u) < R\frac{\psi(u_1)}{\phi(u_1)}.$$

Then for each ρ satisfying

$$\sup_{u\in\phi^{-1}(]-\infty,R])} \psi(u) < \rho < R\frac{\psi(u_1)}{\phi(u_1)},$$

one has

$$\sup_{\alpha\geqslant 0}\inf_{u\in E} [\phi(u) + \alpha(\rho - \psi(u))] < \inf_{u\in E}\sup_{\alpha\geqslant 0} [\phi(u) + \alpha(\rho - \psi(u))] .$$

Lemma 2.4. *(see [18]) Let E be a reflexive real Banach space, $S \subset \mathbb{R}$ an interval, let $\phi : E \longrightarrow \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of E and whose derivative admits a continuous inverse on E^*; $\psi : E \longrightarrow \mathbb{R}$ a C^1 functional with compact derivative. Assume that*

i) $\lim_{\|u\|\to\infty} [\phi(u) - \alpha\psi(u)] = \infty$, *for all $\alpha \in S$;*

ii) there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\alpha\in S}\inf_{u\in E} [\phi(u) + \alpha(\rho - \psi(u))] < \inf_{u\in E}\sup_{\alpha\in S} [\phi(u) + \alpha(\rho - \psi(u))] .$$

Then there exist a nonempty open set $\Lambda \subset S$ and a positive number γ, with the following property: for every $\alpha \in \Lambda$ and every C^1 functional $\Gamma : E \longrightarrow \mathbb{R}$ with compact derivative, there exists $\xi > 0$ such that, for each $\beta \in]0, \xi]$, the functional $\phi - \alpha\psi - \beta\Gamma$ has at least three distinct critical point in E, whose norms are less than γ.

Lemma 2.5. *(see [17]) Let E be a separable and reflexive real Banach space with the norm $\|.\|_E$, E^* be the dual space of E. Let $\phi : E \longrightarrow \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional that is bounded on bounded subsets of E and whose Gâteaux derivative admits a continuous inverse on E^* and $\psi : E \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.*
Assume that ϕ has a strict local minimum u_0 with $\phi(u_0) = \psi(u_0) = 0$.
Let

$$\delta = \max\left\{0, \lim_{\|u\|\to\infty} \sup\frac{\psi(u)}{\phi(u)}, \lim_{u\to u_0} \sup\frac{\psi(u)}{\phi(u)}\right\},$$

and

$$\eta = \sup_{u\in\phi^{-1}(]0,\infty[)} \frac{\psi(u)}{\phi(u)},$$

and assume that $\delta < \eta$. Then, for each compact interval $[a, b] \subset \left]\frac{1}{\eta}, \frac{1}{\delta}\right[$ (with the conventions $(\frac{1}{0} = \infty)$, $(\frac{1}{\infty} = 0)$), there exists $\omega > 0$ with the following property: for each $\beta \in [a, b]$ and every C^1 functional $\Gamma : E \longrightarrow \mathbb{R}$ with compact derivative, there exists $\zeta > 0$ such that, for each $\alpha \in]0, \zeta]$, the functional $\phi - \alpha\psi - \beta\Gamma$ has at least three distinct critical point in E whose norms are less than ω.

3 Eigenvalue problem

We consider the nonlinear eigenvalue problem (P_0) corresponding to the BVP (1.1), (1.2):

$$(P_0)\quad \begin{cases} \Delta^2(\varphi_p(\Delta^2 u(t-2))) - \Delta(\varphi_p(\Delta u(t-1))) = \lambda\varphi_p(u(t)),\ t\in[1,N]_{\mathbb{Z}}, \\ u(0)=u(N+1)=\Delta u(-1)=\Delta u(N+1)=0. \end{cases}$$

Definition 3.1. $\lambda\in\mathbb{R}$ is called eigenvalue of (P_0) if there exists $u\in E_N\setminus\{0\}$ such that

$$\sum_{t=1}^{N+2}\varphi_p(\Delta^2 u(t-2))\Delta^2 v(t-2)+\varphi_p(\Delta u(t-1))\Delta v(t-1)=\lambda\sum_{t=1}^{N}\varphi_p(u(t))v(t),\quad \forall v\in E_N\setminus\{0\}.$$

Proposition 3.2. *(see [13]) Let E be a real Banach space, $G,J\in C^1(E,\mathbb{R})$ and a set of constraints $S=\{u\in E \mid G(u)=0\}$. Suppose that for any $u\in S$, $G'(u)\neq 0$ and there exists $u_0\in S$ such that $J(u_0)=\inf_{u\in S}J(u)$. Then there is $\lambda\in\mathbb{R}$ such that $J'(u_0)=\lambda G'(u_0)$.*

Theorem 3.3. *λ_1 and λ_N are the first and the last eigenvalue respectively of the problem (P_0), where*

$$\lambda_1 \text{ (defined in (1.4)) and } \lambda_N=\max_{v\in E_N\setminus\{0\}}\frac{\sum_{t=1}^{N+2}|\Delta^2u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}.$$

Proof. Put

$$J(u)=\sum_{t=1}^{N+2}|\Delta^2 u(t-2)|^p+|\Delta u(t-1)|^p,\ G(u)=\sum_{t=1}^{N}|u(t)|^p-1=\|u\|^p-1,$$

and

$$S=\{u\in E_N \mid G(u)=0\}=\{u\in E_N \mid \|u\|=1\}.$$

It is easy to see that $G'(u)\neq 0$ for any $u\in S$.
The set S is compact and J is continuous on S, then there exists $u_1\in S$ such that

$$J(u_1)=\min_{u\in S}J(u)=\lambda'.$$

Clearly $\lambda'>0$. From the Proposition 3.2, there exists λ_1 such that

$$J'(u_1)=\lambda_1 G'(u_1). \tag{3.1}$$

Which mains that $\Delta^2(\varphi_p(\Delta^2 u_1(t-2)))-\Delta(\varphi_p(\Delta u_1(t-1)))=\lambda_1\varphi_p(u_1(t))$, $t\in[1,N]_{\mathbb{Z}}$.
Multiplying (3.1) by u_1 in the sense of inner product, we obtain

$$\sum_{t=1}^{N+2}|\Delta^2 u_1(t-2)|^p+|\Delta u_1(t-1)|^p=\lambda_1\sum_{t=1}^{N}|u_1(t)|^p,$$

i.e.,

$$J(u_1)=\lambda_1\|u_1\|^p=\lambda_1.$$

Therefore, $\lambda'=\lambda_1$ is an eigenvalue of the problem (P_0).

Thus, we have

$$\begin{aligned}\lambda_1 &= \min_{u\in S}\sum_{t=1}^{N+2}|\Delta^2u(t-2)|^p+|\Delta u(t-1)|^p\\ &= \min_{u\in E_N\setminus\{0\}}\sum_{t=1}^{N+2}|\Delta^2\frac{u(t-2)}{\|u\|}|^p+|\Delta\frac{u(t-1)}{\|u\|}|^p\\ &= \min_{u\in E_N\setminus\{0\}}\frac{\sum_{t=1}^{N+2}|\Delta^2u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}.\end{aligned}$$

Similarly, we show that λ_N is an eigenvalue of the problem (P_0) and

$$\lambda_N=\max_{v\in E_N\setminus\{0\}}\frac{\sum_{t=1}^{N+2}|\Delta^2u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}.$$

If λ is an eigenvalue of the problem (P_0), then there exists $u\in E_N\setminus\{0\}$ such that:

$$\sum_{t=1}^{N+2}\varphi_p(\Delta^2u(t-2))\Delta^2v(t-2)+\varphi_p(\Delta u(t-1))\Delta v(t-1)=\lambda\sum_{t=1}^{N}\varphi_p(u(t))v(t),\quad\forall v\in E_N\setminus\{0\}.$$

In particular for $v=u$, we get $\lambda=\dfrac{\sum_{t=1}^{N+2}|\Delta^2u(t-2)|^p+|\Delta u(t-1)|^p}{\sum_{t=1}^{N}|u(t)|^p}$.

So, we deduce that $\lambda_1\le\lambda\le\lambda_N$.
Then, λ_1 and λ_N are the first and the least eigenvalue respectively of the problem (P_0).
The proof of Theorem 3.3 is complete. □

It is clear to see that

$$\frac{\lambda_1}{p}\|u\|^p\le\Phi(u)\le\frac{\lambda_N}{p}\|u\|^p,\quad\forall u\in E_N.\tag{3.2}$$

4 Proofs of the main results

Proof of Theorem 1.1. We will apply Lemma 2.4, with $\phi=\Phi$, $\psi=\Psi_1$ and $\Gamma=\Psi_2$.
Firsty, we show that the functionals Φ, Ψ_1 and Ψ_2 satisfy the regularity assumptions of Lemma 2.4.
It is easy to see that Ψ_1, Ψ_2 are continuously Gâteaux differentiable functional whose Gateaux derivative is compact.
Clearly, by (3.2) that Φ is coercive, sequentially weakly lower semicontinuous functional and is bounded on each bounded subset of E_N, Gâteaux differentiable. We continue to prove the existence of the inverse function $(\Phi')^{-1}:E_N^*\longrightarrow E_N$. At first, we show the strict monotonicity of Φ'. Indeed, let $u_1,u_2\in E_N$ with $u_1\neq u_2$, then

$$\begin{aligned}(\Phi'(u_1)-\Phi'(u_2))\,(u_1-u_2)\ge&\sum_{t=1}^{N+2}\left(|\Delta^2u_1(t-2)|^{p-2}\Delta^2u_1(t-2)-|\Delta^2u_2(t-2)|^{p-2}\Delta^2u_2(t-2)\right)\\ &\left(\Delta^2u_1(t-2)-\Delta^2u_2(t-2)\right)\\ &+\sum_{t=1}^{N+2}\left(|\Delta u_1(t-1)|^{p-2}\Delta u_1(t-1)-|\Delta^2u_2(t-1)|^{p-2}\Delta u_2(t-1)\right)\\ &\left(\Delta u_1(t-1)-\Delta u_2(t-1)\right).\end{aligned}$$

By the well-known inequality, for any $v, w \in \mathbb{R}^N$,

$$\left(|v|^{m-2}v - |w|^{m-2}w\right)(v-w) \geq C_m|v-w|^m, \quad m \geq 2, C_m > 0,$$

we obtain,

$$(\Phi'(u_1) - \Phi'(u_2))(u_1 - u_2) \geq C_1 \sum_{t=1}^{N+2} |\Delta^2 u_1(t-2) - \Delta^2 u_2(t-2)|^{p-2} + C_2 \sum_{t=1}^{N+2} |\Delta u_1(t-1) - \Delta u_2(t-1)|^{p-2} > 0,$$

where C_1 and C_2 are two positive constants. Therefore Φ' is strictly monotone. Moreover,

$$\Phi'(u).u = \sum_{t=1}^{N+2} |\Delta^2 u(t-2)|^p + |\Delta u(t-1)|^p \geq \lambda_1 \|u\|^p.$$

So, $\Phi'(u) \longrightarrow \infty$ as $\|u\| \to \infty$.
Then, from Lemma 2.2, Φ' has an inverse mapping $(\Phi')^{-1} : E_N^* \longrightarrow E_N$.
Now we prove that $(\Phi')^{-1}$ is continuous. Let $(u_n^*), u^* \in E_N^*$ with $u_n^* \longrightarrow u^*$, and let $(\Phi')^{-1}(u_n^*) = u_n$, $(\Phi')^{-1}(u^*) = u$. Then, $\Phi'(u_n) = u_n^*$ and $\Phi'(u) = u^*$, which means that (u_n) is bounded in E_N. Hence there exists $u_0 \in E_N$ and a subsequence, again denoted by (u_n) such that $u_n \longrightarrow u_0$ in E_N. Thus $\Phi'(u_n) \longrightarrow \Phi'(u_0)$. Since the limit is unique, it follows that $\Phi'(u_0) = \Phi'(u)$. Therefore $u_0 = u$, then $(\Phi')^{-1}(u_n^*) \longrightarrow (\Phi')^{-1}(u^*)$. Hence $(\Phi')^{-1}$ is continuous.
Next, from (H_1) there exists $\rho > 0$ such that

$$\frac{pF(t,x)}{|x|^p} \leq \delta + \varepsilon \quad \text{for} \quad (t,|x|) \in [1,N]_{\mathbb{Z}} \times]\rho, +\infty[,$$

where $0 < \varepsilon < \lambda_1 - \delta$, i.e.,

$$F(t,x) \leq \frac{1}{p}(\eta + \varepsilon)|x|^p \qquad \text{for } (t,|x|) \in [1,N]_{\mathbb{Z}} \times]\rho, +\infty[. \tag{4.1}$$

Then, by (4.1) and the continuity of $x \longrightarrow F(t,x)$, there exists $c > 0$ such that

$$F(t,x) \leq \frac{1}{p}(\delta + \varepsilon)|x|^p + c, \quad \forall (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.2}$$

According to (1.4), we have

$$\sum_{t=1}^{N+2} |\Delta^2 u(t-2)|^p + |\Delta u(t-1)|^p \geq \lambda_1 ||u||^p. \tag{4.3}$$

Let $\alpha \in [0,1]$, using the preceding inequality and (4.2), we obtain

$$\Phi(u) - \alpha\Psi_1(u) \geq \frac{1}{p}\lambda_1\|u\|^p - \frac{1}{p}\alpha(\delta+\varepsilon)\sum_{t=1}^{N}|u(t)|^p - cN$$
$$\geq \frac{1}{p}\left[\lambda_1 - (\delta+\varepsilon)\right]\|u\|^p - cN.$$

Since $\varepsilon < \lambda_1 - \delta$, then

$$\lim_{\|u\|\to\infty} (\Phi(u) - \alpha\Psi_1(u)) = \infty.$$

The condition i) in Lemma 2.4 is satisfied.
Next, for $u_0 = 0$ we have $\Phi(0) = \Psi_1(0) = 0$.

Put $R = \frac{1}{p}\lambda_1 c^p$ and choose $u_1 \in E_N$ defined by: $u_1(t) = \begin{cases} d & , \ t \in [1,N]_{\mathbb{Z}}, \\ 0 & , \ \text{otherwise.} \end{cases}$
By (3.2) with $u = u_1$ and $c^p < Nd^p$, we have

$$\begin{aligned} \Phi(u_1) &\geq \frac{\lambda_1}{p}\|u\|^p \\ &= \frac{\lambda_1}{p} N d^p \\ &> \frac{\lambda_1}{p} c^p = R. \end{aligned}$$

If $\Phi(u) \leqslant R$, then for all $t \in [1,N]_{\mathbb{Z}}$, we have

$$\begin{aligned} |u(t)| \leq \|u\| &\leqslant \left(\frac{p\Phi(u)}{\lambda_1}\right)^{\frac{1}{p}} \\ &\leqslant \left(\frac{pR}{\lambda_1}\right)^{\frac{1}{p}} = c. \end{aligned}$$

Which implies that,

$$\Phi^{-1}(]-\infty, R]) \subseteq \{u \in E_N / |u(t)| \leqslant c, \ t \in [1,N]_{\mathbb{Z}}\}.$$

So, we get

$$\begin{aligned} \sup_{u \in \Phi^{-1}(]-\infty,R])} \Psi_1(u) &= \sup_{u \in \Phi^{-1}(]-\infty,R])} \sum_{t=1}^{N} F(t,u(t)) \\ &\leqslant N \max_{(t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,c]} F(t,x). \end{aligned}$$

Therefore, from (H_2) we deduce that

$$\sup_{u \in \Phi^{-1}(]-\infty,R])} \Psi_1(u) < \frac{\lambda_1}{6}\left(\frac{c}{d}\right)^p \sum_{t=1}^{N} F(t,d).$$

On the other hand, it is easy to verify that $\Phi(u_1) = \frac{6}{p} d^p$, $\Psi_1(u_1) = \sum_{t=1}^{N} F(t,d)$,
and

$$R\frac{\Psi_1(u_1)}{\Phi(u_1)} = \frac{\lambda_1}{6}\left(\frac{c}{d}\right)^p \sum_{t=1}^{N} F(t,d).$$

Consequently,

$$\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi_1(u) < R\frac{\Psi_1(u_1)}{\Phi(u_1)}.$$

Then from Lemma 2.3, for each ρ satisfying

$$\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi_1(u) < \rho < r\frac{\Psi_1(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\alpha \geqslant 0} \inf_{u \in E} \left[\Phi(u) + \alpha(\rho - \Psi_1(u))\right] < \inf_{u \in E} \sup_{\alpha \geqslant 0} \left[\Phi(u) + \alpha(\rho - \Psi_1(u))\right].$$

Therefore, all the assumptions of Lemma 2.4 are satisfied. Then the functional $I_{\alpha,\beta}$ admits at least three distinct critical points, which are solutions of BVP (1.1), (1.2). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We know that the functional Φ, Ψ_1 and Ψ_2 satisfy the regularity assumptions of Lemma 2.5. For $u_0 = 0$, Φ has a strict local minimum and $\Phi(0) = \Psi_2(0) = 0$.
In view of (H_3), there exist $0 < R_1 < R_2$ such that

$$G(t,x) \leqslant \frac{1}{p}\nu|x|^p, \quad \text{for any } (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,R_1[\, \cup\,]R_2,\infty[\,. \tag{4.4}$$

Together with the continuity of g , this implies that there exists $c_1 > 0$ and $q > p$ such that

$$G(t,x) \leqslant c_1|x|^q \;\text{ for all } (t,x) \in [1,N]_{\mathbb{Z}} \times [-R_2,-R_1] \cup [R_1,R_2].$$

Using the preceding inequality and (4.4), we have

$$G(t,x) \leqslant \frac{1}{p}\nu|x|^p + c_1|x|^q \;\text{ for all } (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}.$$

So, from (2.1), it follows that

$$\begin{aligned}\Psi_2(u) &\leqslant \sum_{t=1}^{N} \frac{1}{p}\nu|u(t)|^p + c_1|u(t)|^q \\ &\leqslant \frac{1}{p}\nu\|u\|^p + c_1N\|u\|_\infty^q \\ &\leqslant \frac{1}{p}\nu\|u\|^p + c_1N\|u\|^q,\end{aligned}$$

for any $u \in E_N$.
Hence, using (3.2), we have

$$\lim_{\|u\|\to 0} \sup \frac{\Psi_2(u)}{\Phi(u)} \leqslant \frac{\nu}{\lambda_1}. \tag{4.5}$$

On the other hand, if $|u(t)| > R_2$ for any $t \in [1,N]_{\mathbb{Z}}$ then the inequality (4.4) implies that

$$G(t,u(t)) \leqslant \frac{1}{p}\nu|u(t)|^p. \tag{4.6}$$

And if $|u(t)| \leqslant R_2$, by continuity of g there exist c_2 such that

$$|G(t,u(t))| \leqslant c_2 \;\text{ for any } t \in [1,N]_{\mathbb{Z}}. \tag{4.7}$$

Using the preceding inequality, (4.6) and (2.1), we obtain

$$\begin{aligned}\Psi_2(u) = \sum_{t=1}^{N} G(t,u(t)) &= \sum_{|u(t)|\leq R_2} G(t,u(t)) + \sum_{|u(t)|>R_2} G(t,u(t)) \\ &\leqslant Nc_2 + \frac{1}{p}\nu\sum_{t=1}^{N}|u(t)|^p \\ &\leqslant Nc_2 + \frac{1}{p}\nu\|u\|^p.\end{aligned}$$

Therefore, we get

$$\frac{\Psi_2(u)}{\Phi(u)} \leqslant \frac{pNc_2}{\lambda_1\|u\|^p} + \frac{\nu}{\lambda_1}.$$

So, we deduce that

$$\lim_{\|u\|\to \infty} \sup \frac{\Psi_2(u)}{\Phi(u)} \leqslant \frac{\nu}{\lambda_1}. \tag{4.8}$$

Consequently, from (4.5) and (4.8), we obtain

$$\delta = \max\{0, \lim_{\|u\|\to\infty} \sup \frac{\Psi_2(u)}{\Phi(u)}, \lim_{\|u\|\to 0} \sup \frac{\Psi_2(u)}{\Phi(u)}\} \leqslant \frac{\nu}{\lambda_1}.$$

For $\rho > 0$, choose $u_2 \in E_N$ such that $u_2(t) = \rho$ for any $t \in [1, N]_{\mathbb{Z}}$,
Clearly $\Phi(u_2) = \frac{6}{p}\rho^p > 0$, therefore $u_2 \in \Phi^{-1}(]0,\infty[)$. So, it is easy to see that

$$\eta = \sup_{u \in \Phi^{-1}(]0,\infty[)} \frac{\Psi_2(u)}{\Phi(u)} \geqslant \frac{\Psi_2(u_2)}{\Phi(u_2)} = \frac{p \sum_{t=1}^{N} G(t,\rho)}{6\rho^p}.$$

From (H_4), we deduce that

$$\eta > \frac{\nu}{\lambda_1} \geqslant \delta.$$

Hence, all the assumptions of Lemma 2.5 are satisfied. Therefore the functional $I_{\alpha,\beta}$ admits at least three distinct critical points, which are solutions of the BVP (1.1), (1.2). The proof of Theorem 1.2 is complete. □

Example 4.1. We consider the problem

$$\begin{cases} \Delta^2\left(\varphi_{10}(\Delta^2 u(t-2))\right) - \Delta\left(\varphi_{10}(\Delta u(t-1))\right) = \alpha f(t,u(t)) + \beta g(t,u(t)), \ t \in [1,6]_{\mathbb{Z}}, \\ u(0) = u(7) = \Delta u(-1) = \Delta u(7) = 0, \end{cases} \tag{4.9}$$

where $f : [1,6]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ and for $t \in [1,6]_{\mathbb{Z}}$, $x \in \mathbb{R}$ let

$$g(t,x) = t \begin{cases} x & , \ |x| > 1 \\ x^7 & , \ |x| \leqslant 1. \end{cases}$$

It is easy to see that

$$G(t,x) = t \begin{cases} \frac{1}{2}x^2 - \frac{3}{8}, & |x| > 1 \\ \frac{1}{8}x^8 & , \ |x| \leqslant 1. \end{cases}$$

Clearly $\lim_{|x|\to\infty} \sup \max_{t \in [1,N]_{\mathbb{Z}}} \frac{pG(t,x)}{|x|^p} = 0$ and $\lim_{x \to 0} \sup \max_{t \in [1,N]_{\mathbb{Z}}} \frac{pG(t,x)}{|x|^p} = 0$, thus (H_3) holds for any $\nu > 0$, (for example $\nu = \lambda_1 \times 10^{-6}$).
We choose $\rho = 1$, then we have $\sum_{t=1}^{6} G(t,1) = \frac{21}{8}$ and $\frac{6\nu\rho^p}{p\lambda_1} = 6 \times 10^{-7}$, therefore (H_4) is satisfied.
We deduce for each interval compact $[a,b] \subset \left]\frac{8}{35}, 10^6\right[$, there exist $\zeta > 0$ such that for each $\beta \in [a,b]$, there exists $\eta > 0$ such that for each $\alpha \in]0,\eta]$, the problem (4.9) has at least three distinct solutions in E_N, whose norms are less than ζ.

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