**Palestine Journal of Mathematics**

**Vol 13(2)(2024)** , 156–159 

# CHARACTERIZATION OF RICCI CURVATURE ALONG SOME GEODESICS OF A RICCI-BOURGUIGNON SOLITON

Puja Sarkar and Mampi Howlader

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53C20, 53C21, 53C44; Secondary 53B30, 53D25.

Keywords and phrases: Ricci-Bourguignon soliton soliton, Busemann function, Riemannian manifold.

The authors would like to thank the anonymous referee for carefully reading the paper and for thoughtful criticism of the manuscript, which helped improve the manuscript considerably. The second author expresses gratitude to the UGC-NFSC for the Junior Research Fellow award.

**Abstract** The Ricci-Bourguignon soliton is a self-similar solution of the Ricci-Bourguignon flow. This article is dedicated to the study of Ricci-Bourguignon soliton in a complete Riemannian manifold. In particular, it is shown that under certain conditions, the Ricci curvature of the Ricci-Bourguignon soliton vanishes along some geodesics.

## 1 Introduction

In 1982, Hamilton [8, 9] presented the concept of Ricci flow in a complete Riemannian manifold $(M, g_0)$ to investigate the various geometric and topological properties of a Riemannian manifold. The Ricci flow is defined by an evolution equation for the Riemannian metrics on $(M, g_0)$:

$$\frac{\partial}{\partial t}g(t) = -2Ric, \quad g(0) = g_0.$$

A Ricci soliton generalizes the Einstein metric and is defined as

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where $X$ denotes a smooth vector field on $M$, $\mathcal{L}$ indicates the Lie-derivative operator and $\lambda \in \mathbb{R}$. Ricci soliton is called shrinking, steady or expanding according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, resp. The vector field $X$ is known as the potential vector field of the Ricci soliton. If $X$ is either Killing or vanishing vector field, then the Ricci soliton is called trivial Ricci soliton, and (1.1) reduces to an Einstein metric. If $X$ becomes the gradient of a smooth function $f \in C^\infty(M)$, the ring of smooth functions on $M$, then the Ricci soliton is called gradient Ricci soliton and (1.1) reduces to the form

$$Ric + \nabla^2 f = \lambda g, \tag{1.2}$$

where $\nabla^2 f$ is the Hessian of $f$. The study of Ricci soliton reveals many geometrical and topological properties of a manifold; for more information, see [12, 10, 13, 5, 6, 1].

Jean-Pierre Bourguignon [2] introduced a more generalized type of Ricci flow, which is called Ricci-Bourguignon flow:

$$\frac{\partial}{\partial t}g(t) = -2(Ric - \rho R g),$$

where $\rho$ is a real nonzero constant.

If $\rho = 0$, then the Ricci-Bourguignon flow reduces to the Ricci flow. Just like Ricci soliton, a similar kind of self-similar solution can be defined in the case of Ricci-Bourguignon flow, which is called gradient Ricci-Bourguignon soliton [7] or gradient $\rho$-Einstein soliton [3].

**Definition 1.1.** [3] A Riemannian manifold $(M, g)$ of dimension $n \geq 3$ is said to be the gradient $\rho$-Einstein Ricci soliton if

$$Ric + \nabla^2 f = \lambda g + \rho R g, \quad \rho \in \mathbb{R},\ \rho \neq 0,$$

for some function $f \in C^\infty(M)$ and some constant $\lambda \in \mathbb{R}$. The function $f$ is called Einstein potential. The gradient $\rho$-Einstein soliton is called expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$.

Ricci-Bourguignon solitons are extensively studied, for reference see [11, 14, 3, 7].

In this article, we have deduced the behavior of the Ricci curvature along some geodesics. The main result of our article is the following:

**Theorem 1.2.** *Let $(M, g)$ be a Riemannian manifold, where $g$ is gradient Ricci-Bourguignon soliton with non negative Ricci curvature. If $M$ contains a line, then there exists a geodesic $\varrho$ such that the Ricci curvature of $g$ vanishes along $\varrho$, i.e.,*

$$Ric(\varrho'(t), \varrho'(t)) = 0, \textit{ for all } t > 0.$$

## 2 Main proof

*Proof of Theorem 1.2.* Let $c : (-\infty, +\infty) \to M$ be a line in $M$. Suppose that $\beta^+$ and $\beta^-$ are the Busemann functions which correspond to the rays $c^+ = c|_{[0,\infty)}$ and $c^- = c|_{(-\infty,0]}$ respectively, i.e.,

$$\beta^+(x) = \lim_{t\to\infty} (d(x, c^+(t)) - t),$$

$$\beta^-(x) = \lim_{t\to\infty} (d(x, c^-(-t)) - t).$$

Now using the Cheeger-Gromoll splitting theorem, see [4, Theorem 1], we get that $\beta^+$ and $\beta^-$ are superharmonic functions. Again, using the triangle inequality, we obtain for all $x \in M$

$$\beta^+(x) + \beta^-(x) = \lim_{t\to\infty} (d(x, c^+(t)) + d(x, c^-(-t)) - 2t) \geq 0.$$

Furthermore,

$$(\beta^+ + \beta^-)(c(t)) = 0, \tag{2.1}$$

since $c$ is a line. Let $y \in c$ and $D$ be an arbitrary connected region in $M$ with smooth boundary $\partial D$ and containing $y$ in its interior. The (2.1) implies that

$$(\beta^+ + \beta^-)(y) = 0.$$

Now consider two continuous functions $h^+$ and $h^-$ on $D$ which are harmonic on $int(D)$ with $h^+|_{\partial D} = \beta^+|_{\partial D}$ and $h^-|_{\partial D} = \beta^-|_{\partial D}$. Since $(h^+ + h^-)|_{\partial D}$ is non-negative, it follows from the minimum principle for harmonic functions that $h^+(y) + h^-(y) \geq 0$. Now $\beta^+ \geq h^+$ and $\beta^- \geq h^-$, so we must have $\beta^+(y) = h^+(y)$ and $\beta^-(y) = h^-(y)$. Then it follows that $\beta^+ = h^+$ and $\beta^- = h^-$ on $D$. Since, $D$ is arbitrary, it indicates that $\beta^+$ and $\beta^-$ are differentiable and harmonic in $M$. The property of Busemann function implies that for any $x, y \in M$,

$$|b_t(x) - b_t(y)| \leq d(x, y).$$

Letting $t \to \infty$ ,it shows that $|\nabla\beta^+| \leq 1$. Again, for a given $x$, take a minimal geodesic $\varrho_t$ from $x$ to $c(t)$. Let $\{t_n\}$ be a sequence such that $\varrho'_{t_n}(0) \to \varrho'(0)$. Then for all $y \in \varrho$, it implies that $|\beta^+(x) - \beta^+(y)| = d(x, y)$. Therefore, it is clear that $|\nabla\beta^+| = 1$, and $\varrho$ is the integral curve of $\nabla\beta^+$ through the point $x$. Let $\nabla\beta^+ = \nu$. Now an orthonormal frame $e_1, e_2, \cdots, e_{n-1}, \nu$ can be constructed in a neighborhood of $x$ that is parallel along $\varrho$. And it shows that $\nabla_\nu \nu = 0$ at $x$.

Now, we calculate

$$
\begin{aligned}
\nabla^2 f(\nu,\nu) &= (\rho R+\lambda)\langle \nu,\nu\rangle - Ric(\nu,\nu)\\
&= (\rho R+\lambda) - \sum_{i=1}^{n-1}\langle R(e_i,\nu)\nu, e_i\rangle\\
&= (\rho R+\lambda) - \sum_{i=1}^{n-1}\langle \nabla_{e_i}\nabla_\nu \nu - \nabla_\nu\nabla_{e_i}\nu - \nabla_{[e_i,\nu]}\nu, e_i\rangle\\
&= (\rho R+\lambda) + \sum_{i=1}^{n-1}\langle \nabla_\nu\nabla_{e_i}\nu, e_i\rangle + \sum_{i=1}^{n-1}\langle \nabla_{\nabla_{e_i}\nu}\nu, e_i\rangle\\
&= (\rho R+\lambda) + \sum_{i=1}^{n-1}\nu\langle \nabla_{e_i}\nu, e_i\rangle + \sum_{i,j=1}^{n-1}\langle \nabla_{e_i}\nu, e_j\rangle\langle \nabla_{e_j}\nu, e_i\rangle\\
&= (\rho R+\lambda) - \sum_{i=1}^{n-1}\nu\langle \nu, \nabla_{e_i}e_i\rangle + \|\nabla\nu\|^2\\
&= (\rho R+\lambda) + \nu(\Delta\beta^+) + \|\nabla\nu\|^2 = (\rho R+\lambda) + \|\nabla\nu\|^2.
\end{aligned}
$$

It shows that along the geodesic $\varrho$, $\nabla^2 f(\nu,\nu) \geq (\rho R+\lambda)$. Again, the equation of Ricci-Bourguignon soliton implies that along $\varrho$,

$$
\int_0^t \nabla^2 f(\varrho'(t),\varrho'(t))dt = \int_0^t (\rho R+\lambda)dt - \int_0^t Ric(\varrho'(t),\varrho'(t))dt \leq \int_0^t (\rho R+\lambda)dt.
$$

Hence, we obtain

$$
\int_0^t \nabla^2 f(\varrho'(t),\varrho'(t))dt = \int_0^t (\rho R+\lambda)dt.
$$

Integrating the Ricci-Bourguignon equation and putting the above value, it yields

$$
\begin{aligned}
\int_0^t Ric(\varrho'(t),\varrho'(t))dt &= -\int_0^t \nabla^2 f(\varrho'(t),\varrho'(t))dt + \int_0^t (\rho R+\lambda)\langle \varrho'(t),\varrho'(t)\rangle dt\\
&= -\int_0^t (\rho R+\lambda)dt + \int_0^t (\rho R+\lambda)dt = 0.
\end{aligned}
$$

Since, $Ric$ is non-negative everywhere, the above inequality implies that Ricci curvature vanishes along the geodesic $\varrho$. □

## References


[1] A. M. Blaga and M. Crasmareanu, *Inequalities for gradient Einstein and Ricci solitons*, Facta. Univ. Ser. Math. Inform., **35(2)**, 351–356 (2020).

[2] J. P. Bourguignon, *Ricci curvature and Einstein metrics, in: Global Differential Geometry and Global Analysis*, Lecture Notes in Math. **838**, Springer-Verlag, Berlin (1981).

[3] G. Catino and L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal., **132**, 66–94 (2016).

[4] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom., **6**, 119–128 (1971).

[5] H. G. Cao and D. Zhou, *On complete gradient Ricci solitons*, J. Diff. Geom., **85**, 175–185 (2010).

[6] B. Y. Chen and S. Deskmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl., **19(1)**, 13–21 (2014).

[7] S. Dwivedi, *Some results on Ricci-Bourguignon solitons and almost solitons*, Canadian Math. Bull., **64(3)**, 591–604 (2021).

[8] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom., **17**, 255–306 (1982).

[9] R. S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math., **71**, 237–261 (1988).

[10] C. K. Mondal, *Characterization of the potential function of a Ricci soliton*, Proc. Indian Acad. Sci. (Math. Sci.), **133(6 )**, 1–6 (2023).

[11] C. K. Mondal and A. A. Shaikh, *Some results in $\eta$-Ricci soliton and gradient $\rho$-Einstein soliton in a complete Riemannian manifold*, Comm. Korean Math. Soc., **34(4)**, 1279–1287 (2019).

[12] C. K. Mondal and A. A. Shaikh, *On Ricci solitons whose potential is convex*, Proc. Indian Acad. Sci. (Math. Sci.), **130(55)**, 1–7 (2020).

[13] A. A. Shaikh and C. K. Mondal, *Isometry theorem of gradient Shrinking Ricci solitons*, J. Geom. Phys., **163**, 104110 (2021).

[14] A. A. Shaikh, P. Mandal and C. K. Mondal, *Diameter estimation of gradient $\rho$-Einstein solitons*, J. Geom. Phys., **177**, 104518 (2022).

## Author information

Puja Sarkar, Department of Education, The University of Burdwan, Golapbag, Purba Barddhaman- 713104, West Bengal, India.
E-mail: psarkar@edu.buruniv.ac.in

Mampi Howlader, Department of Education, The University of Burdwan, Golapbag, Purba Barddhaman-713104, West Bengal, India.
E-mail: mampihowlader99@gmail.com

Received: 2023-02-18
Accepted: 2023-12-06