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On the unit group of the semisimple group algebras of groups up to order 144

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Abstract In this paper, we determine the structure of the unit groups of the semisimple group algebras of non-metabelian groups of order 144. Up to isomorphism, there are 197 non-isomorphic groups of order 144, and only 28 are non-metabelian. Mittal and Sharma [19] studied the unit groups of the semisimple group algebras of non-metabelian groups of order 144 that have exponent either 36 or 72. In this work, we characterize the unit groups of the group algebras of non-metabelian groups of order 144 having exponent 12 and 24. This paper completes the study of the unit groups of the group algebras of non-metabelian groups up to order 144.

1 Introduction

Let $\mathbb{K}G$ denote the group algebra generated by the finite group G of order n over the finite field $\mathbb{K}$ of order $q = p^k$ having characteristics p. Let $\mathcal{U}(\mathbb{K}G)$ denote the collection of all elements in $\mathbb{K}G$ having multiplicative inverses. The set $\mathcal{U}(\mathbb{K}G)$ is known as the unit group of $\mathbb{K}G$. Owing to the applications of the units in various fields like coding theory (see [8],[9],[10]), number theory ([6]), cryptography [17] etc., the classification of the unit groups of the group algebras has become a salient research area [7, 13, 20, 21, 25, 26, 27, 28].

For the group algebras generated by metabelian groups (recall that a group G is metabelian if its derived subgroup is abelian), Bakshi et al. [3] completely characterized the unit groups. Therefore, most of the researchers in this area focus on the unit groups of the group algebras of non-metabelian groups. Thanks to Pazderski, it is possible to explicitly calculate the possible orders of non-metabelian groups (see [23]). One can note that the smallest possible order of a non-metabelian group is 24. The unit groups of the semisimple group algebras of non-metabelian groups of order 24 are studied in [11, 12]. One of the notable works in this direction is due to Mittal and Sharma [15], where the authors studied the unit groups of the semisimple group algebras of non-metabelian groups up to order 72. Furthermore, Mittal and Sharma also characterized the unit groups of the semisimple group algebras of non-metabelian groups up to order 120 (see [16, 18, 22, 24]), except that of the symmetric group S_5. Arvind and Panja study the unit group of the semisimple group algebra of S_5 in [2]. In continuation, Abhilash et al. [1] considered all the non-metabelian groups of order 128 and studied the unit groups of their corresponding semisimple group algebras.

Next, using [23], it is straight-forward to note that there are non-metabelian groups of order 144. Up to isomorphism, there are 197 groups of order 144, and only 28 are non-metabelian. Moreover, the possible exponents of these 28 non-metabelian groups are $12, 24, 36$, and 72. Recently, Mittal and Sharma [19] computed the unit groups of the semisimple group algebras of non-metabelian groups of order 144 that have exponents 36 or 72. In this work, we consider the remaining non-metabelian groups of order 144, i.e., the groups having exponents 12 or 24 (a total of 11 such groups), and compute the unit groups of their corresponding semisimple group algebras. This paper will complete the study of the unit groups of the semisimple group algebras of groups up to order 144.

This paper is organized as follows. The preliminaries needed in this paper are collected in section 2. Moreover, in the same section, we discuss the non-metabelian groups of order 144. Sections 3 and 4 deal with our main results on the structure of the unit groups of semisimple group algebras of 11 non-metabelian groups of order 144. Finally, section 5 concludes the paper.

2 Preliminaries

Throughout this paper, let $\mathbb{K}$ denote the finite field of order $q = p^k$ with characteristic p and let G denote the finite group of order n. The definitions given below are as in [5].

Definition 2.1. An element $x \in G$ is called p'-element, if $p \nmid |x|$, where $|x|$ is the order of x.

Let the least common multiple of the orders of all p'-elements in G be denoted by s. Let the primitive s^{th} root of unity over $\mathbb{K}$ be denoted by ω. Therefore, $\mathbb{K}(\omega)$ is the splitting field over $\mathbb{K}$. Next, we define the set

$$T_{G,\mathbb{K}} = \{t \mid \sigma(\omega) = \omega^t, \text{ where } \sigma \in \text{Gal}(\mathbb{K}(\omega)/\mathbb{K})\},$$

where Gal$(\mathbb{K}(\omega)/\mathbb{K})$ denotes the Galois group of $\mathbb{K}(\omega)$ over $\mathbb{K}$.

Definition 2.2. For any p'-element $x \in G$, let $\gamma_x = \sum_{h \in C_x} h$. Then, the cyclotomic $\mathbb{K}$-class of γ_x is the set

$$S_{\mathbb{K}}(\gamma_x) = \{\gamma_{x^t} \mid t \in T_{G,\mathbb{K}}\}.$$

Proposition 2.1. [5] The set of simple components of $\frac{\mathbb{K}G}{J(\mathbb{K}G)}$ and the set of cyclotomic $\mathbb{K}$-classes in G, where $J(\mathbb{K}G)$ is the Jacobson radical of $\mathbb{K}G$, are in 1-1 correspondence.

Proposition 2.2. [5] Let l be the number of cyclotomic $\mathbb{K}$-classes in G. If $K_1, K_2, \cdots, K_l$ are the simple components of $Z\big(\frac{\mathbb{K}G}{J(\mathbb{K}G)}\big)$ and $S_1, S_2, \cdots, S_l$ are the cyclotomic $\mathbb{K}$-classes of G, then $|S_i| = [K_i : \mathbb{K}]$ with a suitable ordering of the indices, assuming that the Galois group Gal$(\mathbb{K}(\omega) : \mathbb{K})$ is cyclic.

Lemma 2.1. [14] Let $\mathbb{K}G$ be a semi-simple group algebra and let G' be the derived subgroup of G. Then,

$$\mathbb{K}G \cong \mathbb{K}Ge_{G'} \oplus \triangle(G, G'),$$

where $\mathbb{K}Ge_{G'} = \mathbb{K}\left(\frac{G}{G'}\right)$ is the sum of all commutative simple components of $\mathbb{K}G$ and $\triangle(G, G')$ is the sum of all others.

Proposition 2.3. [14] The number of irreducible representations of $\mathbb{K}G$ is equal to the number of conjugacy classes of G.

We end this section by discussing the non-metabelian groups of order 144.

2.1 Non-metabelian groups of order 144

From [19, section 2], we know that there are 28 non-metabelian groups of order 144. These are listed as follows: We write all the 28 non-metabelian groups of order 144 in the following list:

1. $(Q_8 \rtimes C_9).C_2$
2. $(Q_8 \rtimes C_9) \rtimes C_2$
3. $((C_2 \times C_2) \rtimes C_9) \rtimes C_4$
4. $C_2 \times (Q_8 \rtimes C_9)$
5. $((C_4 \times C_2) \rtimes C_2) \rtimes C_9$
6. $C_2 \times (((C_2 \times C_2) \rtimes C_9) \rtimes C_2)$
7. $(C_3 \times C_3) \rtimes ((C_4 \times C_2) \rtimes C_2)$
8. $(C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$
9. $(C_3 \times C_3) \rtimes D_{16}$
10. $(C_3 \times C_3) \rtimes QD_{16}$
11. $(C_3 \times C_3) \rtimes Q_{16}$
12. $(C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$
13. $C_3 \times (C_2 \cdot S_4)$
14. $C_3 \times GL(2, 3)$
15. $C_3 \times (A_4 \rtimes C_4)$
16. $C_3 \rtimes (C_2.S_4)$
17. $(C_3 \times SL(2, 3)) \rtimes C_2$
18. $(C_3 \times A_4) \rtimes C_4$
19. $((C_4 \times S_3) \rtimes C_2) \rtimes C_3$
20. $S_3 \times SL(2, 3)$
21. $C_6 \times SL(2, 3)$
22. $C_3 \times (((C_4 \times C_2) \rtimes C_2) \rtimes C_3)$
23. $(C_3 \times C_3) \rtimes QD_{16}$
24. $S_3 \times S_4$
25. $C_2 \times ((S_3 \times S_3) \rtimes C_2)$
26. $C_2 \times ((C_3 \times C_3) \rtimes Q_8)$
27. $C_6 \times S_4$
28. $C_2 \times ((C_2 \times A_4) \rtimes C_2)$.

Among these 28 groups, the groups at the serial numbers 7, 8, 12, 18 and 19 have exponent 12 (total 5) and the groups at the serial numbers 9, 10, 11, 16, 17 and 23 have exponent 24 (total 6). In the subsequent sections, we study the unit groups of group algebras corresponding to these 11 groups.

3 Groups of exponent 12

As discussed in section 2, we know that there are 5 non-metabelian groups of order 144. In this section, we characterize the unit group of the semisimple group algebra generated by these 5 groups. Throughout this paper, let $x^{-1}y^{-1}xy = [x, y]$ denote the commutator of $x, y \in G$. The 5 non-metabelian groups of order 144 with exponent 12 are given below. We remark that, in order to be consistent with the list given in section 2, we represent the groups with the same serial numbers as appearing earlier.

7. $(C_3 \times C_3) \rtimes ((C_4 \times C_2) \rtimes C_2)$
8. $(C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$
12. $(C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$
18. $(C_3 \times A_4) \rtimes C_4$
19. $((C_4 \times S_3) \rtimes C_2) \rtimes C_3$

3.1 $G_7 = (C_3 \times C_3) \rtimes ((C_4 \times C_2) \rtimes C_2)$.

The group G_7 has the following presentation:

$$G_7 = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2, [x_2, x_1]x_3^{-1}, [x_3, x_1], [x_4, x_1], [x_5, x_1]x_5^{-1}, [x_6, x_1], x_2^2x_4^{-1}, [x_3, x_2], [x_4, x_2], [x_5, x_2]x_6^{-1}x_5^{-2}, [x_6, x_2]x_5^{-1}, x_3^2, [x_4, x_3], [x_5, x_3]x_5^{-1}, [x_6, x_3]x_6^{-1}, x_4^2, [x_5, x_4], [x_6, x_4], x_5^3, [x_6, x_5], x_6^3\rangle$$

The sizes, orders and the representatives of the 18 conjugacy classes of G_7 are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_1x_6	x_2x_4	x_2x_5
Size	1	6	6	9	1	4	18	6	12	6	12
Order	1	2	4	2	2	3	4	2	6	4	12

x_3x_4	x_4x_5	x_5x_6	$x_1x_2x_4$	$x_1x_4x_6$	$x_2x_4x_5$	$x_4x_5x_6$
9	4	4	18	12	12	4
2	6	3	4	6	12	6

Theorem 3.1. Let G_7 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1,5\}$ mod 12, $\mathcal{U}(\mathbb{K}_q G_7) \cong (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^8$.
2) for $p^k \equiv \{7,11\}$ mod 12, $\mathcal{U}(\mathbb{K}_q G_7) \cong (\mathbb{K}_q^*)^4 \oplus (\mathbb{K}_{q^2}^*)^2 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^6 \oplus GL_4(\mathbb{K}_{q^2})$.

Proof. The group G_7 is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_q G_7) = 0$. Also, the commutator subgroup $G_7' \cong (C_3 \times C_3) \rtimes C_2$ and $\frac{G_7}{G_7'} \cong C_4 \times C_2$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition. Let us discuss the Wedderburn decomposition in the following 2 cases.
Case 1: k is even in $q = p^k$ or $p^k \equiv \{1,5\}$ mod 12.
In this case, $\mid S_{\mathbb{K}}(\gamma_g) \mid = 1, \forall g \in G_7$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition of $\mathbb{K}_q G_7$ is given by,

$$\mathbb{K}_q G_7 \cong (\mathbb{K}_q)^8 \bigoplus_{i=1}^{10} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \Rightarrow 136 = \sum_{i=1}^{10} n_i^2.$$

The choices of n_i's can be

$$(2^9,10), (2^7,6^3), (2^6,4^3,8), (2^5,3^3,5,8), (2^5,4,5^4), (2^4,3^2,4,5^2,6), (2^3,3^4,4,6^2),$$

$$(2^3,3^3,4^3,7), (2^2,3^6,5,7), (2^2,4^8), (2,3^3,4^5,5) \text{ and } (3^6,4^2,5^2).$$

In the direction of finding n_i's uniquely, we consider the normal subgroup $N = \langle x_4 \rangle$ of G_7. The Wedderburn decomposition of the factor group $F = \frac{G_7}{N} \cong (S_3 \times S_3) \rtimes C_2$ is due to [15] and is given below: $\mathbb{K}_q F \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_4(\mathbb{K}_q)^4$. With this information, we can conclude that the choices for n_i's can either be $(2^2,4^8)$ or $(2,3^3,4^5,5)$. Suppose, if $p = 5$, then by proposition 1 of [4], $(2,3^3,4^5,5)$ cannot be the choice in the decomposition of $\mathbb{K}_q G_7$. Therefore, we have

$$\mathbb{K}_q G_7 \cong (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^8.$$

Case 2: k is odd and $p^k \equiv \{7,11\}$ mod 12.
In this case, $S_{\mathbb{K}}(\gamma_{x_2}) = \{\gamma_{x_2}, \gamma_{x_2x_4}\}, S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}, S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2x_4x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_7$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_q G_7 \cong (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{i=1}^{8} M_{n_i}(\mathbb{K}_q) \oplus M_{n_9}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 136 = \sum_{i=1}^{8} n_i^2 + 2 \cdot n_9^2.$$

Therefore, by repeating the same process as in case 1, we get that $(2^2,4^6,4)$ is the only possibility for n_i's. Thus, we have

$$\mathbb{K}_q G_7 \cong (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^6 \oplus M_4(\mathbb{K}_{q^2}).$$

This completes the proof. □

3.2 $G_8 = (C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$.

The group G_8 has the following presentation:

$$G_8 = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2 x_4^{-1}, [x_2,x_1]x_3^{-1}, [x_3,x_1], [x_4,x_1], [x_5,x_1]x_5^{-1}, [x_6,x_1], x_2^2 x_4^{-1}, [x_3,x_2], [x_4,x_2],$$
$$[x_5,x_2]x_6^{-1}x_5^{-2}, [x_6,x_2]x_6^{-2}x_5^{-1}, x_3^2, [x_4,x_3], [x_5,x_3]x_5^{-1}, [x_6,x_3]x_6^{-1}, x_4^2, [x_5,x_4], [x_6,x_4], x_5^3, [x_6,x_5], x_6^3 \rangle.$$

The sizes, orders and the representatives of the 18 conjugacy classes of G_8 are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_1x_6	x_2x_4	x_2x_5
Size	1	6	6	9	1	4	18	6	12	6	12
Order	1	4	4	2	2	3	4	4	12	4	12

x_3x_4	x_4x_5	x_5x_6	$x_1x_2x_4$	$x_1x_4x_6$	$x_2x_4x_5$	$x_4x_5x_6$
9	4	4	18	12	12	4
2	6	3	4	12	12	6

Theorem 3.2. Let G_8 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1,5\}$ mod 12, $\mathcal{U}(\mathbb{K}_q G_8) \cong (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^8$.
2) for $p^k \equiv \{7,11\}$ mod 12, $\mathcal{U}(\mathbb{K}_q G_8) \cong (\mathbb{K}_q^*)^4 \oplus (\mathbb{K}_{q^2}^*)^2 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^4 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

Proof. The group G_8 is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_q G_8) = 0$. Also, the commutator subgroup $G_8' \cong (C_3 \times C_3) \rtimes C_2$ and $\frac{G_8}{G_8'} \cong C_4 \times C_2$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition. As in theorem 3.1, we further discuss the following two cases.
Case 1: k is even in $q = p^k$ or $p^k \equiv \{1,5\}$ mod 12. The proof is same as case 1 in theorem 3.1.

Case 2: k is odd and $p^k \equiv \{7, 11\}$ mod 12.
In this case, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_2}) = \{\gamma_{x_2}, \gamma_{x_2x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_6}) = \{\gamma_{x_1x_6}, \gamma_{x_1x_4x_6}\}$, $S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2x_4x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_8$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_8 \cong (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{i=1}^{6} M_{n_i}(\mathbb{K}_q) \bigoplus_{i=7}^{8} M_{n_i}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 136 = \sum_{i=1}^{6} n_i^2 + 2 \cdot n_7^2 + 2 \cdot n_8^2.$$

By repeating the same process as in theorem 3.1, we conclude that $(2^2, 4^4, 4, 4)$ is the only possibility for n_i's. Therefore, we have

$$\mathbb{K}_qG_8 \cong (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^4 \oplus (M_4(\mathbb{K}_{q^2}))^2.$$

This completes the proof. □

3.3 $G_{12} = (C_3 \times C_3) \rtimes (C_4 \rtimes C_4)$.

Here, note that the structure of the groups G_8 and G_{12} are same, but they are not isomorphic because of the small variation in the presentations. The group G_{12} has the following presentation:

$$G_{12} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2x_3^{-1}, [x_2, x_1]x_3^{-1}, [x_3, x_1], [x_4, x_1], [x_5, x_1]x_6^{-2}, [x_6, x_1]x_6^{-1}x_5^{-2}, x_2^2x_4^{-1}x_3^{-1}, [x_3, x_2],$$
$$[x_4, x_2], [x_5, x_2]x_6^{-1}x_5^{-2}, [x_6, x_2]x_6^{-2}x_5^{-2}, x_3^2, [x_4, x_3], [x_5, x_3]x_5^{-1}, [x_6, x_3]x_6^{-1}, x_4^2, [x_5, x_4], [x_6, x_4], x_5^3, [x_6, x_5], x_6^3\rangle$$

The sizes, orders and the representatives of the 12 conjugacy classes of G_{12} are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_2x_4	x_3x_4	x_4x_5	$x_1x_2x_4$
Size	1	18	18	9	1	8	18	18	18	9	8	18
Order	1	4	4	2	2	3	4	4	4	2	6	4

Theorem 3.3. Let G_{12} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1, 5\}$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{12}) \cong (\mathbb{K}_q^*)^8 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_8(\mathbb{K}_q))^2$.
2) for $p^k \equiv \{7, 11\}$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{12}) \cong (\mathbb{K}_q^*)^4 \oplus (\mathbb{K}_{q^2}^*)^2 \oplus (GL_2(\mathbb{K}_q))^2 \oplus (GL_8(\mathbb{K}_q))^2$.

Proof. The group G_{12} is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_qG_{12}) = 0$. Also, the commutator subgroup $G'_{12} = (C_3 \times C_3) \rtimes C_2$ and $\frac{G_{12}}{G'_{12}} = C_4 \times C_2$. Therefore, lemma 2.1 can be applied to the Wedderburn decomposition. Let us discuss the decomposition in 2 cases.
Case 1: k is even in $q = p^k$ or $p^k \equiv \{1, 5\}$ mod 12.
In this case, $| \ S_{\mathbb{K}}(\gamma_g) \ | = 1, \forall g \in G_{12}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition of $\mathbb{K}_qG_{12}$ is given by,

$$\mathbb{K}_qG_{12} \cong (\mathbb{K}_q)^8 \bigoplus_{i=1}^{4} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \ \Rightarrow 136 = \sum_{i=1}^{4} n_i^2.$$

The choices of n_i's can be $(2^2, 8^2)$ and $(2, 4^2, 10)$. In the direction of finding n_i's uniquely, we consider the normal subgroup $N = \langle x_5, x_6 \rangle$ of G_{12}. The factor group $F = \frac{G_{12}}{N} \cong C_4 \rtimes C_4$. Using [3], we note that $\mathbb{K}_qF \cong (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^2$. With this information, we can conclude that the Wedderburn decomposition of $\mathbb{K}_qG_{12}$ is given by

$$\mathbb{K}_qG_{12} \cong (\mathbb{K}_q)^8 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_8(\mathbb{K}_q))^2.$$

Case 2: k is odd and $p^k \equiv \{7, 11\}$ mod 12.
In this case, $S_{\mathbb{K}}(\gamma_{x_2}) = \{\gamma_{x_2}, \gamma_{x_2x_4}\}, S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{12}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_{12} \cong (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{i=1}^{4} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \Rightarrow 136 = \sum_{i=1}^{4} n_i^2.$$

Further, by repeating the same process as in theorem 3.1, we get that $(2^2, 8^2)$ is the only possibility for n_i. Therefore, we have

$$\mathbb{K}_qG_{12} \cong (\mathbb{K}_q)^4 \oplus (\mathbb{K}_{q^2})^2 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_8(\mathbb{K}_q))^2.$$

This completes the proof. □

3.4 $G_{18} = (C_3 \times A_4) \rtimes C_4$.

The group G_{18} has the following presentation:

$$G_{18} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2x_2^{-1}, [x_2, x_1], [x_3, x_1]x_3^{-1}, [x_4, x_1]x_4^{-1}, [x_5, x_1]x_6^{-1}x_5^{-1}, [x_6, x_1]x_6^{-1}x_5^{-1}, x_2^2, [x_3, x_2],$$
$$[x_4, x_2], [x_5, x_2], [x_6, x_2], x_3^3, [x_4, x_3], [x_5, x_3]x_6^{-1}x_5^{-1}, [x_6, x_3]x_5^{-1}, x_4^3, [x_5, x_4], [x_6, x_4], x_5^2, [x_6, x_5], x_6^2\rangle$$

The sizes, orders and the representatives of the 18 conjugacy classes of G_{18} are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_5	x_2x_3	x_2x_4	x_2x_5	x_3x_4
Size	1	18	1	8	2	3	18	18	8	2	3	8
Order	1	4	2	3	3	2	4	4	6	6	2	3

x_4x_5	$x_1x_2x_5$	$x_2x_3x_4$	$x_2x_4x_5$	$x_3^2x_4$	$x_2x_3^2x_4$
6	18	8	6	8	8
6	4	6	6	3	6

Theorem 3.4. Let G_{18} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1,5\}$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{18}) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^8 \oplus (GL_3(\mathbb{K}_q))^4 \oplus (GL_6(\mathbb{K}_q))^2$.
2) for $p^k \equiv \{7,11\}$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{18}) \cong (\mathbb{K}_q^*)^2 \oplus \mathbb{K}_{q^2}^* \oplus (GL_2(\mathbb{K}_q))^8 \oplus (GL_3(\mathbb{K}_q))^2 \oplus GL_3(\mathbb{K}_{q^2}) \oplus (GL_6(\mathbb{K}_q))^2$.

Proof. The group G_{18} is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_qG_{18}) = 0$. Also, the commutator subgroup $G'_{18} = C_3 \times A_4$ and $\frac{G_{18}}{G'_{18}} = C_4$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition.
Case 1: k is even in $q = p^k$ or $p^k \equiv \{1,5\}$ mod 12.
In this case, $|\ S_{\mathbb{K}}(\gamma_g)\ | = 1, \forall g \in G_{18}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition of $\mathbb{K}_qG_{18}$ is given by,

$$\mathbb{K}_qG_{18} \cong (\mathbb{K}_q)^4 \bigoplus_{i=1}^{14} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{14} n_i^2.$$

The choices of n_i's can be

$$(2^{11}, 4^2, 8), (2^{10}, 5^4), (2^9, 3^2, 5^2, 6), (2^8, 3^4, 6^2), (2^8, 3^3, 4^2, 7), (2^7, 4^7),$$

$$(2^6, 3^3, 4^4, 5), (2^5, 3^6, 4, 5^2), (2^4, 3^8, 4, 6) \text{ and } (3^{12}, 4^2).$$

In order to find the value of n_i's uniquely, we consider the normal subgroups $N_1 = \langle x_5, x_6\rangle$ and $N_2 = \langle x_4\rangle$ of G_{18}. The factor group $F_1 = \frac{G_{18}}{N_1} \cong (C_3 \times C_3) \rtimes C_4$. Using [3], we know that $\mathbb{K}_qF_1 \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^8$. By theorem 3.3 from [15], $\mathbb{K}_qF_2 \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_3(\mathbb{K}_q))^4$. With this information, we conclude that the Wedderburn decomposition of $\mathbb{K}_qG_{18}$ is given by

$$\mathbb{K}_qG_{18} \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^8 \oplus (M_3(\mathbb{K}_q))^4 \oplus (M_6(\mathbb{K}_q))^2.$$

Case 2: k is odd and $p^k \equiv \{7,11\}$ mod 12.
In this case, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_2}\}, S_{\mathbb{K}}(\gamma_{x_1x_5}) = \{\gamma_{x_1x_5}, \gamma_{x_1x_2x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{18}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_{18} \cong (\mathbb{K}_q)^2 \oplus \mathbb{K}_{q^2} \bigoplus_{i=1}^{14} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{14} n_i^2.$$

Furthermore, using [3, 15] we know that $\mathbb{K}_qF_1 \cong (\mathbb{K}_q)^2 \oplus \mathbb{K}_{q^2} \oplus (M_2(\mathbb{K}_q))^8$ and $\mathbb{K}_qF_2 \cong (\mathbb{K}_q)^2 \oplus \mathbb{K}_{q^2} \oplus (M_2(\mathbb{K}_q))^2 \oplus (M_3(\mathbb{K}_q))^2 \oplus M_3(\mathbb{K}_{q^2})$. This means that

$$\mathbb{K}_qG_{18} \cong (\mathbb{K}_q)^2 \oplus \mathbb{K}_{q^2} \oplus (M_2(\mathbb{K}_q))^8 \oplus (M_3(\mathbb{K}_q))^2 \oplus M_3(\mathbb{K}_{q^2}) \oplus (M_6(\mathbb{K}_q))^2.$$

This completes the proof. □

3.5 $G_{19} = ((C_4 \times S_3) \rtimes C_2) \rtimes C_3$.

The group G_{19} has the following presentation:

$$G_{19} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2x_6^{-1}, [x_2, x_1], [x_3, x_1], [x_4, x_1], [x_5, x_1]x_5^{-1}, [x_6, x_1], x_2^3, [x_3, x_2]x_4^{-1}, [x_4, x_2]x_4^{-1}x_3^{-1}, [x_5, x_2], [x_6, x_2], x_3^2x_6^{-1}, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], x_4^2x_6^{-1}, [x_5, x_4], [x_6, x_4], x_5^3, [x_6, x_5], x_6^2\rangle$$

The sizes, orders and the representatives of the 21 conjugacy classes of G_{19} are given below:

Representative	e	x_1	x_2	x_3	x_5	x_6	x_1x_2	x_1x_3	x_1x_6	x_2^2	x_2x_3	x_2x_5	x_3x_5
Size	1	3	4	6	2	1	12	18	3	4	4	8	12
Order	1	4	3	4	3	2	12	2	4	3	6	3	12

x_5x_6	$x_1x_2^2$	$x_1x_2x_3$	$x_2^2x_5$	$x_2^2x_6$	$x_2x_3x_5$	$x_1x_2^2x_6$	$x_2^2x_5x_6$
2	12	12	8	4	8	12	8
6	12	12	3	6	6	12	6

Theorem 3.5. Let G_{19} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv 1$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{19}) \cong (\mathbb{K}_q^*)^6 \oplus (GL_2(\mathbb{K}_q))^9 \oplus (GL_3(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^3 \oplus GL_6(\mathbb{K}_q)$.
2) for $p^k \equiv 5$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{19}) \cong (\mathbb{K}_q^*)^2 \oplus (\mathbb{K}_{q^2}^*)^2 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_2(\mathbb{K}_{q^2}))^3 \oplus (GL_3(\mathbb{K}_q))^2 \oplus GL_4(\mathbb{K}_q) \oplus GL_4(\mathbb{K}_{q^2}) \oplus GL_6(\mathbb{K}_q)$.
3) for $p^k \equiv 7$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{19}) \cong (\mathbb{K}_q^*)^6 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_2(\mathbb{K}_{q^2}))^3 \oplus (GL_3(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^3 \oplus GL_6(\mathbb{K}_q)$.
4) for $p^k \equiv 11$ mod 12, $\mathcal{U}(\mathbb{K}_qG_{19}) \cong (\mathbb{K}_q^*)^2 \oplus (\mathbb{K}_{q^2}^*)^2 \oplus GL_2(\mathbb{K}_q) \oplus (GL_2(\mathbb{K}_{q^2}))^4 \oplus (GL_3(\mathbb{K}_q))^2 \oplus GL_4(\mathbb{K}_q) \oplus GL_4(\mathbb{K}_{q^2}) \oplus GL_6(\mathbb{K}_q)$.

Proof. The group G_{19} is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_qG_{19}) = 0$. Also, the commutator subgroup $G'_{19} \cong C_3 \times Q_8$ and $\frac{G_{19}}{G'_{19}} \cong C_6$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition. We discuss the decomposition in the following 4 cases.
Case 1: k is even in $q = p^k$ or $p^k \equiv 1 \bmod 12$.
In this case, $|\ S_{\mathbb{K}}(\gamma_g)\ | = 1, \forall g \in G_{19}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition of $\mathbb{K}_qG_{19}$ is given by,

$$\mathbb{K}_qG_{19} \cong (\mathbb{K}_q)^6 \bigoplus_{i=1}^{15} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \ \Rightarrow 138 = \sum_{i=1}^{15} n_i^2.$$

The choices of n_i's can be

$$(2^{12}, 4, 5, 7), (2^{10}, 4^3, 5^2), (2^9, 3^3, 5^3), (2^9, 3^2, 4^3, 6), (2^8, 3^5, 5, 6), (2^5, 3^6, 4^4) \text{ and } (2^4, 3^9, 4, 5)$$

In the direction of finding n_i's uniquely, we consider the normal subgroup $N = \langle x_5 \rangle$ of G_{19}. The Wedderburn decomposition of the factor group $F = \frac{G_{19}}{N} \cong ((C_4 \times C_2) \rtimes C_2) \rtimes C_3$ is $\mathbb{K}_qF \cong (\mathbb{K}_q)^6 \oplus M_2(\mathbb{K}_q)^6 \oplus M_3(\mathbb{K}_q)^2$ (see [15]). With this information, we can conclude that the choices for n_i's are reduced to $(2^9, 3^3, 5^3)$, $(2^9, 3^2, 4^3, 6)$ and $(2^8, 3^5, 5, 6)$. Suppose, if $p = 5$, then by proposition 1 of [4], $(2^9, 3^2, 4^3, 6)$ can be the only choice in the Wedderburn decomposition of $\mathbb{K}_qG_{19}$. Thus, we have

$$\mathbb{K}_qG_{19} \cong (\mathbb{K}_q)^6 \oplus (M_2(\mathbb{K}_q))^9 \oplus (M_3(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^3 \oplus M_6(\mathbb{K}_q).$$

Case 2: k is odd and $p^k \equiv 5 \bmod 12$.
In this case, $S_{\mathbb{K}}(\gamma_{x_2}) = \{\gamma_{x_2}, \gamma_{x_2^2}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2^2}\}$, $S_{\mathbb{K}}(\gamma_{x_2x_3}) = \{\gamma_{x_2x_3}, \gamma_{x_2^2x_6}\}$, $S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2^2x_5}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2x_3}) = \{\gamma_{x_1x_2x_3}, \gamma_{x_1x_2^2x_6}\}$, $S_{\mathbb{K}}(\gamma_{x_2x_3x_5}) = \{\gamma_{x_2x_3x_5}, \gamma_{x_2^2x_5x_6}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{19}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_{19} \cong (\mathbb{K}_q)^2 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{i=1}^{7} M_{n_i}(\mathbb{K}_q) \bigoplus_{i=8}^{11} M_{n_i}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 138 = \sum_{i=1}^{7} n_i^2 + 2 \cdot \sum_{i=8}^{11} n_i^2.$$

Therefore, repeat the same process as in case 1, we get that $(2^9, 3^2, 4^3, 6)$ is the only possibility for n_i's. Hence, we have

$$\mathbb{K}_qG_{19} \cong \mathbb{K}_q^2 \oplus \mathbb{K}_{q^2}^2 \oplus M_2(\mathbb{K}_q)^3 \oplus M_2(\mathbb{K}_{q^2})^3 \oplus M_3(\mathbb{K}_q)^2 \oplus M_4(\mathbb{K}_q) \oplus M_4(\mathbb{K}_{q^2}) \oplus M_6(\mathbb{K}_q).$$

Case 3: k is odd and $p^k \equiv 7 \bmod 12$.
In this case, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_6}\}, S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_3}\}, S_{\mathbb{K}}(\gamma_{x_1x_2^2}) = \{\gamma_{x_1x_2^2}, \gamma_{x_1x_2^2x_6}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{19}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_{19} \cong (\mathbb{K}_q)^6 \bigoplus_{i=1}^{9} M_{n_i}(\mathbb{K}_q) \bigoplus_{i=10}^{12} M_{n_i}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 138 = \sum_{i=1}^{9} n_i^2 + 2 \cdot \sum_{i=10}^{12} n_i^2.$$

On repeating the same process as in case 1, we get that $(2^9, 3^2, 4^3, 6)$ is the only possibility for n_i. Therefore, we have

$$\mathbb{K}_qG_{19} \cong (\mathbb{K}_q)^6 \oplus (M_2(\mathbb{K}_q))^3 \oplus (M_2(\mathbb{K}_{q^2}))^3 \oplus (M_3(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^3 \oplus M_6(\mathbb{K}_q).$$

Case 4: k is odd and $p^k \equiv 11 \bmod 12$.
In this case, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_6}\}, S_{\mathbb{K}}(\gamma_{x_2}) = \{\gamma_{x_2}, \gamma_{x_2^2}\}, S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2^2x_6}\}, S_{\mathbb{K}}(\gamma_{x_2x_3}) = \{\gamma_{x_2x_3}, \gamma_{x_2^2x_6}\}$, $S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2^2x_5}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_2^2}) = \{\gamma_{x_1x_2x_3}, \gamma_{x_1x_2^2}\}$, $S_{\mathbb{K}}(\gamma_{x_2x_3x_5}) = \{\gamma_{x_2x_3x_5}, \gamma_{x_2^2x_5x_6}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{19}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_{19} \cong (\mathbb{K}_q)^2 \oplus (\mathbb{K}_{q^2})^2 \bigoplus_{i=1}^{5} M_{n_i}(\mathbb{K}_q) \bigoplus_{i=6}^{10} M_{n_i}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 138 = \sum_{i=1}^{5} n_i^2 + 2 \cdot \sum_{i=6}^{10} n_i^2.$$

W eepeat the same process as in case 1 to note that $(2^9, 3^2, 4^3, 6)$ is the only possibility for n_i's. Consequently, we get

$$\mathbb{K}_qG_{19} \cong (\mathbb{K}_q)^2 \oplus (\mathbb{K}_{q^2})^2 \oplus M_2(\mathbb{K}_q) \oplus (M_2(\mathbb{K}_{q^2}))^4 \oplus (M_3(\mathbb{K}_q))^2 \oplus M_4(\mathbb{K}_q) \oplus M_4(\mathbb{K}_{q^2}) \oplus M_6(\mathbb{K}_q).$$

This completes the proof. □

4 Groups of exponent 24

In this section, we characterize the unit group of group algebra generated by 6 non-metabelian groups of order 144 with exponent 24. We use the same numbers for these groups as in section 2. The 6 non-metabelian groups of order 144 with exponent 24 are given below:

9. $(C_3 \times C_3) \rtimes D_{16}$
10. $(C_3 \times C_3) \rtimes QD_{16}$
11. $(C_3 \times C_3) \rtimes Q_{16}$
16. $C_3 \rtimes (C_2 \cdot S_4)$
17. $(C_3 \times SL(2, 3)) \rtimes C_2$
23. $(C_3 \times C_3) \rtimes QD_{16}$

4.1 $G_9 = (C_3 \times C_3) \rtimes D_{16}$.

The group G_9 has the following presentation:

$$G_9 = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2, [x_2, x_1]x_3^{-1}, [x_3, x_1]x_4^{-1}, [x_4, x_1], [x_5, x_1]x_5^{-1}, [x_6, x_1], x_2^2, [x_3, x_2]x_4^{-1}, [x_4, x_2],$$
$$[x_5, x_2]x_6^{-1}x_5^{-2}, [x_6, x_2]x_6^{-2}x_5^{-1}, x_3^2x_4^{-1}, [x_4, x_3], [x_5, x_3]x_5^{-1}, [x_6, x_3]x_6^{-1}, x_4^2, [x_5, x_4], [x_6, x_4], x_5^3, [x_6, x_5], x_6^3\rangle$$

The sizes, orders and the representatives of the 15 conjugacy classes of G_9 are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_6	x_2x_5	x_4x_5	x_5x_6
Size	1	12	12	18	1	4	18	12	12	4	4
Order	1	2	2	4	2	3	8	6	6	6	3

$x_1x_2x_4$	$x_1x_3x_5$	$x_2x_3x_5$	$x_4x_5x_6$
18	12	12	4
8	6	6	6

Theorem 4.1. Let G_9 be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1, 7\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_9) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_q))^8$.
2) for $p^k \equiv \{5, 11\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_9) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_4(\mathbb{K}_q))^4 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.
3) for $p^k \equiv \{13, 19\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_9) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_4(\mathbb{K}_q))^8$.
4) for $p^k \equiv \{17, 23\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_9) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_q))^4 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

Proof. The group G_9 is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_qG_9) = 0$. Also, the commutator subgroup $G_9' = (C_3 \times C_3) \rtimes C_4$ and $\frac{G_9}{G_9'} = C_2 \times C_2$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition. We discuss the Wedderburn decomposition in the following 4 cases.
Case 1: k is even in $q = p^k$ or $p^k \equiv \{1, 7\}$ mod 24.
In this case, $\mid S_{\mathbb{K}}(\gamma_g) \mid = 1, \forall g \in G_9$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition of $\mathbb{K}_qG_9$ is given by

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \bigoplus_{i=1}^{11} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{11} n_i^2.$$

The choices of n_i's can be

$$(2^9, 10), (2^8, 6^3), (2^7, 4^3, 8), (2^6, 3^3, 5, 8), (2^6, 4, 5^4), (2^5, 3^2, 4, 5^2, 6), (2^4, 3^4, 4, 6^2), (2^4, 3^3, 4^3, 7),$$
$$(2^3, 3^6, 5, 7), (2^3, 4^8), (2^2, 3^3, 4^5, 5), (2, 3^6, 4^2, 5^2) \text{ and } (3^8, 4^2, 6).$$

In the direction of finding n_i's uniquely, we consider the normal subgroup $N_1 = \langle x_4 \rangle$ of G_9. The Wedderburn decomposition of the factor group $F_1 = \frac{G_9}{N_1} \cong (S_3 \times S_3) \rtimes C_2$ is $\mathbb{K}_qF_1 \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_4(\mathbb{K}_q)^4$ (see [15]). With this information, we can conclude that the choices of n_i's can either be $(2^3, 4^8)$ or $(2^2, 3^3, 4^5, 5)$. Suppose, if $p = 5$, then by proposition 1 of [4], $(2^2, 3^3, 4^5, 5)$ cannot be the choice in the decomposition of $\mathbb{K}_qG_9$. Hence

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^3 \oplus (M_4(\mathbb{K}_q))^8.$$

Case 2: k is odd and $p^k \equiv \{5, 11\}$ mod 24.
In this case, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$, $S_{\mathbb{K}}(\gamma_{x_1x_6}) = \{\gamma_{x_1x_6}, \gamma_{x_1x_3x_5}\}$, $S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2x_3x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_9$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \bigoplus_{i=1}^{5} M_{n_i}(\mathbb{K}_q) \bigoplus_{i=6}^{8} M_{n_i}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{5} n_i^2 + 2 \cdot \sum_{i=6}^{8} n_i^2.$$

On repeating the same process as in case 1, we get that $(2, 4^4, 2, 4^2)$ is the only possibility for n_i. Hence, we have

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_2(\mathbb{K}_{q^2}) \oplus (M_4(\mathbb{K}_q))^4 \oplus (M_4(\mathbb{K}_{q^2}))^2.$$

Case 3: k is odd and $p^k \equiv \{13, 19\}$ mod 24.
In this case, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_9$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \bigoplus_{i=1}^{9} M_{n_i}(\mathbb{K}_q) \oplus M_{n_{10}}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{9} n_i^2 + 2 \cdot n_{10}^2.$$

Next, we consider $N_2 = \langle x_5, x_6 \rangle \trianglelefteq G_9$. Accordingly, $F_2 = \frac{G_9}{N_2} \cong D_{16}$. Using [3], we note that $\mathbb{K}_qF_2 \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_2(\mathbb{K}_{q^2})$. Hence, we note that

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_2(\mathbb{K}_{q^2}) \oplus (M_4(\mathbb{K}_q))^8.$$

Case 4: k is odd and $p^k \equiv \{17, 23\}$ mod 24.
In this case, $S_{\mathbb{K}}(\gamma_{x_1x_6}) = \{\gamma_{x_1x_6}, \gamma_{x_1x_3x_5}\}, S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2x_3x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_9$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \bigoplus_{i=1}^{7} M_{n_i}(\mathbb{K}_q) \bigoplus_{i=8}^{9} M_{n_i}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{7} n_i^2 + 2 \cdot \sum_{i=8}^{9} n_i^2.$$

By repeating the procedure as in case 3, we note that

$$\mathbb{K}_qG_9 \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^3 \oplus (M_4(\mathbb{K}_q))^4 \oplus (M_4(\mathbb{K}_{q^2}))^2.$$

This completes the proof. □

4.2 $G_{10} = (C_3 \times C_3) \rtimes QD_{16}$.

The group G_{10} has the following presentation:

$$G_{10} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2, [x_2, x_1]x_3^{-1}, [x_3, x_1]x_4^{-1}, [x_4, x_1], [x_5, x_1]x_5^{-1}, [x_6, x_1], x_2^2 x_4^{-1}, [x_3, x_2]x_4^{-1}, [x_4, x_2],$$
$$[x_5, x_2]x_6^{-1}x_5^{-2}, [x_6, x_2]x_6^{-2}x_5^{-1}, x_3^2 x_4^{-1}, [x_4, x_3], [x_5, x_3]x_5^{-1}, [x_6, x_3]x_6^{-1}, x_4^2, [x_5, x_4], [x_6, x_4], x_5^3, [x_6, x_5], x_6^3 \rangle$$

The sizes, orders and the representatives of the 15 conjugacy classes of G_{10} are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_6	x_2x_5	x_4x_5	x_5x_6	$x_1x_2x_4$	$x_1x_3x_5$	$x_2x_3x_5$	$x_4x_5x_6$
Size	1	12	12	18	1	4	18	12	12	4	4	18	12	12	4
Order	1	2	4	4	2	3	8	6	12	6	3	8	6	12	6

Theorem 4.2. Let G_{10} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv 1 \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{10}) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_q))^8$.

2) for $p^k \equiv 5 \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{10}) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_4(\mathbb{K}_q))^4 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

3) for $p^k \equiv \{7, 23\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{10}) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_4(\mathbb{K}_q))^6 \oplus GL_4(\mathbb{K}_{q^2})$.

4) for $p^k \equiv \{11, 19\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{10}) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_q))^6 \oplus GL_4(\mathbb{K}_{q^2})$.

5) for $p^k \equiv 13 \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{10}) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_4(\mathbb{K}_q))^8$.

6) for $p^k \equiv 17 \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{10}) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_q))^4 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

Proof. The group G_{10} is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_q G_{10}) = 0$. Also, the commutator subgroup $G'_{10} \cong (C_3 \times C_3) \rtimes C_4$ and $\frac{G_{10}}{G'_{10}} \cong C_2 \times C_2$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition. Let us discuss the Wedderburn decomposition in the following 6 cases.
Case 1: k is even or $p^k \equiv 1 \bmod 24$. The proof follows from case 1 of theorem 4.1.
Case 2: k is odd and $p^k \equiv 5 \bmod 24$. The proof follows from case 2 of theorem 4.1.
Case 3: k is odd and $p^k \equiv \{7, 23\} \bmod 24$.
For $p^k \equiv 7 \bmod 24$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}, S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2x_3x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{10}$. For $p^k \equiv 23 \bmod 24$, $S_{\mathbb{K}}(\gamma_{x_1x_2}) = \{\gamma_{x_1x_2}, \gamma_{x_1x_2x_4}\}, S_{\mathbb{K}}(\gamma_{x_1x_6}) = \{\gamma_{x_1x_6}, \gamma_{x_1x_3x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{10}$. For both the cases, the Wedderburn decomposition and the choices of n_i's are same as in case 3 of theorem 4.1. Then, as in case 3 of theorem 4.1, we note that $\mathbb{K}_q F_2 \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_2(\mathbb{K}_{q^2})$. Thus, we have

$$\mathbb{K}_q G_{10} \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_2(\mathbb{K}_{q^2}) \oplus (M_4(\mathbb{K}_q))^6 \oplus M_4(\mathbb{K}_{q^2}).$$

Case 4: k is odd and $p^k \equiv \{11, 19\} \bmod 24$.
For $p^k \equiv 11 \bmod 24$, $S_{\mathbb{K}}(\gamma_{x_1x_6}) = \{\gamma_{x_1x_6}, \gamma_{x_1x_3x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{10}$. For $p^k \equiv 19 \bmod 24$, $S_{\mathbb{K}}(\gamma_{x_2x_5}) = \{\gamma_{x_2x_5}, \gamma_{x_2x_3x_5}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{10}$. For both the cases, similar to case 3, we can see that the Wedderburn decomposition is given by $\mathbb{K}_q F_2 \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^3$

$$\mathbb{K}_q G_{10} \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^3 \oplus (M_4(\mathbb{K}_q))^6 \oplus M_4(\mathbb{K}_{q^2}).$$

Case 5: k is odd and $p^k \equiv 5 \bmod 24$. The proof follows from case 3 of theorem 4.1.
Case 6: k is odd and $p^k \equiv 5 \bmod 24$. The proof follows from case 4 of theorem 4.1. This completes the proof. □

4.3 $G_{11} = (C_3 \times C_3) \rtimes Q_{16}$.

The group G_{11} has the following presentation:

$$G_{11} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2 x_4^{-1}, [x_2, x_1]x_3^{-1}, [x_3, x_1]x_4^{-1}, [x_4, x_1], [x_5, x_1]x_5^{-1}, [x_6, x_1], x_2^2 x_4^{-1}, [x_3, x_2]x_4^{-1}, [x_4, x_2],$$
$$[x_5, x_2]x_6^{-1}x_5^{-2}, [x_6, x_2]x_6^{-2}x_5^{-1}, x_3^2 x_4^{-1}, [x_4, x_3], [x_5, x_3]x_5^{-1}, [x_6, x_3]x_6^{-1}, x_4^2, [x_5, x_4], [x_6, x_4], x_5^3, [x_6, x_5], x_6^3 \rangle$$

The sizes, orders and the representatives of the 15 conjugacy classes of G_9 are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_6	x_2x_5	x_4x_5	x_5x_6	$x_1x_2x_4$	$x_1x_3x_5$	$x_2x_3x_5$	$x_4x_5x_6$
Size	1	12	12	18	1	4	18	12	12	4	4	18	12	12	4
Order	1	4	4	4	2	3	8	12	12	6	3	8	12	12	6

Theorem 4.3. Let G_{11} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1, 23\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{11}) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_q))^8$.

2) for $p^k \equiv \{5, 19\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{11}) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_4(\mathbb{K}_q))^4 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

3) for $p^k \equiv \{7, 17\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{11}) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_4(\mathbb{K}_q))^4 \oplus (GL_4(\mathbb{K}_{q^2}))^2$.

4) for $p^k \equiv \{11, 13\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{11}) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_4(\mathbb{K}_q))^8$.

Proof. The proof can be done on the similar lines of theorem 4.1. □

4.4 $G_{16} = C_3 \rtimes (C_2 \cdot S_4)$.

The group G_{16} has the following presentation:

$$\begin{aligned} G_{16} = \langle & x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2 x_6^{-1}, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_3^{-1}, [x_4, x_1]x_6^{-1}x_5^{-1}x_4^{-1}, [x_5, x_1]x_5^{-1}x_4^{-1}, [x_6, x_1], \\ & x_2^3, [x_3, x_2], [x_4, x_2]x_5^{-1}x_4^{-1}, [x_5, x_2]x_6^{-1}x_4^{-1}, [x_6, x_2], x_3^3, [x_4, x_3], [x_5, x_3], [x_6, x_3], x_4^2 x_6^{-1}, \\ & [x_5, x_4]x_6^{-1}, [x_6, x_4], x_5^2 x_6^{-1}, [x_6, x_5], x_6^2 \rangle \end{aligned}$$

The sizes, orders and the representatives of the 15 conjugacy classes of G_{16} are given below:

Representative	e	x_1	x_2	x_3	x_4	x_6	x_1x_4	x_2x_3	x_2x_5	x_3x_4	x_3x_6	$x_1x_2x_4$	$x_2^2x_3$	$x_2x_3x_5$	$x_2^2x_3x_4$
Size	1	36	8	2	6	1	18	8	8	12	2	18	8	8	8
Order	1	4	3	3	4	2	8	3	6	12	6	8	3	6	6

Theorem 4.4. Let G_{16} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1, 7, 17, 23\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{16}) \cong (\mathbb{K}_q^*)^2 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_3(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^4 \oplus GL_6(\mathbb{K}_q)$.

2) for $p^k \equiv \{5, 11, 13, 19\} \bmod 24$, $\mathcal{U}(\mathbb{K}_q G_{16}) \cong (\mathbb{K}_q^*)^2 \oplus (GL_2(\mathbb{K}_q))^4 \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_3(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^4 \oplus GL_6(\mathbb{K}_q)$.

Proof. The group G_{16} is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_q G_{16}) = 0$. Also, the commutator subgroup $G'_{16} \cong C_3 \times SL(2, 3)$ and $\frac{G_{16}}{G'_{16}} \cong C_2$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition. Let us discuss the decomposition in the following 2 cases.
Case 1: k is even in $q = p^k$ or $p^k \equiv \{1, 7, 17, 23\} \bmod 24$.
In this case, $\mid S_{\mathbb{K}}(\gamma_g) \mid = 1, \forall g \in G_{16}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition of $\mathbb{K}_q G_{16}$ is given by,

$$\mathbb{K}_q G_{16} \cong (\mathbb{K}_q)^2 \bigoplus_{i=1}^{13} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \ \Rightarrow 142 = \sum_{i=1}^{13} n_i^2.$$

The choices of n_i's can be

$$(2^{11}, 7^2), (2^9, 3, 5, 6^2), (2^9, 4^2, 5, 7), (2^7, 4^4, 5^2), (2^6, 3^6, 8), (2^6, 3^3, 4, 5^3), (2^6, 3^2, 4^4, 6),$$

$$(2^5, 3^5, 4, 5, 6), (2^3, 3^9, 7), (2^2, 3^6, 4^5) \text{ and } (2, 3^9, 4^2, 5).$$

In the direction of finding n_i's uniquely, we consider the normal subgroup $N = C_3$ of G_{16}. The Wedderburn decomposition of the factor group $F = \frac{G_{16}}{N} \cong C_2 \cdot S_4$ is $\mathbb{K}_q F \cong (\mathbb{K}_q)^2 \oplus M_2(\mathbb{K}_q)^3 \oplus M_3(\mathbb{K}_q)^2 \oplus M_4(\mathbb{K}_q)$ (see [15, theorem 3.1]). With this information, we can conclude that the choices of n_i's can be $(2^6, 3^3, 4, 5^3)$, $(2^6, 3^2, 4^4, 6)$ or $(2^5, 3^5, 4, 5, 6)$. Suppose, if $p = 5$, then by proposition 1 of [4], $(2^6, 3^2, 4^4, 6)$ is the only choice for the decomposition of $\mathbb{K}_q G_{16}$. Therefore, we have

$$\mathbb{K}_q G_{16} \cong (\mathbb{K}_q)^2 \oplus (M_2(\mathbb{K}_q))^6 \oplus (M_3(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^4 \oplus M_6(\mathbb{K}_q).$$

Case 2: k is odd and $p^k \equiv \{5, 11, 13, 19\} \bmod 24$.
In this case, $S_{\mathbb{K}}(\gamma_{x_1x_4}) = \{\gamma_{x_1x_4}, \gamma_{x_1x_2x_4}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{16}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_q G_{16} \cong (\mathbb{K}_q)^2 \bigoplus_{i=1}^{11} M_{n_i}(\mathbb{K}_q) \oplus M_{n_{12}}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 142 = \sum_{i=1}^{11} n_i^2 + 2 \cdot n_{12}^2.$$

On repeating the same process as in case 1, we get that $(2^6, 3^2, 4^4, 6)$ is the only possibility for n_i's. Hence, we get

$$\mathbb{K}_q G_{16} \cong (\mathbb{K}_q)^2 \oplus (M_2(\mathbb{K}_q))^4 \oplus M_2(\mathbb{K}_{q^2}) \oplus (M_3(\mathbb{K}_q))^2 \oplus (M_4(\mathbb{K}_q))^4 \oplus M_6(\mathbb{K}_q).$$

This completes the proof. □

4.5 $G_{17} = (C_3 \times SL(2,3)) \rtimes C_2$.

The group G_{17} has the following presentation:

$$G_{17} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_3^{-1}, [x_4, x_1]x_6^{-1}x_5^{-1}x_4^{-1}, [x_5, x_1]x_5^{-1}x_4^{-1}, [x_6, x_1], x_2^3, [x_3, x_2], [x_4, x_2]x_5^{-1}x_4^{-1}, [x_5, x_2]x_6^{-1}x_4^{-1}, [x_6, x_2], x_3^3, [x_4, x_3], [x_5, x_3], [x_6, x_3], x_4^2x_6^{-1}, [x_5, x_4]x_6^{-1}, [x_6, x_4], x_5^2x_6^{-1}, [x_6, x_5], x_6^2 \rangle$$

The sizes, orders and the representatives of the 15 conjugacy classes of G_{17} are given below:

Representative	e	x_1	x_2	x_3	x_4	x_6	x_1x_4	x_2x_3	x_2x_5	x_3x_4	x_3x_6
Size	1	36	8	2	6	1	18	8	8	12	2
Order	1	2	3	3	4	2	8	3	6	12	6

$x_1x_2x_4$	$x_2^2x_3$	$x_2x_3x_5$	$x_2^2x_3x_4$
18	8	8	8
8	3	6	6

Theorem 4.5. Let G_{17} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1, 11, 17, 19\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_{17}) \cong (\mathbb{K}_q^*)^2 \oplus (GL_2(\mathbb{K}_q))^6 \oplus (GL_3(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^4 \oplus GL_6(\mathbb{K}_q)$.
2) for $p^k \equiv \{5, 7, 13, 23\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_{17}) \cong (\mathbb{K}_q^*)^2 \oplus (GL_2(\mathbb{K}_q))^4 \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_3(\mathbb{K}_q))^2 \oplus (GL_4(\mathbb{K}_q))^4 \oplus GL_6(\mathbb{K}_q)$.

Proof. The proof is similar to that of theorem 4.4. □

4.6 $G_{23} = (C_3 \times C_3) \rtimes QD_{16}$.

The group G_{23} has the following presentation:

$$G_{23} = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2x_3^{-1}, [x_2, x_1]x_4^{-1}x_3^{-1}, [x_3, x_1], [x_4, x_1], [x_5, x_1]x_6^{-1}x_5^{-2}, [x_6, x_1]x_6^{-1}x_5^{-1}, x_2^2, [x_3, x_2]x_4^{-1}, [x_4, x_2], [x_5, x_2]x_6^{-2}, [x_6, x_2]x_6^{-1}, x_3^2x_4^{-1}, [x_4, x_3], [x_5, x_3]x_6^{-2}, [x_6, x_3]x_6^{-1}x_5^{-2}, x_4^2, [x_5, x_4]x_5^{-1}, [x_6, x_4]x_6^{-1}, x_5^3, [x_6, x_5], x_6^3 \rangle$$

The sizes, orders and the representatives of the 9 conjugacy classes of G_{23} are given below:

Representative	e	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_4	x_2x_5
Size	1	18	12	18	9	8	36	18	24
Order	1	8	2	4	2	3	4	8	6

Theorem 4.6. Let G_{23} be the group defined above and $\mathbb{K}_q$ be the finite field of characteristic $p > 3$. Then

1) for k even or $p^k \equiv \{1, 11, 17, 19\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_{23}) \cong (\mathbb{K}_q^*)^4 \oplus (GL_2(\mathbb{K}_q))^3 \oplus (GL_8(\mathbb{K}_q))^2$.
2) for $p^k \equiv \{5, 7, 13, 23\}$ mod 24, $\mathcal{U}(\mathbb{K}_qG_{23}) \cong (\mathbb{K}_q^*)^4 \oplus GL_2(\mathbb{K}_q) \oplus GL_2(\mathbb{K}_{q^2}) \oplus (GL_8(\mathbb{K}_q))^2$.

Proof. The group G_{23} is finite and so, Artinian. Thus, by Maschke's theorem, $J(\mathbb{K}_qG_{23}) = 0$. Also, the commutator subgroup $G_{23}' = (C_3 \times C_3) \rtimes C_4$ and $\frac{G_{23}}{G_{23}'} = C_2 \times C_2$. Therefore, lemma 2.1 can be applied to compute the Wedderburn decomposition. Let us discuss the Wedderburn decomposition in the following 2 cases.
Case 1: k is even in $q = p^k$ or $p^k \equiv \{1, 11, 17, 19\}$ mod 24.
In this case, $\mid S_{\mathbb{K}}(\gamma_g) \mid = 1, \forall g \in G_{23}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition of $\mathbb{K}_qG_{23}$ is given by

$$\mathbb{K}_qG_{23} \cong (\mathbb{K}_q)^4 \bigoplus_{i=1}^{5} M_{n_i}(\mathbb{K}_q), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{5} n_i^2.$$

The choices of n_i's can be

$$(2^3, 8^2), (2^2, 4^2, 10), (3^3, 7, 8), (3^2, 4, 5, 9), (4^2, 6^3) \text{ and } (4, 5^3, 7).$$

In the direction of finding n_i's uniquely, we consider the normal subgroup $N = \langle x_5, x_6 \rangle$ of G_{23}. Using [3], the Wedderburn decomposition of the group $F = \frac{G_{23}}{N} \cong QD_{16}$ is given by $\mathbb{K}_qF \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^3$. With this information, we can conclude that the only choice for the decomposition of $\mathbb{K}_qG_{23}$ is $(2^3, 8^2)$. Therefore, we get

$$\mathbb{K}_qG_{23} \cong (\mathbb{K}_q)^4 \oplus (M_2(\mathbb{K}_q))^3 \oplus (M_8(\mathbb{K}_q))^2.$$

Case 2: k is odd and $p^k \equiv \{5, 11, 13, 19\}$ mod 24.
In this case, $S_{\mathbb{K}}(\gamma_{x_1}) = \{\gamma_{x_1}, \gamma_{x_1x_4}\}$ and $S_{\mathbb{K}}(\gamma_g) = \{\gamma_g\}$, for the remaining $g \in G_{23}$. By lemma 2.1 and propositions 2.2 and 2.3, the Wedderburn decomposition is given by

$$\mathbb{K}_qG_{23} \cong (\mathbb{K}_q)^4 \bigoplus_{i=1}^{3} M_{n_i}(\mathbb{K}_q) \oplus M_{n_4}(\mathbb{K}_{q^2}), n_i \geq 2 \Rightarrow 140 = \sum_{i=1}^{3} n_i^2 + 2 \cdot n_4^2.$$

Therefore, we repeat the same process as in case 1 to get that $(2, 8^2, 2)$ is the only possibility for n_i's. Therefore, we get

$$\mathbb{K}_qG_{23} \cong (\mathbb{K}_q)^4 \oplus M_2(\mathbb{K}_q) \oplus M_2(\mathbb{K}_{q^2}) \oplus (M_8(\mathbb{K}_q))^2.$$

This completes the proof. □

5 Conclusion remarks

We have characterized the unit groups of the semisimple group algebras of non-metabelian groups of order 144 that have exponent either 12 or 24. In all, we have considered 11 group algebras in this paper. With this paper, the study of the unit groups of semisimple group algebras of groups up to order 144 is completed. This paper further motivates the researchers to compute the unit groups of the semisimple group algebras of non-metabelian groups of order greater than 144.

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