Palestine Journal of Mathematics

Vol 13(2)(2024) , 172–180

A NOTE ON X-log CONVEXITY IN $\mathbb{R}^n$

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 26A51; Secondary 26B25.

Keywords and phrases: X-log convex function, X-log quasi convex function, Local log-minimum, Global log-minimum.

Abstract We define and look into the fundamental features of X-log convex, X-log quasi convex, X-log semi-strictly quasi convex, and X-log-pseudo convex functions. This paper provides several examples and arguments in favour of the concept. Also, we develope another X-log convex function as the sum of an X-log convex and X-log affine convex function. Additionally, we define local and global log-minimum. Under one condition, it is possible to demonstrate that the local log-minimum of an X-log convex and X-log quasi-convex function is also the global log-minimum.

1 Introduction

In all areas of mathematics as well as related fields like physics, economics, and engineering, convex functions play a significant role. The underlying reason for this is because those become supportive in the presence of convexity as necessary conditions for the existence of a minimum, which is particularly important to extremum problems. Around the turn of the 20th century, Jensen [7] pioneered the concept of the convex functions, and Fenchel [19, 20] began a systematic investigation of the conjugate function more than forty years later. Fenchel's [19] lecture notes become the foundation for Rockfellar's [17] renowned book, convex analysis.

Recent years have seen a number of generalizations of convex functions and convex sets being addressed and investigated utilizing the concepts and methods. It is well understood that logarithmically convex functions provide more accuracy and inequality than convex functions. The notion of exponentially convex functions is closely connected to log-convex functions. As can be seen in [1, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 18, 21, 22], a large number of experts and researchers have recently worked on generalized convexity, generalized log-convexity and exponential-convexity with fruitful applications. Recently, Ali and Akhter developed the idea of X-convex sets and functions [8].

Motivated and inspired by continuing research in this intriguing, practical, and dynamic field, we explore the idea of log-convex functions. We give some new definitions, namely local log-minimum, global log-minimum and X-log convex, X-log quasi-convex, X-log semi-strictly quasi-convex, X-log affine convex functions. The fundamental features of X-log convex functions are discussed. It has been demonstrated that X-log convex functions have the same pleasant features as convex functions. Several novel notions have been introduced and researched. We see that the difference (sum) between the X-log convex and X-log affine convex functions is another X-log convex function. The local log-minimum of X-log convex functions is shown to be the global log-minimum.

2 Preliminaries

Definition 2.1. [8] A subset G of $\mathbb{R}^n$ is said to be an X**-convex set** if for all $e, e_o \in G$, $c \in [0, 1]$, and if there exists a vector valued map $q : G \to \mathbb{R}^n$ such that $c(e - e_o) + q(e_o) \in G$.

Definition 2.2. [8] A function $\mathcal{L} : G \to \mathbb{R}$, defined on a nonempty X-convex subset G of $\mathbb{R}^n$, is said to be

(a) X**-convex** if $\forall\ e, e_o \in G$ and $0 \leq c \leq 1$, $\exists$ a vector valued map $q : G \to \mathbb{R}^n$ such that,

$$\mathcal{L}(c(e - e_o) + q(e_o)) \leq c\mathcal{L}(e) + (1 - c)\mathcal{L}(e_o),$$

(b) **strictly** X**-convex** if $\forall\ e, e_o \in B$, $e \neq e_o$ and $0 < c < 1$, $\exists$ a vector valued map $q : G \to \mathbb{R}^n$ such that,

$$\mathcal{L}(c(e - e_o) + q(e_o)) < c\mathcal{L}(e) + (1 - c)\mathcal{L}(e_o).$$

A function $\mathcal{L} : G \to \mathbb{R}$ is said be (strictly) X-concave if $-\mathcal{L}$ is (strictly) X-convex.

Definition 2.3. [8] A function $\mathcal{L} : G \to \mathbb{R}$, defined on a nonempty X-convex subset G of $\mathbb{R}^n$, is said to be

(a) **quasi-**X**-convex** if $\forall\ e, e_o \in G$ and $0 \leq c \leq 1$, $\exists$ a vector valued map $q : G \to \mathbb{R}^n$ such that,

$$\mathcal{L}(c(e - e_o) + q(e_o)) \leq \max\{\mathcal{L}(e), \mathcal{L}(e_o)\},$$

(b) **strictly quasi-**X**-convex** if $\forall\ e, e_o \in G$, $e \neq e_o$ and $0 < c < 1$, $\exists$ a vector valued map $q : G \to \mathbb{R}^n$ such that,

$$\mathcal{L}(c(e - e_o) + q(e_o)) < \max\{\mathcal{L}(e), \mathcal{L}(e_o)\},$$

(c) **semi-strictly quasi-**X**-convex** if $\forall\ e, e_o \in G$ and $0 < c < 1$, $\exists$ a vector valued map $q : G \to \mathbb{R}^n$ such that, $\mathcal{L}(e) \neq \mathcal{L}(e_o)$

$$\mathcal{L}(c(e - e_o) + q(e_o)) < \max\{\mathcal{L}(e), \mathcal{L}(e_o)\}.$$

A function $\mathcal{L} : G \to \mathbb{R}$ is said to be (strictly, semi-strictly) quasi-X-concave if $-\mathcal{L}$ is (strictly, semi-strictly) quasi-X-convex.

3 X-log Convexity

Motivated by the research work [8, 10, 11], we offer the following definitions.

Definition 3.1. Let G be a non-empty X-convex subset of $\mathbb{R}^n$. A function $\mathcal{L} : G \to (0, \infty)$ is said to be

(a) X-log **convex** function, if $\forall\ e, e_o \in G, c_o \in [0, 1]$, and $\exists$ a vector valued map $f : G \to \mathbb{R}^n$, such that

$$\log(\mathcal{L}(c_o(e - e_o) + f(e_o))) \leq (1 - c_o)\log\mathcal{L}(e_o) + c_o\log\mathcal{L}(e),$$

(b) X-log **strictly convex** if $\forall\ e, e_o \in G, e \neq e_o, 0 < c_o < 1$, $\exists$ a vector valued map $f : G \to \mathbb{R}^n$, such that

$$\log(\mathcal{L}(c_o(e - e_o) + f(e_o))) < (1 - c_o)\log\mathcal{L}(e_o) + c_o\log\mathcal{L}(e).$$

Example 3.2. Let $\mathcal{P} = (-\infty, -11] \cup [-2, -1]$ be an X-convex set w.r.t. the map $f : \mathcal{P} \to \mathbb{R}$ be defined as: $f(e_o) = e_o - 13$. Also, let the function $\mathcal{L} : \mathcal{P} \to (0, \infty)$ be defined as:

$$\mathcal{L}(e) = \exp(e - c),\ c > 0.$$

So, $\mathcal{L}$ is not log-convex function because their domain is disconnected. But $\mathcal{L}$ is an X-log convex function with respect to map f.

Remark 3.3. By the above definitions it is obvious that every X-log strictly convex function is X-log convex. But the converse is may not true in general. See the following example.

Example 3.4. Let $\mathcal{P} = \mathbb{R}$ be an X-convex set w.r.t. the map $f : \mathbb{R} \to \mathbb{R}$ be defined as: $f(e_o) = e_o - |[e_o]|$, where $[,]$ is the greatest integer and let $\mathcal{L} : \mathbb{R} \to (0, \infty)$ be defined as:

$$\mathcal{L}(e) = \exp(e - c),\ c \leq -1.$$

So, $\mathcal{L}$ is X-log convex but not X-log strictly convex function w.r.t. the map f.

Definition 3.5. Let G be a non-empty X-convex subset of $\mathbb{R}^n$. A function $\mathcal{L} : G \to (0, \infty)$ is said to be

(a) X-log **quasi-convex** if $\forall\, e, e_o \in G$ and $0 \leq c_o \leq 1$,

$$\log(\mathcal{L}(c_o(e - e_o) + f(e_o))) \leq \max\{\log \mathcal{L}(e_o), \log \mathcal{L}(e)\}$$

(b) X-log **strictly quasi-convex** if $\forall\, e, e_o \in G$, $e \neq e_o$ and $0 < c_o < 1$,

$$\log(\mathcal{L}(c_o(e - e_o) + f(e_o))) < \max\{\log \mathcal{L}(e_o), \log \mathcal{L}(e)\},$$

(c) X-log **semi-strictly quasi-convex** if $\forall\, e, e_o \in G$ and $0 < c_o < 1$, $\log \mathcal{L}(e_o) \neq \log \mathcal{L}(e)$

$$\log(\mathcal{L}(c_o(e - e_o) + f(e_o)) < \max\{\log \mathcal{L}(e_o), \log \mathcal{L}(e)\}.$$

Example 3.6. Let $\mathcal{P} = [1, 10] \cup [20, \infty)$ be an X-convex set w.r.t. the map $f : \mathcal{P} \to \mathbb{R}$ be defined as: $f(e_o) = e_o + 29$ and let $\mathcal{L} : \mathcal{P} \to \mathbb{R}$ be defined as:

$$\mathcal{L}(e_o) = \begin{cases} e^2, & \text{if } e_o = 1 \\ e, & \text{otherwise} \end{cases}.$$

So, $\mathcal{L}$ is X-log quasi-convex and X-log semi-strictly quasi-convex with respect to f.

Example 3.7. Let $\mathcal{P} = [1, 11] \cup [23, \infty)$ be an X-convex set w.r.t. the map $f : \mathcal{P} \to \mathbb{R}$ be defined as: $f(e_o) = e_o + 29$ and let $\mathcal{L} : \mathcal{P} \to (0, \infty)$ be defined as:

$$\mathcal{L}(e_o) = \begin{cases} e^e, & \text{if } e_o = 1 \\ e, & \text{otherwise} \end{cases}.$$

So, $\mathcal{L}$ is X-log quasi-convex and X-log semi-strictly quasi-convex with respect to f, but it is not X-log convex w.r.t. the map f.

Remark 3.8. An X-log convex function is X-log semi-strictly quasi-convex function, but the converse is may not true in general, (See the Example (3.7)).

Remark 3.9. By the above definitions it is clear that every X-log strictly quasi-convex function is an X-log quasi-convex. But the converse is may not true in general, (See the Example (3.6)).

4 Main Results

Theorem 4.1. *Let G be a non-empty X-convex subset of $\mathbb{R}^n$. If the function $\mathcal{L} : G \to (0, \infty)$ is an X-log convex, then the level set $Z_\gamma = \{e \in G : \log \mathcal{L}(e) \leq \gamma,\ \gamma \in \mathbb{R}\}$ is also an X-convex set.*

Proof. Consider $e, e_o \in Z_\gamma$. Then $\log \mathcal{L}(e) \leq \gamma$ and $\log \mathcal{L}(e_o) \leq \gamma$. Since G is an X-convex set, then $\forall\, e, e_o \in G$, $c_o \in [0, 1]$ and there exits a vector-valued map $f : G \to \mathbb{R}^n$ s.t. $c_o(e - e_o) + f(e_o) \in G$. So by X-log convexity of $\mathcal{L}$, we obtain

$$\begin{aligned} \log(\mathcal{L}(c_o(e - e_o) + f(e_o))) &\leq (1 - c_o) \log \mathcal{L}(e_o) + c_o \log \mathcal{L}(e) \\ &\leq (1 - c_o)\gamma + c_o\gamma \\ &\leq \gamma, \end{aligned}$$

from which is follows that $c_o(e - e_o) + f(e_o) \in G$. Hence Z_γ is an X-convex set. □

Theorem 4.2. *If the function $\mathcal{L} : G \to (0, \infty)$ is an X-log convex, defined on a non-empty X-convex subset G of $\mathbb{R}^n$, then*

$$epi(\mathcal{L}) = \{(e, \gamma) : e \in G, \log \mathcal{L}(e) \leq \gamma,\ \gamma \in \mathbb{R}\}$$

is an X-convex set.

Proof. Let us consider $\mathcal{L}$ is X-log convex. Suppose that $(e,\gamma),(e_o,\delta) \in epi(\mathcal{L})$. So, we find $\log \mathcal{L}(e) \leq \gamma$ and $\log \mathcal{L}(e_o) \leq \delta$. Thus, $\forall\, c_o \in [0,1],\ e, e_o \in G$, and $\exists$ a vector-valued function $f : G \to \mathbb{R}^n$ such that

$$\begin{aligned} \log(\mathcal{L}(c_o(e-e_o)+f(e_o))) &\leq (1-c_o)\log \mathcal{L}(e_o) + c_o \log \mathcal{L}(e) \\ &\leq (1-c_o)\gamma + c_o\delta \end{aligned}$$

which implies that

$$(c_o(e-e_o)+f(e_o),\ (1-c_o)\gamma + c_o\delta) \in epi(\mathcal{L}).$$

This proves $epi(\mathcal{L})$ is an X-convex set. □

Theorem 4.3. *Let G be a non-empty X-convex subset of $\mathbb{R}^n$. The function $\mathcal{L} : G \to (0,\infty)$ is an X*-log *quasi-convex, if and only if the level set*

$$Z_\gamma = \{e \in G : \log \mathcal{L}(e) \leq \gamma,\ \gamma \in \mathbb{R}\}$$

is an X-convex set.

Proof. For $e, e_o \in Z_\gamma$, and $\max\{\log \mathcal{L}(e), \log \mathcal{L}(e_o)\} = \gamma$. For a vector-valued map $f : G \to \mathbb{R}^n$, and for any $c_o \in [0,1]$, $c_o(e-e_o)+f(e_o) \in G$, then we have to prove that $c_o(e-e_o)+f(e_o) \in Z_\gamma$. By X-log quasi-convexity of $\mathcal{L}$, we obtain

$$\begin{aligned} \log(\mathcal{L}(c_o(e-e_o)+f(e_o))) &\leq \max\{\log \mathcal{L}(e_o), \log \mathcal{L}(e)\} \\ &= \gamma \end{aligned}$$

which implies that $c_o(e-e_o)+f(e_o) \in Z_\gamma$, therefore the level set Z_γ is an X-convex set.
Conversely, suppose that Z_γ is an X-convex set. So, for any $e, e_o \in Z_\gamma$, $c_o \in [0,1]$, and there exists a vector-valued map $f : G \to \mathbb{R}^n$ such that $c_o(e-e_o)+f(e_o) \in Z_\gamma$. Let $e, e_o \in Z_\gamma$ for $\gamma = \max\{\log \mathcal{L}(e), \log \mathcal{L}(e_o)\}$.
Then from the definition of the set Z_γ, it gives

$$\log(\mathcal{L}(c_o(e-e_o)+f(e_o))) \leq \max\{\log \mathcal{L}(e_o), \log \mathcal{L}(e)\} = \gamma.$$

Hence $\mathcal{L}$ is an X-log quasi convex function. □

Theorem 4.4. *Let G be a non-empty X-convex subset of $\mathbb{R}^n$. Let us consider $\mathcal{L}$, an X*-log *convex function. Let $\eta = \inf_{e\in G} \log \mathcal{L}(e)$. Then the set $Q = \{e \in G : \log \mathcal{L}(e) = \eta\}$ is an X-convex subset G of $\mathbb{R}^n$. If $\mathcal{L}$ is X*-log *strictly convex, then Q is singleton.*

Proof. Since $\mathcal{L}$ is an X-log convex function, then for any $e, e_o \in G, c_o \in [0,1]$ and $\exists$ a vector-valued map $f : G \to \mathbb{R}^n$, s.t.

$$\begin{aligned} \log(\mathcal{L}(c_o(e-e_o)+f(e_o))) &\leq (1-c_o)\log \mathcal{L}(e_o) + c_o \log \mathcal{L}(e) \\ &= (1-c_o)\eta + c_o\eta \\ &= \eta. \end{aligned}$$

Which implies that $c_o(e-e_o)+f(e_o) \in Q$. And hence Q is an X-convex set.
Now, for the other part, suppose on contrary that $\log \mathcal{L}(e) = \log \mathcal{L}(e_o) = \eta$. As G is an X-convex set, then for $0 < c_o < 1, c_o(e-e_o)+f(e_o) \in G$. Also, since $\mathcal{L}$ is an X-log strictly convex,

$$\begin{aligned} \log(\mathcal{L}(c_o(e-e_o)+f(e_o))) &< (1-c_o)\log \mathcal{L}(e_o) + c_o \log \mathcal{L}(e) \\ &= (1-c_o)\eta + c_o\eta \\ &= \eta. \end{aligned}$$

This contradicts the fact $\eta = \inf_{e\in G} \log \mathcal{L}(e)$ and hence the result is proved. □

Theorem 4.5. *Let G be a non-empty X-convex subset of $\mathbb{R}^n$. If $\mathcal{L}$ is an X*-log *convex function such that* $\log \mathcal{L}(e) \leq \log \mathcal{L}(e_o)$, $\forall\, e, e_o \in G$, *then $\mathcal{L}$ is an X*-log *quasi convex function.*

Proof. Since $\mathcal{L}$ is an X-log convex function then, for any $e, e_o \in G, c_o \in [0,1]$, and $\exists$ a vector-valued map $f : G \to \mathbb{R}^n$, we have

$$\begin{aligned}
\log(\mathcal{L}(c_o(e-e_o)+f(e_o))) &\leq (1-c_o)\log\mathcal{L}(e_o)+c_o\log\mathcal{L}(e)\\
&\leq (1-c_o)\log\mathcal{L}(e_o)+c_o\log\mathcal{L}(e_o)\\
&= \log\mathcal{L}(e_o)\\
&= \max\{\log\mathcal{L}(e_o),\log\mathcal{L}(e)\},
\end{aligned}$$

therefore, the function $\mathcal{L}$ is X-log quasi convex. □

Definition 4.6. Let G be a non-empty X-convex subset of $\mathbb{R}^n$. A function $\mathcal{L} : G \to (0,\infty)$ is said to be an X-log pseudo convex, if $\exists$ a strictly positive bi-function $b : G \times G \to \mathbb{R}^+$ s.t.

$$\log\mathcal{L}(e) < \log\mathcal{L}(e_o)$$

$$\Longrightarrow$$

$$\log(\mathcal{L}(c_o(e-e_o)+f(e_o))) \leq \log\mathcal{L}(e_o)+c_o(c_o-1)b(e,e_o),\ \forall\, e,e_o \in G, c_o \in [0,1].$$

Theorem 4.7. *Let G be a non-empty X-convex subset of $\mathbb{R}^n$. A function $\mathcal{L} : G \to (0,\infty)$ is an X-log convex function such that $\log\mathcal{L}(e) < \log\mathcal{L}(e_o)$, then the function $\mathcal{L}$ is an X-log pseudo convex.*

Proof. Since $\log\mathcal{L}(e) < \log\mathcal{L}(e_o)$, and $\mathcal{L}$ is an X-log convex function, then $\forall\, e, e_o \in G, c_o \in [0,1]$ and $\exists$ a vector-valued function $f : G \to \mathbb{R}^n$, we obtain

$$\begin{aligned}
\log(\mathcal{L}(c_o(e-e_o)+f(e_o))) &\leq (1-c_o)\log\mathcal{L}(e_o)+c_o\log\mathcal{L}(e)\\
&= \log\mathcal{L}(e_o)+c_o\{\log\mathcal{L}(e)-\log\mathcal{L}(e_o)\}\\
&< \log\mathcal{L}(e_o)+c_o\{\log\mathcal{L}(e)-\log\mathcal{L}(e_o)\}\\
&\quad -c_o^2\{\log\mathcal{L}(e)-\log\mathcal{L}(e_o)\}\\
&= \log\mathcal{L}(e_o)+c_o(c_o-1)\{\log\mathcal{L}(e_o)-\log\mathcal{L}(e)\}\\
&= \log\mathcal{L}(e_o)+c_o(c_o-1)b(e,e_o),
\end{aligned}$$

where $b(e_o,e) = \log\mathcal{L}(e)-\log\mathcal{L}(e_o) > 0$. □

Theorem 4.8. *Let $\mathcal{L} : G \to (0,\infty)$ be an X-log pseudo convex function on G and let $K : I \to \mathbb{R}$ be strictly increasing convex function such that range$(\log\mathcal{L}) \subseteq I$. Then the composite function $K(\log\mathcal{L})$ is an X-log pseudo convex function on G.*

Proof. Since $\mathcal{L}$ is an X-log pseudo convex function on G, we have

$$\log\mathcal{L}(e) < \log\mathcal{L}(e_o)$$

$$\Longrightarrow$$

$$\log(\mathcal{L}(c_o(e-e_o)+f(e_o))) \leq \log\mathcal{L}(e_o)+c_o(c_o-1)b(e,e_o),\ \forall\, e,e_o \in G, c_o \in [0,1],$$

where $b(e,e_o)$ is strictly positive function.

Since K is strictly increasing convex function, so

$$K(\log\mathcal{L}(e)) < K(\log\mathcal{L}(e_o))$$

$$\Longrightarrow$$

$$\begin{aligned}
K(\log(\mathcal{L}(c_o(e-e_o)+f(e_o))) &\leq K((1-c_o)\log\mathcal{L}(e_o)+c_o(c_o-1)b(e,e_o))\\
&< (1-c_o)K(\log\mathcal{L}(e_o)+c_o(c_o-1)K(b(e,e_o))
\end{aligned}$$

for every $c_o \in [0,1]$, $e, e_o \in G$ and $K(b(e,e_o))$ is strictly positive function. Which shows that $K(\log(\mathcal{L}))$ is X-log pseudo convex function on G. □

Definition 4.9. Let G be a non-empty X-convex subset of $\mathbb{R}^n$. A function $\mathcal{L} : G \to (0, \infty)$ is said to be X**-**log **affine convex** function, if $\forall\, e, e_o \in G, c_o \in [0, 1]$, and $\exists$ a vector valued function $f : G \to \mathbb{R}^n$, such that

$$\log(\mathcal{L}(c_o(e - e_o) + f(e_o)) = (1 - c_o) \log \mathcal{L}(e_o) + c_o \log \mathcal{L}(e).$$

Theorem 4.10. *Let G be a non-empty X-convex subset of $\mathbb{R}^n$. Let $\mathcal{L}_1 : G \to (0, \infty)$ be an X-log affine convex function. Then $\mathcal{L}_2 : G \to (0, \infty)$ is an X-log convex function, if and only if, $\log \mathcal{L} = \log \mathcal{L}_2 - \log \mathcal{L}_1$ is an X-log convex function.*

Proof. Let $\mathcal{L}_1$ be X-log affine convex function. Then $\forall\, e, e_o \in G, c_o \in [0, 1]$ and $\exists$ a vector-valued function $f : G \to \mathbb{R}^n$, s.t.

$$\log(\mathcal{L}_1(c_o(e - e_o) + f(e_o))) = (1 - c_o) \log \mathcal{L}_1(e_o) + c_o \log \mathcal{L}_1(e). \tag{4.1}$$

Since $\mathcal{L}_2$ is X-log convex function. Then $\forall\, e, e_o \in G, c_o \in [0, 1]$ and $\exists$ a vector-valued function $f : G \to \mathbb{R}^n$, such that

$$\log(\mathcal{L}_2(c_o(e - e_o) + f(e_o))) \leq (1 - c_o) \log \mathcal{L}_2(e_o) + c_o \log \mathcal{L}_2(e). \tag{4.2}$$

From (4.1) and (4.2), we obtain

$$\begin{aligned} \log \mathcal{L}_2(w) - \log \mathcal{L}_1(w) \leq (1 - c_o)&[\log \mathcal{L}_2(e_o) - \log \mathcal{L}_1(e_o)] \\ &+ c_o[\log \mathcal{L}_2(e) - \log \mathcal{L}_1(e)], \end{aligned}$$

where $w = c_o(e - e_o) + f(e_o)$.

$$\log(\mathcal{L}(c_o(e - e_o) + f(e_o))) \leq (1 - c_o) \log \mathcal{L}(e_o) + c_o \log \mathcal{L}(e)$$

which shows that $\log \mathcal{L} = \log \mathcal{L}_2 - \log \mathcal{L}_1$ is an X-log convex function.

The converse part is obvious. □

Theorem 4.11. *Let G be a non-empty X-convex subset of $\mathbb{R}^n$. If $\mathcal{L}_1,\ \mathcal{L}_2 : G \to (0, \infty)$ are two X-log convex functions with respect to the same map f, then the sum $\mathcal{L}_1 + \mathcal{L}_2$ is an X-log convex function on G.*

Proof. Since $\mathcal{L}_1,\ \mathcal{L}_2$ are X-log convex functions, then for any $e, e_o \in G, c_o \in [0, 1]$, $\exists$ a vector-valued function $f : G \to \mathbb{R}^n$, we obtain

$$\log(\mathcal{L}_1(c_o(e - e_o) + f(e_o))) \leq (1 - c_o) \log \mathcal{L}_1(e_o) + c_o \log \mathcal{L}_1(e) \tag{4.3}$$

and

$$\log(\mathcal{L}_2(c_o(e - e_o) + f(e_o))) \leq (1 - c_o) \log \mathcal{L}_2(e_o) + c_o \log \mathcal{L}_2(e) \tag{4.4}$$

multiplying inequalities (4.3) and (4.4), we have

$$\begin{aligned} \log((\mathcal{L}_1 + \mathcal{L}_2)(w) &\leq (1 - c_o)^2 \log(\mathcal{L}_1 + \mathcal{L}_2)(e_o) + c_o^2 \log(\mathcal{L}_1 + \mathcal{L}_2)(e) \\ &\quad + c_o(1 - c_o)[\log(\mathcal{L}_1(e_o) + \mathcal{L}_2(e)) + \log(\mathcal{L}_1(e) + \mathcal{L}_2(e_o))] \\ &\leq (1 - c_o) \log(\mathcal{L}_1 + \mathcal{L}_2)(e_o) + c_o \log\,(L_1 + \mathcal{L}_2)(e) + \mathcal{C}. \end{aligned}$$

Where $w = c_o(e - e_o) + f(e_o)$ and $c_o(1 - c_o)[\log(\mathcal{L}_1(e_o) + \mathcal{L}_2(e)) + \log(\mathcal{L}_1(e) + \mathcal{L}_2(e_o))] = \mathcal{C} > 0$. This implies that $\mathcal{L}_1 + \mathcal{L}_2$ is also an X-log convex function. □

Definition 4.12. Let $\mathcal{L} : G \subseteq \mathbb{R}^n \to (0, \infty)$ be a function. A point $e \in G$ is said to be

(a) **local** log**-minimum** if

$$\log \mathcal{L}(e) \leq \log \mathcal{L}(e_o), \quad \forall\, e_o \in G \cap B_\delta(e) \tag{4.5}$$

(b) **strict local** log**-minimum** if

$$\log \mathcal{L}(e) < \log \mathcal{L}(e_o), \quad \forall\, e_o \in G \cap B_\delta(e) \tag{4.6}$$

(c) **global** log**-minimum** if

$$\log \mathcal{L}(e) \leq \log \mathcal{L}(e_o), \quad \forall\, e_o \in G \tag{4.7}$$

(d) **strict global** log**-minimum** if

$$\log \mathcal{L}(e) < \log \mathcal{L}(e_o), \quad \forall\, e_o \in G. \tag{4.8}$$

Theorem 4.13. *Let G be an X-convex subset of $\mathbb{R}^n$ with condition $\|c_o(e_0 - e) + f(e) - e\| < \delta$ for each $e_0, e \in G$, $c_o \in [0,1]$, where $f : G \to \mathbb{R}^n$ is a vector valued map associated with X-convexity and $\delta > 0$. Every local* log*-minimum of an X-*log *(strictly) convex function $\mathcal{L} : G \to (0,\infty)$ defined on a nonempty X-convex subset G of R^n is a (unique) global* log*-minimum of $\mathcal{L}$ over G. Moreover, the set of points at which an X-convex function attains its global* log*-minimum on G is an X-convex set.*

Proof. Let $e \in G$ be local log-minimum of $\mathcal{L}$ over G. Then there exists $\delta > 0$ such that

$$\log \mathcal{L}(e) \leq \log \mathcal{L}(e_o), \;\; \forall\, e_o \in G \cap B_\delta(e) \tag{4.9}$$

Suppose, on contrary, e is not a global log-minimum over G, then there exists $e \in G, e_0 \neq e$ such that $\log \mathcal{L}(e_0) < \log \mathcal{L}(e)$.

By X-log convexity of $\mathcal{L}$, we have for $c_o \in [0,1]$,

$$\begin{aligned} \log(\mathcal{L}(c_o(e_0 - e) + f(e)) &\leq c_o \log \mathcal{L}(e_0) + (1 - c_o) \log \mathcal{L}(e) \\ &< c_o \log \mathcal{L}(e) + (1 - c_o) \log \mathcal{L}(e), \\ &< \log \mathcal{L}(e). \end{aligned}$$

This contradicts inequality (4.9), since $\|c_o(e_0 - e) + f(e) - e\| < \delta$ for each $e_0, e \in G$, $c_o \in [0,1]$ then $c_o(e_0 - e) + f(e) \in G \cap B_\delta(e)$ for c_o.

If $\mathcal{L}$ is an X-log strictly convex function, we need to show that e is a unique log-minimum. On the contrary, suppose there exists $e_0 \neq e$ such that e_0 is also a global log-minimum of $\mathcal{L}$ over G. Then, it is clear that $\log \mathcal{L}(e) = \log \mathcal{L}(e_0)$. By X-log strictly convexity of $\mathcal{L}$, we have for $c_o \in (0,1)$

$$\begin{aligned} \log(\mathcal{L}(c_o(e_0 - e)) + (e))) &< c_o \log \mathcal{L}(e_0) + (1 - c_o) \log \mathcal{L}(e) \\ &< \log \mathcal{L}(e). \end{aligned}$$

This contradicts the fact that e is global log-minimum of $\mathcal{L}$ over G.
Let $B = \{e \in G : \log \mathcal{L}(e) \leq \log \mathcal{L}(e_o)\ \forall\, e_o \in G\}$ be the set of points at which $\mathcal{L}$ attains its global log-minimum. If $e_1, e_2 \in B$, then

$$\log \mathcal{L}(e_1) \leq \log \mathcal{L}(e_o) \;\text{ and }\; \log \mathcal{L}(e_2) \leq \log \mathcal{L}(e_o)\;\; \forall\, e_o \in G.$$

By X-log convexity of $\mathcal{L}$, $\forall\, c_o \in [0,1]$ and for every $e_1, e_2 \in G$, we have

$$\begin{aligned} \log(\mathcal{L}(c_o(e_2 - e_1) + f(e_1))) &\leq c_o \log \mathcal{L}(e_2) + (1 - c_o) \log \mathcal{L}(e_1) \\ &\leq c_o \log \mathcal{L}(e_o) + (1 - c_o) \log \mathcal{L}(e_o) \\ &\leq \log \mathcal{L}(e_o). \end{aligned}$$

This implies that $c_o(e_2 - e_1) + f(e_1) \in B$. □

Theorem 4.14. *Let G be an X-convex subset of $\mathbb{R}^n$ with condition $\|c_o(e_0 - e) + f(e) - e\| < \delta$ for each $e_0, e \in G$, $c_o \in [0,1]$, where $f : G \to \mathbb{R}^n$ is a vector valued map associated with X-convexity and $\delta > 0$. Every strict local* log*-minimum of an X-*log *quasi convex function $\mathcal{L} : G \to (0,\infty)$ defined on a nonempty X-convex subset G of R^n is a strict global* log*-minimum of $\mathcal{L}$ over G. Moreover, the set of points at which an X-convex function attains its global* log*-minimum on G is an X-convex set.*

Proof. Let $e \in G$ be strict local log-minimum of $\mathcal{L}$ over G. Then there exists $\delta > 0$ such that

$$\log \mathcal{L}(e) < \log \mathcal{L}(e_o), \;\; \forall\, e_o \in G \cap B_\delta(e) \tag{4.10}$$

Suppose on the contrary, e is not a strict global log-minimum over G. Then $\exists, e_0 \in G, e \neq e_0$ such that,

$$\log \mathcal{L}(e_0) \leq \log \mathcal{L}(e).$$

By X-log quasi convex function of $\mathcal{L}, \forall c_o \in [0,1]$, $e_0, e \in G$, we obtain

$$\log(\mathcal{L}(c_o(e_0 - e) + f(e))) \leq \log \mathcal{L}(e).$$

This contradicts inequality (4.10), since $\|c_o(e_0-e)+f(e)-e\| < \delta$ for each $e_0, e \in G$, $c_o \in [0,1]$ then $c_o(e_0 - e) + f(e) \in G \cap B_\delta(e)$ for c_o.

Let $B = \{e \in G : \log \mathcal{L}(e) \leq \log \mathcal{L}(e_o) \ \forall\ e_o \in G\}$ be the set of points at which $\mathcal{L}$ attains its global log-minimum. If $e_1, e_2 \in B$, then

$$\log \mathcal{L}(e_1) \leq \log \mathcal{L}(e_o) \ \text{ and } \ \log \mathcal{L}(e_2) \leq \log \mathcal{L}(e_o) \ \ \forall\ e_o \in G.$$

By X-log convexity of $\mathcal{L}$, $\forall\ c_o \in [0,1]$ and for every $e_1, e_2 \in G$, we have

$$\log(\mathcal{L}(c_o(e_2 - e_1) + f(e_1))) \leq \max\{\log \mathcal{L}(e_2), \log \mathcal{L}(e_1)\},$$

$$\log(\mathcal{L}(c_o(e_2 - e_1) + f(e_1))) \leq \log \mathcal{L}(e_o).$$

This implies that $c_o(e_2 - e_1) + f(e_1) \in B$. □

Theorem 4.15. *An* X-log *strictly quasi-convex function* $\mathcal{L} : G \to (0,\infty)$ *defined on a nonempty* X*-convex subset* G *of* R^n *attains its global* log*-minimum on* G *is not more than one point.*

Proof. Assume, on contrary, that $\mathcal{L}$ attains global log-minimum at two distinct points e_1 and $e_2 \in G$. Then

$$\log \mathcal{L}(e_1) \leq \log \mathcal{L}(e_o) \text{ and } \log \mathcal{L}(e_1) \leq \log \mathcal{L}(e_o), \ \forall\ e_o \in G. \tag{4.11}$$

Taking $e_o = e_2$ in the first inequality and $e_o = e_1$ in the second inequality, we get $\log \mathcal{L}(e_1) = \log \mathcal{L}(e_2)$. By strict X-log quasi convexity of $\mathcal{L}$, we have for all $c_o \in (0,1)$, there exists a vector valued function f such that

$$\log(\mathcal{L}(c_o(e_1 - e_2) + f(e_2))) < \log \mathcal{L}(e_1),$$

which contradicts (4.11). □

Theorem 4.16. *Let* G *be an* X*-convex subset of* $\mathbb{R}^n$ *with condition* $\|c_o(e_0 - e) + f(e) - e\| < \delta$ *for each* $e_0, e \in G$, $c_o \in [0,1]$*, where* $f : G \to \mathbb{R}^n$ *is a vector valued map associated with* X*-convexity and* $\delta > 0$*. Suppose that the function* $\mathcal{L} : G \to (0,\infty)$ *is* X*-convex. If* $e \in G$ *is a local* log*-optimal solution to the problem*

$$\begin{aligned} &\textit{minimize } \mathcal{L}(e) \\ &\textit{subject to } e \in G, \end{aligned} \tag{4.12}$$

then, e *is a global* log*-minimum in the problem* (4.12).

Proof. Proof is same as the Theorem (4.13). □

5 Conclusion remarks

In this paper, we have introduced X-log convex, X-log quasi convex, X-log semi-strictly quasi convex, and X-log-pseudo convex functions. Several interesting examples and arguments are established in favour of the concept. Also, we have defined local and global log-minimum. We have developed several interesting results of X-log convexity, X-log quasi convexity, X-log semi-strictly quasi convexity and X-log semi-strictly quasi convexity.

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Received: 2023-02-23
Accepted: 2024-01-10