Palestine Journal of Mathematics

Vol 13(2)(2024) , 181–188

Existence of positive solutions for a $(p, 2)$-Laplacian Steklov problem

A. BOUKHSAS and B. OUHAMOU

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 35J66, Secondary 35D30 .

Keywords and phrases: $(p, 2)$-Laplacian, Steklov problem, nonlinear boundary conditions.

The authors thank the referee and editor for their helpful comments and suggestions that improved the quality of our paper.

Abstract In this paper, we study positive solutions of a Steklov problem driven by the $(p, 2)$-Laplacian operator by using variational method. A sufficient condition of the existence of positive solutions is characterized by the eigenvalues of linear and another nonlinear eigenvalue problems.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Steklov problem:

$$(S_{p,2}) \begin{cases} -\Delta_p u - \Delta u + |u|^{p-2}u + u &= 0 \qquad \text{in } \Omega, \\ \langle |\nabla u|^{p-2}\nabla u + \nabla u, \nu \rangle &= f(x, u) \quad \text{on } \partial\Omega. \end{cases}$$

Here for any $p > 2$ by Δ_p we denote the p-Laplacian differential operator defined by

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) \qquad \text{for all} \qquad u \in W^{1,p}(\Omega).$$

When $p = 2$, we write $\Delta_2 = \Delta$ (the standard Laplace differential operator). ν is the outward unit normal vector on $\partial\Omega$, $\langle ., . \rangle$ is the scalar product of $\mathbb{R}^N$, while the reaction term $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

In problem $(S_{p,2})$, the differential operator $u \mapsto -\Delta_p u - \Delta u$ is non-homogeneous. We mention that equations involving the sum of a p-Laplacian and a Laplacian (also known as $(p, 2)$-equations) arise in mathematical physics, see, for example, the works of Benci et al. [3](quantum physics), Cherfils and Il'yasov [10](plasma physics) and Zhikov [20](homogenization of composites consisting of two different materials with distinct hardening exponents, double phase problems).

In [14], the authors studied the problem $(S_{p,2})$ with the Dirichlet boundary condition, they impose certain conditions on the reaction term $f(x, u)$ to make equation resonant at $\pm\infty$ and zero. Using variational methods and critical groups, they obtain existence and multiplicity results. In [12], the authors consider the case with a reaction term $f(x, u)$ which is superlinear in the positive direction (without satisfying the Ambrosetti-Rabinowitz condition) and sublinear resonant in the negative direction. They apply Morse's theory and variational methods to establish the existence of at least three non-trivial smooth solutions.

A more general problem with a (p, q)-Laplacian equation under a Steklov boundary condition $(1 < q < p < \infty)$, was studied in [5, 6, 7, 8, 9, 17, 18]. Elliptic equations involving differential operators of the form

$$Au := \text{div}(D(u)\nabla u) = \Delta_p u + \Delta_q u,$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$, usually called (p, q)-Laplacian, occurs in many important concrete situations. For instance, this happens when one seeks stationary solutions to the reaction-diffusion system.

$$u_t = Au + c(x,u). \tag{1.1}$$

This system has a wide range of applications in physics and related sciences like chemical reaction design [2], biophysics [11], and plasma physics [16]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.1) corresponds to the diffusion with a diffusion coefficient $D(u)$, whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to the concentration. For some related study see [4, 15].

The energy functional $\varphi \in C^1(W^{1,p}(\Omega),\mathbb{R})$ stemming from the problem $(S_{p,2})$ is defined by

$$\varphi(u) := \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)dx + \frac{1}{2}\int_\Omega (|\nabla u|^2 + |u|^2)dx - \int_{\partial\Omega} F(x,u)d\sigma, \ \ u \in W^{1,p}(\Omega),$$

where $F(x,t) = \int_0^t f(x,s)ds$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$.

We say that $u \in W^{1,p}(\Omega)$ is a weak solution of $(S_{p,2})$ if

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx + \int_\Omega (\nabla u \nabla v + uv)dx - \int_{\partial\Omega} f(x,u)d\sigma = 0,$$

for all $v \in W^{1,p}(\Omega)$. Note that the critical points of the functional φ correspond exactly to the weak solutions of $(S_{p,2})$.

Throughout this paper, we denote by $W^{1,p}(\Omega)$ the usual Sobolev space with the norm $\|u\|_{1,p} := \left(\int_\Omega (|\nabla u|^p + |u|^p)dx\right)^{1/p}$, and by $W^{1,p}(\Omega)^*$ its dual space, and the duality pairing between $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega)^*$ is written as $\langle .,.\rangle$. It is well known that the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is compact for each $r \in [1,p^*)$, where $p^* = \infty$ for $N \le p$ and $p^* = (N-1)p/N-p$ for $N > p$. Hence, for every $r \in [1,p^*)$, there exists $S_r > 0$ such that

$$\|u\|_{L^r(\partial\Omega)} \le S_r \|u\|_{1,p}. \tag{1.2}$$

For each $q \in (p,p^*)$, a vital constant is defined as follows:

$$C_q = \frac{1}{p(q-1)}\left[\frac{q-p}{S_1}\right]^{(q-p)/(q-1)}\left[\frac{q(p-1)}{S_q^q}\right]^{(p-1)/(q-1)}.$$

The asymptotic behaviors of f near zero and infinity lead us to define

$$\begin{aligned} \mu_1 &:= \inf\left\{\int_\Omega (|\nabla u|^2 + |u|^2)dx : u \in H^1(\Omega), \int_{\partial\Omega} |u|^2 d\sigma = 1\right\}, \\ \lambda_1 &:= \inf\left\{\int_\Omega (|\nabla u|^p + |u|^p)dx : u \in W^{1,p}(\Omega), \int_{\partial\Omega} |u|^p d\sigma = 1\right\}. \end{aligned} \tag{1.3}$$

Now, we give our hypothesis on the reaction term $f(x,u)$:

$H(f)_1$ $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x,t) \ge 0$ for any $x \in \partial\Omega$, $t > 0$.

$H(f)_2$ There exist $q \in (p,p^*)$ and $C \in (0,C_q)$ such that for all $x \in \partial\Omega, t \in \mathbb{R}$,

$$|f(x,t)| \le C(1 + |t|^{q-1}).$$

$H(f)_3$ There exist $f_0 > \mu_1$, $f_\infty > \lambda_1$, such that the limits

$$\lim_{t\to 0^+} \frac{f(x,t)}{t} = f_0, \qquad \lim_{t\to\infty} \frac{f(x,t)}{t^{p-1}} = f_\infty, \tag{1.4}$$

exist uniformly for $x \in \partial\Omega$.

Remark 1.1. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we assume that

$$f(x,t) = 0 \ \text{ for a.e. } \ x \in \partial\Omega, \text{for all } \ t \le 0.$$

Our main result is the following theorem.

Theorem 1.2. *Suppose that f satisfies $H(f)_1$-$H(f)_3$. Then, $(S_p, 2)$ yields at least two positive solutions.*

In [7], the authors show that if f satisfies $H(f)_1$ and $H(f)_3$ with $f_0 < \mu_1$, $f_\infty > \lambda_1$, then the problem $(S_{p,2})$ has a positive solution. In this article, Theorem 1.2 is a supplement of the above result. In the process of this work, we require the introduction of the concept of the Fučik spectrum Σ_p of the p-Laplacian operator with the Steklov boundary condition. Specifically, $\Sigma_p = \Sigma_p(m, n)$ is a set that consists of those $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\begin{cases} \Delta_p u &= |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} & \text{on } \partial\Omega, \end{cases}$$

has a nontrivial solution, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, it is shown in [1] that in particular if m and n both change sign in $\partial\Omega$, then each of the four quadrants in the (α, β) plane contains a first (nontrivial) curve of Σ_p.

Remark 1.3. For each $f_0 > \mu_1$, $f_\infty > \lambda_1$ and $q \in (p, p^*)$, we consider the following functions:

$$f(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ tf_0, & t \in (0, \delta], \\ C_1 + C_2 t^{q-1}, & t \in (\delta, R], \\ f_\infty t^{p-1}, & t \in (R, \infty], \end{cases}$$

where $\delta \in (0, 1)$ and $R \in (1, \infty)$. Moreover, $C_1 = (f_0\delta R^{q-1} - f_\infty \delta^{p-1}R^{p-1})/(R^{q-1} - \delta^{q-1})$ and $C_2 = (f_\infty R^{p-1} - f_0\delta/(R^{q-1} - \delta^{q-1})$. One can select sufficiently small δ and sufficiently large R such that $f_0\delta < C_q$, $f_\infty < C_q R^{q-p}$, and $C_1, C_2 > 0$. Considering $C = \max\{f_0\delta, f_\infty R^{p-q}\}$ in the condition $H(f)_2$, we observe that these function f satisfy the hypotheses $H(f)_1$-$H(f)_3$.

2 Preliminaries

Let X be a Banach space and X^* its topological dual while $\langle ., . \rangle$ denotes the duality brackets on the pair (X, X^*).

Definition 2.1. The functional $\varphi \in C^1(X)$ fulfills the Palais-Smale condition (the PS-condition for short) if the following holds:
Every sequence $\{u_n\} \subseteq X$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \longrightarrow 0$ in X^* as $n \longrightarrow \infty$, admits a strongly convergent subsequence.

This compactness-type condition on φ leads to a deformation theorem which is the main ingredient in the minimax theory of the critical values of φ. A basic result in that theory is the so-called mountain pass theorem.

First, we demonstrate that the functional φ satisfies the Palais-Smale condition under the conditions $H(f)_1$-$H(f)_3$. Thus, we only need to prove Lemmas 2.2 and 2.3.

Lemma 2.2. *If $H(f)_1$-$H(f)_3$ hold. $\{u_n\} \subset W^{1,p}(\Omega)$ is bounded, and $\varphi'(u_n) \longrightarrow 0$, as $n \longrightarrow \infty$, then $\{u_n\}$ admits a convergent subsequence.*

Proof. Assume that $\{u_n\}$ is bounded, $\varphi'(u_n) \longrightarrow 0$ in $W^{1,p}(\Omega)^*$, as $n \longrightarrow \infty$. By extracting a subsequence, we may suppose that there exists $\{u_n\} \subset W^{1,p}(\Omega)$ such that, as $n \longrightarrow \infty$

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega), \quad u_n \longrightarrow u \text{ in } L^s(\partial\Omega), \quad s \in [1, p^*). \tag{2.1}$$

It follows from $H(f)_1$-$H(f)_3$ that there exists $C_1 > 0$, such that

$$|f(x, t)| \leq C_1(1 + |t|^{p-1}), \qquad (x, t) \in \partial\Omega \times \mathbb{R}. \tag{2.2}$$

Hence, by Hölder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned}\Big|\int_{\partial\Omega} f(x,u_n)(u_n-u)d\sigma\Big| &\leq C_1\big(\int_{\partial\Omega}|u_n||u_n-u|d\sigma+\int_{\partial\Omega}|u_n|^{p-1}|u_n-n|d\sigma\big)\\ &\leq C_1\Big(\int_{\partial\Omega}|u_n|^2d\sigma\Big)^{\frac{1}{2}}\Big(\int_{\partial\Omega}|u_n-u|^2d\sigma\Big)^{\frac{1}{2}}+C_1\Big(\int_{\partial\Omega}|u_n|^pd\sigma\Big)^{\frac{p-1}{p}}\Big(\int_{\partial\Omega}|u_n-u|^pd\sigma\Big)^{\frac{1}{p}}\\ &\leq C_2\Big(\int_{\partial\Omega}|u_n-u|^2d\sigma\Big)^{\frac{1}{2}}+C_3\Big(\int_{\partial\Omega}|u_n-u|^pd\sigma\Big)^{\frac{1}{p}}\longrightarrow 0,\ \ as\ \ n\longrightarrow\infty.\end{aligned} \tag{2.3}$$

Similarly, we have

$$\Big|\int_{\partial\Omega} f(x,u)(u_n-u)d\sigma\Big|\longrightarrow 0,\ \ as\ \ n\longrightarrow\infty. \tag{2.4}$$

Noting that

$$\begin{aligned}&<\varphi^{'}(u_n)-\varphi^{'}(u),u_n-u>=<\varphi^{'}(u_n),u_n-u>-<\varphi^{'}(u),u_n-u>\\ &=\int_\Omega\nabla u_n.\nabla(u_n-u)dx+\int_\Omega u_n.(u_n-u)dx+\int_\Omega|\nabla u_n|^{p-2}\nabla u_n.\nabla(u_n-u)dx+\int_\Omega|u_n|^{p-2}(u_n-u)dx\\ &\quad-\int_{\partial\Omega}f(x,u_n)(u_n-u)d\sigma-\int_\Omega\nabla u.\nabla(u_n-u)dx-\int_\Omega u.(u_n-u)dx-\int_\Omega|\nabla u|^{p-2}\nabla u.\nabla(u_n-u)dx\\ &\quad-\int_\Omega|u|^{p-2}(u_n-u)dx+\int_{\partial\Omega}f(x,u)(u_n-u)d\sigma\\ &=\int_\Omega\big(|\nabla(u_n-u)|^2+|(u_n-u)|^2\big)dx+\int_\Omega\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big).\nabla(u_n-u)dx\\ &\quad+\int_\Omega\big(|u_n|^{p-2}-|u|^{p-2}\big)(u_n-u)dx-\int_{\partial\Omega}f(x,u_n)(u_n-u)d\sigma+\int_{\partial\Omega}f(x,u)(u_n-u)d\sigma\end{aligned} \tag{2.5}$$

and the inequality deduced from an inequality in Appendix of [13]

$$\int_\Omega\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big).\nabla(u_n-u)dx\geq\frac{2}{p(2^{p-1}-1)}\int_\Omega|\nabla(u_n-u)|^p \tag{2.6}$$

it follows from (2.3) and (2.4) that

$$\begin{aligned}\frac{2}{p(2^{p-1}-1)}\int_\Omega|\nabla(u_n-u)|^p\leq\frac{2}{p(2^{p-1}-1)}\int_\Omega|\nabla(u_n-u)|^p+\int_{\partial\Omega}f(x,u_n)(u_n-u)d\sigma\\ -\int_{\partial\Omega}f(x,u_n)(u_n-u)d\sigma\longrightarrow\infty,\end{aligned} \tag{2.7}$$

where we have used the fact that

$$<\varphi^{'}(u_n)-\varphi^{'}(u),u_n-u>\longrightarrow 0,\ \ as\ \ n\longrightarrow\infty. \tag{2.8}$$

Hence $u_n\longrightarrow u$ in $W^{1,p}(\Omega)$. The proof is completed. □

Lemma 2.3. *If $H(f)_1$-$H(f)_3$ hold, then each Palais-Smale sequence of φ is bounded.*

Proof. Let $\{u_n\}\subset W^{1,p}(\Omega)$ be a Palais-Smale sequence of φ, i.e., there exists $M>0$ such that $|\varphi(u_n)|\leq M$ for all $n\in\mathbb{N}:=\{1,2,...\}$ and $\varphi^{'}(u_n)\longrightarrow 0$, as $n\longrightarrow\infty$.

First, we consider the following problem:

$$\begin{cases}-\Delta_pu-\Delta u+|u|^{p-2}u+u &=0\quad\text{in }\Omega,\\ <|\nabla u|^{p-2}\nabla u+\nabla u,v> &= f_\infty(u^+)^{p-1}+g(x,u)\quad\text{on }\partial\Omega,\end{cases}$$

where $g(x,t) = f(x,t) - f_\infty(t^+)^{p-1}$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$. The operators $A, L, K\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ are defined by

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx + \int_\Omega |u|^{p-2}uv dx - \int_{\partial\Omega} f_\infty (u^+)^{p-1} d\sigma,$$

$$\langle Lu, v\rangle = \int_\Omega \nabla u.\nabla v dx + \int_\Omega uv dx,$$

$$\langle Ku, v\rangle = \int_{\partial\Omega} g(x,u)v d\sigma,$$

$u, v \in W^{1,p}(\Omega)$. For any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|Ku\|_* \le \varepsilon S_p^p \|u\|^{p-1} + C_\varepsilon S_1, \quad u \in W^{1,p}(\Omega). \tag{2.9}$$

Indeed, because $H(f)_1$-$H(f)_3$ hold, for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(x,t)| \le \varepsilon |t|^{p-1} + C_\varepsilon, \quad (x,t) \in \partial\Omega \times \mathbb{R}. \tag{2.10}$$

Using Hölder's inequality and (1.3), we obtain that

$$\begin{aligned}
|\langle Ku, v\rangle| &\le \int_{\partial\Omega} |g(x,u)||v| d\sigma, \\
&\le \varepsilon \int_{\partial\Omega} |u|^{p-1}|v| d\sigma + C_\varepsilon \int_{\partial\Omega} |v| d\sigma \\
&\le \varepsilon \|u\|^{p-1}_{L^p(\partial\Omega)} \|v\|_{L^p(\partial\Omega)} + C_\varepsilon \|v\|_{L^1(\partial\Omega)} \\
&\le \varepsilon S_p^p \|u\|^{p-1}_{1,p} - C_\varepsilon S_1 \|u\|_{1,p}, \quad u, v \in W^{1,p}(\Omega).
\end{aligned}$$

Therefore, (2.9) holds. It follows from Hölder's inequality that

$$\|Lu\|_* \le |\Omega|^{(p-2)/p} \|u\|_{1,p}, \quad u \in W^{1,p}(\Omega). \tag{2.11}$$

Next, we claim that there exists $C_0 > 0$ such that

$$\|Au\|_* \ge C_0 \| \|u\|^{p-1}_{1,p}, \quad u \in W^{1,p}(\Omega). \tag{2.12}$$

Assume for a contradiction that there exists a sequence $\{v_n\} \in W^{1,p}(\Omega)$ with $\|v\|_{1,p} = 1$ such that $\|Av_n\| \le \frac{1}{n}$, i.e.,

$$\left| \int_\Omega |\nabla v_n|^{p-2}\nabla v_n.\nabla w dx + \int_\Omega |v_n|^{p-2} v_n.\nabla w dx - \int_{\partial\Omega} f_\infty (v_n^+)^{p-1} w d\sigma \right| \le \frac{1}{n}\|w\|_{1,p}, \quad w \in W^{1,p}(\Omega), n \in \mathbb{N}. \tag{2.13}$$

Because $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$, we may assume by passing to a subsequence if necessary, which is still denoted by $\{v_n\}$, that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$, $v_n \to v$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. Choosing $w = v_n - v$ in 2.13 and passing to the limit we obtain

$$\lim_{n\to\infty} \left(\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla(v_n - v) dx + \int_\Omega |v_n|^{p-2} v_n (v_n - v) dx \right) = 0.$$

Using Hölder's inequality, we get

$$\begin{aligned}
\int_\Omega |v_n|^{p-2} v_n (v_n - v) dx &\le \int_\Omega |v_n|^{p-1} |v_n - v| dx \\
&\le \left(\int_\Omega |v_n|^{p-1} \right)^{\frac{p-1}{p}} \left(\int_\Omega |v_n - v|^p \right)^{\frac{1}{p}} dx \\
&\le \|v_n\|^{p-1}_{L^p(\Omega)} \|v_n - v\|_{L^p(\Omega)}
\end{aligned}$$

and as v_n converges to v in $L^p(\Omega)$, it follows

$$\int_\Omega |v_n|^{p-2}v_n(v_n-v)dx \to 0$$

and consequently,

$$\lim_{n\to\infty}\bigg(\int_\Omega |\nabla v_n|^{p-2}\nabla v_n\cdot\nabla(v_n-v)dx = 0.$$

Thus, $v_n \to v$ in $W^{1,p}(\Omega)$, and $\|v\|_{L^p(\Omega)} = 1$ by the S_+ property of $-\Delta_p$. Passing to the limit again in (2.13), we have that v is a nontrivial solution of the following problem:

$$(S_p)\begin{cases} -\Delta_p u + |u|^{p-2}u &= 0 \qquad \text{in } \Omega,\\ \langle |\nabla u|^{p-2}\nabla u,\nu\rangle &= f_\infty(u^+)^{p-1} \quad \text{on } \partial\Omega.\end{cases}$$

Therefore, we obtain that $(f_\infty,0)\in\sum_p$. Since $f_\infty\neq\lambda_1$ and $0\neq\lambda_1$ on has $f_\infty>\lambda_1$ and $0>\lambda_1$. This is a contradiction. Thus, from the inequalities (2.11), (2.9), and (2.12), we obtain for all n that

$$C_0\|u_n\|^{p-1} \le \|Au_n\| \le \|\varphi'(u_n)\|+\|Lu_n\|+\|Ku_n\| \le o(1)+||^{(p-2/p}\|u_n\|+\varepsilon S_p^p\|u_n\|^{p-1}+C_\varepsilon S_1.$$

Selecting sufficiently small ε, we obtain the boundedness of $\{u_n\}$. □

Next, we prove that the functional φ satisfies mountain pass geometry.

Lemma 2.4. *Assuming that f satisfies $H(f)_1$-$H(f)_3$, there exist $\rho>0$ and $\alpha>0$ such that*

(i) $\varphi(u)\ge\alpha$ for all $u\in W^{1,p}(\Omega)$ with $\|u\|_{1,p}=\rho$;

(ii) There exists $e\in W^{1,p}(\Omega)$ such that $\|e\|_{1,p}>\rho$. and $\varphi(e)<0$.

Proof. (i) For all $u\in W^{1,p}(\Omega)$, considering $H(f)_2$ and (1.3), we deduce that

$$\begin{aligned}\varphi(u) &\ge \frac{1}{p}\|u\|_{1,p}^p - C\|u\|_{L^1(\partial\Omega)} - \frac{C}{q}\|u\|^q_{L^q(\partial\Omega)}\\ &\ge \frac{1}{p}\|u\|_{1,p}^p - CS_1\|u\|_{1,p} - \frac{C}{q}S_q^q\|u\|_{1,q}^q\\ &= \frac{1}{p}\|u\|_{1,p}^p\Big[\|u\|_{1,p}^{p-1} - a - b\|u\|_{1,q}^{q-1}\Big],\end{aligned}$$

where $a=pCS_1$ and $b=\frac{pC}{q}S_q^q$. We define a function $h:\mathbb{R}_+\to\mathbb{R}$ by $h(t)=t^{p-1}-a-bt^{q-1}$ for all $t\in\mathbb{R}_+$. Then we have that

$$\max_{t\in\mathbb{R}_+} h(t) = h(t_0) = (q-p)b^{\frac{1-p}{q-p}}\left[\frac{(p-1)^{p-1}}{(q-1)^{q-1}}\right]^{1/(q-p)} - a,$$

where $t_0=[b(q-1)/(p-1)]^{-1/(q-p)}>0$. This implies that $h(t_0)>0$ if and only if $a^{q-p}b^{p-1}<(q-p)^{q-p}(p-1)^{p-1}/(q-1)^{q-1}$, i.e., $C<C_q$. Considering $\rho=t_0$, we obtain that $\varphi(u)\ge p^{-1}\rho h(\rho):=\alpha>0$ for all $u\in W^{1,p}(\Omega)$ and $\|u\|_{1,p}=\rho$.

(ii) It follows from $(f)_1$-$(f)_3$, and $f_\infty>\lambda_1$ that for any given $\varepsilon\in(0,f_\infty-\lambda_1)$, there exists $C_\varepsilon>0$ such that

$$f(x,t)\ge(f_\infty-\varepsilon)t^{p-1}-C_\varepsilon, \qquad (x,t)\in\partial\Omega\times\mathbb{R}_+,$$

which implies that

$$F(x,t)\ge\frac{1}{p}(f_\infty-\varepsilon)t^p-C_\varepsilon t, \qquad (x,t)\in\partial\Omega\times\mathbb{R}_+.$$

Hence, for all $t\in\mathbb{R}_+$ and ψ_1 a positive eigenfunction corresponding to λ_1, we have that

$$\varphi(t\psi_1)\le\frac{t^2}{2}\int_\Omega(|\nabla\psi_1|^2+|\psi_1|^2)dx+\frac{t^p}{p}\|\psi_1\|_{1,p}^p-\frac{1}{p\lambda_1}(f_\infty-\varepsilon)t^p+C_\varepsilon t\|\psi_1\|_{L^1(\partial\Omega)}$$

$$\leq \frac{t^2}{2}\int_\Omega (|\nabla \psi_1|^2 + |\psi_1|^2)dx - \frac{1}{p\lambda_1}(f_\infty - \lambda_1 - \varepsilon)t^p + C_\varepsilon t\|\psi_1\|_{L^1(\partial\Omega)}$$

which demonstrates that $\varphi(t\psi_1) \to -\infty$ as $t \to \infty$. Thus, we can select sufficiently large $t_1 > 0$ and $e = t_1\psi_1$ such that $\|e\|_{1,p} > \rho$ and $\varphi(e) < 0$. □

3 Proof of the main result

In this section, we are now ready to prove our main Theorem 1.2.

Proof. According to $H(f)_1$-$H(f)_3$, we obtain for any given $\varepsilon \in (0, f_0 - \mu_1)$ that there exists $C_\varepsilon > 0$ such that

$$f(x, t) \geq (f_0 - \varepsilon)t - C_\varepsilon t^{p-1}, \qquad (x, t) \in \partial\Omega \times \mathbb{R}_+.$$

Subsequently, we have that

$$F(x, t) \geq \frac{1}{2}(f_0 - \varepsilon)t^2 - C_\varepsilon t^p, \qquad (x, t) \in \partial\Omega \times \mathbb{R}_+.$$

Thus, it follows that

$$\varphi(t\phi_1) \leq \frac{t^2}{2}\int_\Omega (|\nabla\phi_1|^2 + |\phi_1|^2)dx + \frac{t^p}{p}\|\phi_1\|_{1,p}^p - \frac{1}{2}(f_0 - \varepsilon)t^2\int_{\partial\Omega}|\phi_1|^2 d\sigma + \frac{C_\varepsilon t^p}{p}\int_{\partial\Omega}|\phi_1|^p d\sigma$$

$$= -\frac{t^2}{2\mu_1}(f_0 - \mu_1 - \varepsilon)\int_\Omega (|\nabla\phi_1|^2 + |\phi_1|^2)dx + \frac{t^p}{p}\|\phi_1\|_{1,p}^p + \frac{C_\varepsilon t^p}{p}\|\phi_1\|_{L^p(\partial\Omega)}^p < 0,$$

for sufficiently small $t > 0$. By Lemma 2.4, there exist $\rho, \alpha > 0$ such that $\varphi(u) > \alpha$ for all $u \in W^{1,p}(\Omega)$ with $\|u\|_{1,p} = \rho$. Hence, we have $m = \inf\{\varphi(u) : u \in \overline{B}_\rho\} < 0$ where $\overline{B}_\rho = \{u \in W^{1,p}(\Omega) : \|u\|_{1,p} \leq \rho\}$. Thus, there exists a minimizing sequence $\{u_n\}$ such that

$$\varphi(u_n) \to m, \quad \varphi'(u_n) \to 0$$

by Ekeland's variational principle. It follows from Lemma 2.2 and Lemma 2.3 that φ satisfies the Palais-Smale condition. Thus, there exists $u_1 \in B_\rho$ such that $\varphi(u_1) = m < 0$ and $\varphi'(u_1) = 0$, which yields that u_1 is a nontrivial critical point of φ. Furthermore, by applying the mountain pass theorem [19] for

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0, 1], W^{1,p}(\Omega) : \gamma(0) = 0, \gamma(1) = e\},$$

there exists $u_2 \in W^{1,p}(\Omega)$ such that $\varphi(u_2) = c > 0$ and $\varphi'(u_2) = 0$, which shows that u_2 is another nontrivial critical point of φ.

Finally, we claim that u_1 and u_2 are positive solutions of the problem $(S_p, 2)$. In fact, if u is a critical point of φ, then by multiplying $(S_p, 2)$ by u^-, we obtain $\|u^-\|_{1,2}^2 + \|u^-\|_{1,p}^p = 0$, which yields $u^- = 0$ and $u \geq 0$. Thus, $u_1 \geq 0$ and $u_2 \geq 0$. Hence, the proof is complete. □

References

[1] A. Anane, O. Chakrone, B. Karim and A. Zerouali, *The beginning of the Fucik spectrum for a Steklov Problem*, Boletim da Sociedade Paranaense de Matemática, 27(1), 21-27, (2009).

[2] R. Aris, *Mathematical Modelling Techniques*, Research Notes in Mathematics, Pitman, London, (1978).

[3] V. Benci, P. D'Avenia, D. Fortunato and L. Pisani, *Solutions in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal., 154, 297-324, (2000).

[4] B. D. Bitim, N. Topal, *Binomial sum formulas from the exponential generating functions of (P, q)-fibonacci and (p, q)-lucas quaternions*, Palestine Journal of Mathematics, Vol. 10(1), 279-289, (2021).

[5] A. Boukhsas, A. Zerouali, O. Chakrone and B. Karim, *On a Positive Solutions for (p, q)-Laplacian Steklov Problem with Two Parameters*, Bol. Soc. Paran. Mat., v.(40), 1-19, doi:10.5269/bspm.46385, (2022).

[6] A. Boukhsas, A. Zerouali, O. Chakrone and B. Karim, *Multiple solutions for a (p, q)-Laplacian Steklov problem*, Annals of the University of Craiova-Mathematics and Computer Science Series, 47(2), 357-368, (2020).

[7] A. Boukhsas, A. Zerouali, O. Chakrone and B. Karim, *Positive solution fot a $(p, 2)$-Laplacian Steklov problem,* Mathematica, 64 (87), No 1, (2022).

[8] A. Boukhsas, A. Zerouali, O. Chakrone and B. Karim, *Steklov eigenvalue problems with indefinite weight for the (p, q)-Laplacian*, Rev. Roumaine Math. Pures Appl., 67(3-4), 127-142, (2022).

[9] A. Boukhsas, B. Ouhamou, *Steklov eigenvalues problems for generalized (p, q)-Laplacian type operators*, Mem. Differ. Equ. Math. Phys., 85, 35-51, (2022).

[10] L. Cherfils, Y. Il'yasov, *On the stationary solutions of generalized reaction diffusion equations with (p, q)-Laplacian*, Commun. Pure Appl. Anal., 4. 1, 922, (2005).

[11] P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems,* Lecture Notes in Biomathematics, 28, Springer Verlag, Berlin-New York, (1979).

[12] L. Gasinski and N. S. Papageorgiou, *Asymmetric $(p, 2)$-equations with double resonance*, Calc. Var. Partial Differential Equations, 56.3, 88, (2017).

[13] P. Lindqvist, *On the equation* $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proceedings of the American Mathematical Society, vol. 109, no.1, pp. 157164, (1990).

[14] N. S. Papageorgiou, V. D. Radulescu and D. D. Repovš, *Existence and multiplicity of solutions for resonant $(p, 2)$-equations*, Advanced Nonlinear Studies, 18.1, 105-129, (2018).

[15] L. G. Romero, *A generalization of the Laplacian differential operator*, Palestine Journal of Mathematics, 5(2), 204-207, (2016).

[16] M. Struwe, *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems,* Springer-Verlag, Berlin, Heidelberg, New York, (1996).

[17] A. Zerouali, B. Karim, O. Chakrone and A. Boukhsas, *Resonant Steklov eigenvalue problem involving the (p, q)-Laplacian*, Afrika Matematika, 30(1), 171-179, (2019).

[18] A. Zerouali, B. Karim, O. Chakrone and A. Boukhsas, *On a positive solution for (p, q)-Laplace equation with Nonlinear Boundary Conditions and indefinite weights*, Bol. Soc. Paran. Mat.(3s.) v, 38(4), 205-219 (2020).

[19] M. Willem, *Minimax theorems*, Springer Science & Business Media, Vol. 24,(1997).

[20] V.V.E. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv., 29, 33-66, (1987).

Author information

A. BOUKHSAS, Moulay Ismail University of Meknes, FST Errachidia, Morocco.
E-mail: `abdelmajidboukhsas@gmail.com`

B. OUHAMOU, Mohammed First University of Oujda, Faculty of Science, Morocco.
E-mail: `ouhamoubrahim@gmail.com`

Received: 2023-02-24
Accepted: 2023-10-28