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# Schur convexity of difference of means

Sandeep Kumar and Sangeetha B. K.

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**Abstract**. In this paper, we study Schur convexity and concavity, its properties like Schur, Schur harmonic convexity(concave)on the ratio of difference of means obtained by arithmetic mean, geometric mean, harmonic mean, contraharmonic mean, heron mean, root square mean, subsidiary means.

## 1 Introduction

In [15] I. J. Taneja refinement of inequalities among means we have the following inequality chain.

$$H(\Phi,\Psi) \leq G(\Phi,\Psi) \leq N_1(\Phi,\Psi) \leq H_e(\Phi,\Psi) \leq N_2(\Phi,\Psi) \leq A(\Phi,\Psi) \leq S(\Phi,\Psi)$$

For the real numbers $\Phi, \Psi > 0$,

$$A = A(\Phi,\Psi) = \frac{\Phi+\Psi}{2}$$
$$G = G(\Phi,\Psi) = \sqrt{\Phi\Psi}$$
$$H = H(\Phi,\Psi) = \frac{2}{\frac{1}{\Phi}+\frac{1}{\Psi}}$$
$$H_e = H_e(\Phi,\Psi) = \frac{\Phi+\sqrt{\Phi*\Psi}+\Psi}{3}$$
$$S = S(\Phi,\Psi) = \sqrt{\frac{\Phi^2+\Psi^2}{2}}$$
$$C = C(\Phi,\Psi) = \frac{\Phi^2+\Psi^2}{\Phi+\Psi}$$
$$N_1 = N_1(\Phi,\Psi) = \left(\frac{\sqrt{\Phi}+\sqrt{\Psi}}{2}\right)^2$$
$$N_2 = N_2(\Phi,\Psi) = \left(\frac{\sqrt{\Phi}+\sqrt{\Psi}}{2}\right)\left(\sqrt{\frac{\Phi+\Psi}{2}}\right)$$

are respectively called arithmetic mean, geometric mean, harmonic mean, root-square mean, contra-harmonic mean. $N_1, N_2$ are called subsidiary means which are discussed in Enormously. With the reference of these well known means, few difference of means are given as below.

$$M_{SA}(\Phi,\Psi) = S(\Phi,\Psi) - A(\Phi,\Psi)$$
$$M_{SN_2}(\Phi,\Psi) = S(\Phi,\Psi) - N_2(\Phi,\Psi)$$
$$M_{SH_e}(\Phi,\Psi) = S(\Phi,\Psi) - H_e(\Phi,\Psi)$$
$$M_{SN_1}(\Phi,\Psi) = S(\Phi,\Psi) - N_1(\Phi,\Psi)$$
$$M_{SG}(\Phi,\Psi) = S(\Phi,\Psi) - G(\Phi,\Psi)$$
$$M_{HN_1}(\Phi,\Psi) = H(\Phi,\Psi) - N_1(\Phi,\Psi)$$

$$M_{GA}(\Phi,\Psi) = G(\Phi,\Psi) - A(\Phi,\Psi)$$
$$M_{AS}(\Phi,\Psi) = A(\Phi,\Psi) - S(\Phi,\Psi)$$
$$M_{N_1N_2}(\Phi,\Psi) = N_1(\Phi,\Psi) - N_2(\Phi,\Psi)$$
$$M_{HN_1}(\Phi,\Psi) = H(\Phi,\Psi) - N_1(\Phi,\Psi)$$
$$M_{N_2N_1}(\Phi,\Psi) = N_2(\Phi,\Psi) - N_1(\Phi,\Psi)$$
$$M_{N_1G}(\Phi,\Psi) = N_1(\Phi,\Psi) - G(\Phi,\Psi)$$

In this paper, we discuss convexity of difference of means list above.

## 2 Preliminaries

This section provides the required definitions and lemmas.

**Lemma 2.1.** *In [21] Jamal Rooin and Mehdi Hassni introduced the homogeneous functions $f(x)$ and $g(x)$, where*

$$f(x) = \frac{\Phi^x - \Psi^x}{\zeta^x - \eta^x},$$
$$g(x) = ln\frac{\Phi^x - \Psi^x}{\zeta^x - \eta^x}$$

for $\Phi \in (-\infty, +\infty)$

- convexity, if $\Phi\Psi - \zeta\eta \geq 0$
- concavity, if $\Phi\eta - \Psi\zeta \leq 0$ and
- equalities holds, if $\Phi\eta - \Psi\zeta = 0$ for $\Phi > \Psi \geq \zeta > \eta > 0$

The notion of Schur-convex function was introduced by I. Schur in 1923 and has interesting applications in quantum theory, analytic inequalities and quantum mechanics. We now present the basic lemmas required to prove the main theorems.

**Lemma 2.2.** *Let $\Omega \subseteq R^n$ be symmetric with non empty interior geometrically convex set and let $\phi : \Omega \to R_+$ be continuous on $\Omega$ and differentiable in $\Omega^0$. If $\phi$ is symmetric on $\Omega$ and*

$$(ln\Phi_1 - ln\Phi_2)\left(\Phi_1\frac{\partial\phi}{\partial\Phi_1} - \Phi_2\frac{\partial\phi}{\partial\Phi_2}\right) \geq 0. \tag{2.1}$$

where $\Phi = (\Phi_1, \Phi_2 \cdots\cdots \Phi_n) \in \Omega^0$, then $\phi$ is Schur-geometrically convex(concave)function.

**Lemma 2.3.** *Let $\Omega \subseteq R^n$ be symmetric with non empty interior harmonically convex set and let $\phi : \Omega \to R_+$ be continous on $\Omega$ and differentiable in $\Omega^0$. If $\phi$ is symmetric on $\Omega$ and*

$$(\Phi_1 - \Phi_2)\left(\Phi_1^2 \frac{\partial \phi}{\partial \Phi_1} - \Phi_2^2 \frac{\partial \phi}{\partial \Phi_2}\right) \geq 0. \tag{2.2}$$

where $\Phi = (\Phi_1, \Phi_2 \ldots\ldots\ldots\ldots \Phi_n) \in \Omega^0$, then $\phi$ is Schur-harmonically convex(concave)function.

An important source of Schur-convex functions can be found in Merkle [16]. Other families of Schur-convex functions are studied in [[11]-[25]].

## 3 Main Results

In this section we set the difference of means which are defined in the equations in the above section.

$$M_{SN_1}(\Phi,\Psi) - M_{SG}(\Phi,\Psi) = G - N_1$$
$$M_{SH_e}(\Phi,\Psi) - M_{SN_1}(\Phi,\Psi) = N_1 - H_e$$
$$M_{SG}(\Phi,\Psi) - M_{SS}(\Phi,\Psi) = S - G$$
$$M_{SA}(\Phi,\Psi) - M_{SG}(\Phi,\Psi) = G - A$$
$$M_{SS}(\Phi,\Psi) - M_{SN_1}(\Phi,\Psi) = N_1 - S$$

The above difference of the means are convex for all positive values. Now, we establish the ratio of difference of above means as follows:

$$\frac{M_{SN_2} - M_{SA}}{M_{SH} - M_{SN_2}} = \frac{A - N_2}{N_2 - H}$$

$$\frac{M_{SG} - M_{SA}}{M_{SG} - M_{SN_1}} = \frac{A - G}{G - N_1}$$

$$\frac{M_{SG} - M_{SN_1}}{M_{SG} - M_{SH_e}} = \frac{N_1 - G}{G - H_e}$$

$$\frac{M_{SS} - M_{SN_1}}{M_{SG} - M_{SS}} = \frac{N_1 - S}{S - G}$$

$$\frac{M_{SS} - M_{SA}}{M_{SG} - M_{SS}} = \frac{A - S}{S - G}$$

$$\frac{M_{SS} - M_{SA}}{M_{SH} - M_{SS}} = \frac{A - S}{S - H}$$

**Theorem 3.1.** *For $\Phi > \Psi > 0$, the ratio of difference of mean*

$$\frac{M_{SN_2} - M_{SA}}{M_{SH} - M_{SN_2}} = \frac{A - N_2}{N_2 - H}$$

*is convex for all positive real values of 't'*

*Proof.*

$$\frac{M_{SN_2}(p,1) - M_{SA}(p,1)}{M_{SH}(p,1) - M_{SN_2}(p,1)} = \frac{A(p,1) - N_2(p,1)}{N_2(p,1) - H(p,1)}$$

from 2.1)

$$\begin{aligned} f(p,1) &= AH - N_2^2 \\ &= \left(\frac{p+1}{2}\right)\left(\frac{2p}{p+1}\right) - \left(\frac{\sqrt{p}+1}{2}\right)^2\left(\sqrt{\frac{p+1}{2}}\right)^2 \\ &= \frac{1}{8}\left[8p - \left(p(p+1+2\sqrt{p}) + p + 1 + 2\sqrt{p}\right)\right] \\ &= \frac{1}{8}\left[8p - \left((p+1)^2 + 2\sqrt{p}(p+1)\right)\right] \\ &= \frac{8p - (p+1)^2 - 2\sqrt{p}(p+1)}{8} \\ f(p,1) &= \frac{6p - p^2 + 1 - 2\sqrt{p}(p+1)}{8} \geq 0 \end{aligned}$$

Therefore, it satisfies convexity property ∀ non negative values of $'p'$. □

**Theorem 3.2.** *For* $\Phi > \Psi > 0$,*the ratio of difference of mean*

$$\frac{M_{SG} - M_{SA}}{M_{SN_1} - M_{SG}} = \frac{A - G}{G - N_1}$$

*is convex for all positive real values of* $'t'$

*Proof.* Let

$$f(p,1) = \frac{M_{SG}(p,1) - M_{SA}(p,1)}{M_{SN_1}(p,1) - M_{SG}(p,1)} = \frac{A(p,1) - G(p,1)}{G(p,1) - N_1(p,1)}$$

$$\begin{aligned} f(p,1) &= AN_1 - G^2 \\ &= \left(\frac{p+1}{2}\right)\left(\frac{\sqrt{p}+1}{2}\right)^2 - p \\ &= \frac{1}{8}\left[p(p+1+2\sqrt{p}) + (p+1+2\sqrt{p})\right] - p \\ &= \frac{1}{8}\left[p^2 + p + 2p\sqrt{p} + p + 1 + 2\sqrt{p}\right] - p \\ &= \frac{1}{8}\left[(p+1)^2 + 2\sqrt{p}(p+1)\right] - p \\ &= \frac{1}{8}\left[\left[(p+1)(p+1+2\sqrt{p}) - 8p\right]\right. \\ &= \frac{1}{8}\left[p^2 + p + 2\sqrt{p}(p+1) + 2p - 8p\right] \\ f(p,1) &= \frac{p^2 + 1 - 6p + 2\sqrt{p}(p+1)}{8} \geq 0 \end{aligned}$$

Therefore, it satisfies convexity property for all non negative values of $'p'$. □

**Theorem 3.3.** *For* $\Phi > \Psi > 0$, *the ratio of difference of mean*

$$\frac{M_{SG} - M_{SN_1}}{M_{SH_e} - M_{SG}} = \frac{N_1 - G}{G - H_e}$$

*is convex for all positive real values of* $'t'$

*Proof.*

$$\frac{M_{SG}(p,1)-M_{SN_1}(p,1)}{M_{SH_e}(p,1)-M_{SG}(p,1)}=\frac{N_1(p,1)-G(p,1)}{G(p,1)-H_e(p,1)}$$

$$f(p,1)=N_1H_e-G^2$$

$$=\frac{1}{12}\left(\frac{\sqrt{p}+1}{2}\right)^2\left(\frac{p+1+\sqrt{p}}{3}\right)-p$$

$$=\frac{1}{12}\left(\frac{p+1+2\sqrt{p}}{4}\right)\left(\frac{p+1+\sqrt{p}}{3}\right)-p$$

$$=\frac{1}{12}\left[p(p+1+2\sqrt{p})+(p+1+2\sqrt{p})+2\sqrt{p}(p+1+2\sqrt{p})-12p\right]$$

$$=\frac{1}{12}[p^2+1+2\sqrt{p}(p+1)+\sqrt{p}(p+1)+4p-12p]$$

$$f(p,1)=\frac{1}{12}[p^2+1-8p+3\sqrt{p}(p+1)]\geq 0$$

Therefore,it satisfies convexity property $\forall$ non negative values of $'p'$. □

**Theorem 3.4.** *For $\Phi>\Psi>0$,the ratio of difference of mean*

$$\frac{M_{SS}-M_{SN_1}}{M_{SG}-M_{SS}}=\frac{N_1-S}{S-G}$$

*is convex for all positive real values of $'t'$*

*Proof.*

$$\frac{M_{SS}(p,1)-M_{SN_1}(p,1)}{M_{SG}(p,1)-M_{SS}(p,1)}=\frac{N_1(p,1)-S(p,1)}{S(p,1)-G(p,1)}$$

$$f=N_1G-S^2$$

$$=\left(\frac{\sqrt{p}+1}{2}\right)^2\sqrt{p}-\left(\sqrt{\frac{p^2+1}{2}}\right)^2$$

$$=\left(\frac{(p+1+2\sqrt{p})\sqrt{p}-2(p+1)^2-2p}{4}\right)$$

$$f(p,1)=\left(\frac{(p+1)\sqrt{p}+6p-2(p+1)^2}{4}\right)\geq 0$$

Therefore, it satisfies convexity property $\forall$ non negative values of '$p$'. □

## 4 Schur convex and combined type properties of difference of means

In this component, we get the Schur convexity(concavity), Schur geometrically convexity(concavity), Schur harmonically convexity(concavity) for the difference of ratio of means in 2 variables.

**Theorem 4.1.** *The ratio of difference of mean*

$$\frac{M_{SN_2}-M_{SA}}{M_{SH}-M_{SN_2}}=\frac{A-N_2}{N_2-H}$$

- *Schur convexity*
- *Schur geometrical convexity*
- *Schur harmonical convexity ,$\forall\ \Phi\geq\Psi$.*

*Proof.*

$$
\begin{aligned}
Let f(\Phi,\Psi) &= \frac{M_{SN_2}(\Phi,\Psi) - M_{SA}(\Phi,\Psi)}{M_{SH} - M_{SN_2}} \\
&= \frac{A(\Phi,\Psi) - N_2(\Phi,\Psi)}{N_2(\Phi,\Psi) - H(\Phi,\Psi)} \\
f(\Phi,\Psi) &= AH - N_2^2 \\
f(\Phi,\Psi) &= \frac{6\Phi\Psi - \Phi^2 - \Psi^2 - 2\sqrt{\Phi\Psi}(\Phi+\Psi)}{8}
\end{aligned}
$$

The partial derivatives of $f(\Phi,\Psi)$ after simple calculations gives.

$$
\begin{aligned}
\frac{\partial f}{\partial \Phi} &= \frac{1}{8}\left[6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi+\Psi)\right] \\
\frac{\partial f}{\partial \Psi} &= \frac{1}{8}\left[6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi+\Psi)\right]
\end{aligned}
$$

□

*Proof.* Case-(1) Schur convexity(concavity) Then

$$
\begin{aligned}
s &= \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} \\
\frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{1}{8}\left[6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi+\Psi)\right] - \frac{1}{8}\left[6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi+\Psi)\right] \\
&= \frac{(\Phi-\Psi)}{8}\left[\frac{(\Phi+\Psi) - 8\sqrt{\Phi\Psi}}{\sqrt{\Phi\Psi}}\right] \\
\frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} &= \frac{(\Phi-\Psi)}{8}\left[\frac{2A - 8G}{G}\right] \geq 0
\end{aligned}
$$

for $\Phi \geq \Psi$. This proves the condition for Schur convexity. □

*Proof.* Case-(2) Schur geometrically convexity(concavity)
Then

$$
\begin{aligned}
s &= \Phi\frac{\partial f}{\partial \Phi} - \Psi\frac{\partial f}{\partial \Psi} \\
\Phi\frac{\partial f}{\partial \Phi} - \Psi\frac{\partial f}{\partial \Psi} &= \frac{\Phi}{8}\left[6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi+\Psi)\right] - \frac{\Psi}{8}\left[6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi+\Psi)\right] \\
&= -\frac{(\Phi-\Psi)}{4}\left[(\Phi+\Psi) + \sqrt{\Phi\Psi}\right] \\
\Phi\frac{\partial f}{\partial \Phi} - \Psi\frac{\partial f}{\partial \Psi} &= -\frac{(\Phi-\Psi)}{4}\left[2A + G\right] \leq 0
\end{aligned}
$$

for $\Phi \geq \Psi$. This proves the condition for Schur geometrically concavity. □

*Proof.* Case-(3) Schur harmonically convexity(concavity)

Then

$$s = \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi}$$
$$= \frac{\Phi^2}{8}\left[6\Psi - 2\Phi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Psi}{\Phi}}(\Phi+\Psi)\right] - \frac{\Psi^2}{8}\left[6\Phi - 2\Psi - 2\sqrt{\Phi\Psi} - \sqrt{\frac{\Phi}{\Psi}}(\Phi+\Psi)\right]$$
$$= \frac{(\Phi-\Psi)}{8}\left[8\Phi\Psi - 2(\Phi+\Psi)^2 - 3(\Phi+\Psi)\sqrt{\Phi\Psi}\right]$$
$$\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} = \frac{(\Phi-\Psi)}{4}\left[4(G^2 - A^2) - 3AG\right] \geq 0$$

for $\Phi \geq \Psi$. This proves the condition for Schur harmonically convexity. □

**Theorem 4.2.** *For $\Phi > \Psi > 0$, the difference of ratio of mean*

$$\frac{M_{SG} - M_{SA}}{M_{SN_1} - M_{SG}} = \frac{A - G}{G - N_1}$$

*is*

*(1) Schur convexity*

*(2) Schur geometrical convexity*

*(3) Schur harmonical convexity ,$\forall\ \Phi \geq \Psi$.*

*Proof.* Let

$$f(\Phi,\Psi) = \frac{M_{SN_1}(\Phi,\Psi) - M_{SG}(\Phi,\Psi)}{M_{SH_e(\Phi,\Psi)} - M_{SN_1(\Phi,\Psi)}} = \frac{G(\Phi,\Psi) - N_1(\Phi,\Psi)}{N_1(\Phi,\Psi) - H_e(\Phi,\Psi)}$$

Hence,

$$f(\Phi,\Psi) = AN_1 - G^2$$
$$f(\Phi,\Psi) = \left(\frac{\Phi+\Psi}{2}\right)\left(\frac{\Phi+\Psi+2\sqrt{\Phi\Psi}}{4}\right) - \Phi\Psi$$
$$f(\Phi,\Psi) = \frac{\Phi^2 + \Psi^2 - 6\Phi\Psi + 2\sqrt{\Phi\Psi}(\Phi+\Psi)}{8}$$

the partial derivatives of $f(\Phi,\Psi)$ and simple computations gives us,

$$\frac{\partial f}{\partial \Phi} = \frac{1}{8}\left[2\Phi - 6\Psi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Psi}{\Phi}}(\Phi+\Psi)\right]$$
$$\frac{\partial f}{\partial \Psi} = \frac{1}{8}\left[2\Psi - 6\Phi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Phi}{\Psi}}(\Phi+\Psi)\right]$$

□

*Proof.* Case-(1) Schur convex Then

$$s = \frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi}$$
$$= \frac{\Phi-\Psi}{8}\left[\frac{8\sqrt{\Phi\Psi} + (\Phi+\Psi)}{\sqrt{\Phi\Psi}}\right] \geq 0$$
$$\frac{\partial f}{\partial \Phi} - \frac{\partial f}{\partial \Psi} = \frac{\Phi-\Psi}{8}\left[\frac{8G+2A}{G}\right] \geq 0.$$

for $\Phi \geq \Psi$. This verifies the condition for Schur convex from 5 □

*Proof.* Case-(2) Schur geometrically convex. we have

$$\Phi\frac{\partial f}{\partial \Phi} = \frac{\Phi}{8}\left[2\Phi - 6\Psi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Psi}{\Phi}}(\Phi+\Psi)\right]$$

$$\Psi\frac{\partial f}{\partial \Psi} = \frac{\Psi}{8}\left[2\Psi - 6\Phi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Phi}{\Psi}}(\Phi+\Psi)\right]$$

$$\Phi\frac{\partial f}{\partial \Phi} = \frac{1}{8}\left[2\Phi^2 - 6\Phi\Psi + 2\Phi\sqrt{\Phi\Psi} + \Phi(\Phi+\Psi)\sqrt{\frac{\Psi}{\Phi}}\right]$$

$$\Psi\frac{\partial f}{\partial \Psi} = \frac{1}{8}\left[2\Psi^2 - 6\Phi\Psi + 2\Psi\sqrt{\Phi\Psi} + \Psi(\Phi+\Psi)\sqrt{\frac{\Phi}{\Psi}}\right]$$

$$\Phi\frac{\partial f}{\partial \Phi} - \Psi\frac{\partial f}{\partial \Psi} = \frac{\Phi-\Psi}{4}\left[(\Phi+\Psi) + \sqrt{\Phi\Psi}\right]$$

$$\Phi\frac{\partial f}{\partial \Phi} - \Psi\frac{\partial f}{\partial \Psi} = \frac{\Phi-\Psi}{4}\left[2A + G\right] \geq 0$$

for $\Phi \geq \Psi$. This proves Schur geometrically convex. □

*Proof.* Case-(3) Schur harmonic convexity. we have

$$s = \Phi^2\frac{\partial f}{\partial \Phi} - \Psi^2\frac{\partial f}{\partial \Psi}$$

$$\Phi^2\frac{\partial f}{\partial \Phi} = \frac{\Phi^2}{8}\left[2\Phi - 6\Psi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Psi}{\Phi}}(\Phi+\Psi)\right]$$

$$\Psi^2\frac{\partial f}{\partial \Psi} = \frac{\Psi^2}{8}\left[2\Psi - 6\Phi + 2\sqrt{\Phi\Psi} + \sqrt{\frac{\Phi}{\Psi}}(\Phi+\Psi)\right]$$

$$= \frac{\Phi-\Psi}{4}\left[(\Phi+\Psi)^2 - 4\Phi\Psi + \sqrt{\Phi\Psi}(\Phi+\Psi) + \sqrt{\Phi\Psi}\left(\frac{\Phi+\Psi}{2}\right)\right]$$

$$\Phi^2\frac{\partial f}{\partial \Phi} - \Psi^2\frac{\partial f}{\partial \Psi} = \left(\frac{\Phi-\Psi}{4}\right)\left[4A^2 - 4G + 3GA\right] \geq 0$$

for $\Phi \geq \Psi$. This verifies the condition for Schur harmonic convex. □

**Theorem 4.3.** *The ratio of difference of mean*

$$\frac{M_{SG} - M_{SN_1}}{M_{SH_e} - M_{SG}} = \frac{N_1 - G}{G - H_e}$$

*(1) Schur convexity*

*(2) Schur geometrical convexity*

*(3) Schur harmonical convexity ,for all* $\Phi \geq \Psi$.

.

*Proof.*

$$let f(\Phi,\Psi) = \frac{M_{SG}(\Phi,\Psi) - M_{SN_1}(\Phi,\Psi)}{M_{SH_e}(\Phi,\Psi) - M_{SG}(\Phi,\Psi)} = \frac{N_1 - G}{G - H_e}$$

Hence,

$$
\begin{aligned}
f(\Phi,\Psi) &= N_1 H_e - G^2\\
f(\Phi,\Psi) &= \frac{1}{12}\left(\Phi^2+\Psi^2-8\Phi\Psi+3\sqrt{\Phi\Psi}(\Phi+\Psi)\right)\\
\frac{\partial f}{\partial \Phi} &= \frac{1}{12}\left(2\Phi-8\Psi+3\sqrt{\Phi\Psi}+\frac{3\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}\right)\\
\frac{\partial f}{\partial \Psi} &= \frac{1}{12}\left(2\Psi-8\Phi+3\sqrt{\Phi\Psi}+\frac{3\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}\right)
\end{aligned}
$$

□

*Proof.* Case-(1) Schur convexity.
Then

$$
\begin{aligned}
\frac{\partial f}{\partial \Phi} &= \frac{1}{12}\left(2\Phi-8\Psi+3\sqrt{\Phi\Psi}+\frac{3\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}\right)\\
\frac{\partial f}{\partial \Psi} &= \frac{1}{12}\left(2\Psi-8\Phi+3\sqrt{\Phi\Psi}+\frac{3\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}\right)\\
\frac{\partial f}{\partial \Phi}-\frac{\partial f}{\partial \Psi} &= \frac{(\Phi-\Psi)}{12}\left[\frac{10\sqrt{\Phi\Psi}-3(\Phi+\Psi)}{\sqrt{\Phi\Psi}}\right]\\
\frac{\partial f}{\partial \Phi}-\frac{\partial f}{\partial \Psi} &= \frac{(\Phi-\Psi)}{12}\left[\frac{10G-3A}{G}\right]\geq 0
\end{aligned}
$$

for $\Phi \geq \Psi$. This proves the condition of Schur convexity from 5 □

*Proof.* Case-(2) Schur geometrically convex. we have

$$
\begin{aligned}
s &= \Phi\frac{\partial f}{\partial \Phi}-\Psi\frac{\partial f}{\partial \Psi}\\
\Phi\frac{\partial f}{\partial \Phi}-\Psi\frac{\partial f}{\partial \Psi} &= \frac{\Phi}{12}\left(2\Phi-8\Psi+3\sqrt{\Phi\Psi}+\frac{3\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}\right)\\
&\quad -\frac{\Psi}{12}\left(2\Psi-8\Phi+3\sqrt{\Phi\Psi}+\frac{3\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}\right)\\
&= \frac{(\Phi-\Psi)}{12}\left(2(\Phi+\Psi)+3\sqrt{\Phi\Psi}\right)\\
\Phi\frac{\partial f}{\partial \Phi}-\Psi\frac{\partial f}{\partial \Psi} &= \frac{(\Phi-\Psi)}{12}\left(4A+3G\right)\geq 0
\end{aligned}
$$

for $\Phi \geq \Psi$. This proves the condition of Schur geometrically convex. □

*Proof.* Case-(3) Schur harmonically convex. we have

$$s = \Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi}$$

$$\begin{aligned}
\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{\Phi^2}{12}\left(2\Phi - 8\Psi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}\right) \\
&- \frac{\Psi^2}{12}\left(2\Psi - 8\Phi + 3\sqrt{\Phi\Psi} + \frac{3\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}\right) \\
&= \frac{\Phi^2}{12}\Big[2\big((\Phi+\Psi)^2 - \Phi\Psi\big) - 8\Phi\Psi + 3\sqrt{\Phi\Psi}(\Phi+\Psi) \\
&+ 3\left(\frac{(\Phi+\Psi)}{2}\right)\sqrt{\Phi\Psi}\Big] \\
\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi} &= \frac{(\Phi-\Psi)}{12}\left[8A^2 - 10G^2 + 9AG\right] \\
(\Phi-\Psi)\left[\Phi^2 \frac{\partial f}{\partial \Phi} - \Psi^2 \frac{\partial f}{\partial \Psi}\right] &= \frac{(\Phi-\Psi)^2}{12}\left[8A^2 - 10G^2 + 9AG\right] \geq 0
\end{aligned}$$

for $\Phi \geq \Psi$. This proves the condition of Schur harmonically convexity. □

**Theorem 4.4.** *The ratio of difference of mean*

$$\frac{M_{SS} - M_{SN_1}}{M_{SG} - M_{SS}} = \frac{N_1 - S}{S - G}$$

*(1) Schur convexity/concavity.*

*(2) Schur geometrically convexity/concavity.*

*(3) Schur harmonically convexity/concavity.* $,\forall\ \Phi \geq \Psi$.

*Proof.*

$$\begin{aligned}
f(\Phi,\Psi) &= \frac{M_{SS}(\Phi,\Psi) - M_{SN_1}(\Phi,\Psi)}{M_{SG}(\Phi,\Psi) - M_{SS}(\Phi,\Psi)} = \frac{N_1 - S}{S - G} \\
f &= \left(\frac{\sqrt{\Phi}+\sqrt{\Psi}}{2}\right)^2 \sqrt{\Phi\Psi} - \left(\sqrt{\frac{\Phi^2+\Psi^2}{2}}\right)^2 \\
&= \left(\frac{\Phi+\Psi+2\sqrt{\Phi\Psi}}{4}\right)\sqrt{\Phi\Psi} - \frac{\Phi^2+\Psi^2}{2} \\
&= \frac{(\Phi+\Psi+2\sqrt{\Phi\Psi})\sqrt{\Phi\Psi} - 2(\Phi^2+\Psi^2+2\Phi\Psi-2\Phi\Psi)}{4}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\Phi+\Psi+2\sqrt{\Phi\Psi})\sqrt{\Phi\Psi} - 2\big((\Phi+\Psi)^2 - 2\Phi\Psi\big)}{4} \\
&= \frac{(\Phi+\Psi)\sqrt{\Phi\Psi} + 2\Phi\Psi - 2(\Phi+\Psi)^2 + 4\Phi\Psi}{4} \\
f &= \frac{(\Phi+\Psi)\sqrt{\Phi\Psi} - 2(\Phi+\Psi)^2 + 6\Phi\Psi}{4} \\
\frac{\partial f}{\partial \Phi} &= \frac{1}{4}\left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}} + 6\Psi - 4(\Phi+\Psi)\right] \\
\frac{\partial f}{\partial \Psi} &= \frac{1}{4}\left[\sqrt{\Phi\Psi} + \frac{\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}} + 6\Phi - 4(\Phi+\Psi)\right]
\end{aligned}$$

Case-(1) Schur convexity.

$$
\begin{aligned}
\frac{\partial f}{\partial \Phi}-\frac{\partial f}{\partial \Psi} &= \frac{1}{4}\left[\frac{\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}-\frac{\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}+6\Psi-6\Phi\right] \\
&= \frac{1}{4}\left[\frac{(\Phi+\Psi)}{2\sqrt{\Phi\Psi}}(\Psi-\Phi)+6(\Psi-\Phi)\right] \\
&= \frac{1}{4}\left[\frac{-(\Phi+\Psi)(\Phi-\Psi)}{2\sqrt{\Phi\Psi}}-6(\Phi-\Psi)\right] \\
&= \frac{-(\Phi-\Psi)}{4}\left[\frac{(\Phi+\Psi)}{2\sqrt{(\Phi\Psi)}}+6\right] \\
\frac{\partial f}{\partial \Phi}-\frac{\partial f}{\partial \Psi} &= \frac{-(\Phi-\Psi)}{4}\left[\frac{A+6G}{G}\right]\leq 0. \\
(\Phi-\Psi)\frac{\partial f}{\partial \Phi}-\frac{\partial f}{\partial \Psi} &= \frac{-(\Phi-\Psi)^2}{4}\left[\frac{A+6G}{G}\right]\leq 0.
\end{aligned}
$$

This proves f is Schur concave.
Case-(2) Schur geometrically convexity.

$$
\begin{aligned}
\Phi\frac{\partial f}{\partial \Phi}-\Psi\frac{\partial f}{\partial \Psi} &= \frac{\Phi}{4}\left[\sqrt{\Phi\Psi}+\frac{\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}+6\Psi-4(\Phi+\Psi)\right] \\
&- \frac{\Psi}{4}\left[\sqrt{\Phi\Psi}+\frac{\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}+6\Phi-4(\Phi+\Psi)\right] \\
&= \frac{1}{4}\left[\Phi\sqrt{\Phi\Psi}+\Phi\frac{\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}+6\Phi\Psi-4\Phi(\Phi+\Psi)\right] \\
&- \frac{1}{4}\left[\Psi\sqrt{\Phi\Psi}+\Psi\frac{\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}+6\Phi\Psi-4\Psi(\Phi+\Psi)\right] \\
&= \frac{1}{4}\left[\Phi\sqrt{\Phi\Psi}+\frac{\sqrt{\Phi\Psi}(\Phi+\Psi)}{2}+6\Phi\Psi-4\Phi(\Phi+\Psi)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{4}\left[\Psi\sqrt{\Phi\Psi}+\frac{\sqrt{\Phi\Psi}(\Phi+\Psi)}{2}+6\Phi\Psi-4\Psi(\Phi+\Psi)\right] \\
&= \frac{1}{4}\left[\Phi\sqrt{\Phi\Psi}-4\Phi(\Phi+\Psi)\right]-\frac{1}{4}\left[\Psi\sqrt{\Phi\Psi}-4\Psi(\Phi+\Psi)\right] \\
&= \frac{1}{4}\left[(\Phi-\Psi)\sqrt{\Phi\Psi}-4(\Phi-\Psi)(\Phi+\Psi)\right] \\
&= \frac{(\Phi-\Psi)}{4}\left[\sqrt{\Phi\Psi}-4(\Phi+\Psi)\right] \\
\Phi\frac{\partial f}{\partial \Phi}-\Psi\frac{\partial f}{\partial \Psi} &= \frac{(\Phi-\Psi)}{4}[G-8A] \\
(\ln\Phi-\ln\Psi)(\frac{\partial f}{\partial \Phi}-\frac{\partial f}{\partial \Psi}) &= (\ln\Phi-\ln\Psi)\frac{(\Phi-\Psi)}{4}[G-8A]\geq 0
\end{aligned}
$$

This proves $f$ is Schur geometrically convex.

Case-(3) Schur harmonically convex.

$$\Phi^2\frac{\partial f}{\partial\Phi}-\Psi^2\frac{\partial f}{\partial\Psi}=\frac{\Phi^2}{4}\left[\sqrt{\Phi\Psi}+\frac{\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}+6\Psi-4(\Phi+\Psi)\right]$$
$$-\frac{\Psi^2}{4}\left[\sqrt{\Phi\Psi}+\frac{\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}+6\Phi-4(\Phi+\Psi)\right]$$
$$=\frac{1}{4}\left[\Phi^2\sqrt{\Phi\Psi}+\Phi^2\frac{\sqrt{\Psi}(\Phi+\Psi)}{2\sqrt{\Phi}}+6\Phi^2\Psi-4\Phi^2(\Phi+\Psi)\right]$$
$$-\frac{1}{4}\left[\Psi^2\sqrt{\Phi\Psi}+\Psi^2\frac{\sqrt{\Phi}(\Phi+\Psi)}{2\sqrt{\Psi}}+6\Phi\Psi^2-4\Psi^2(\Phi+\Psi)\right]$$
$$=\frac{1}{4}\left[(\Phi^2-\Psi^2)\sqrt{\Phi\Psi}+\frac{(\Phi+\Psi)}{2}\left(\frac{\Phi^2\sqrt{\Psi}}{\sqrt{\Phi}}-\frac{\Psi^2\sqrt{\Phi}}{\sqrt{\Psi}}\right)\right.$$
$$\left.+6\Phi^2\Psi-6\Psi^2\Phi-4\Phi^2(\Phi+\Psi)+4\Psi^2(\Phi+\Psi)\right]$$

$$=\frac{1}{4}\left[(\Phi-\Psi)(\Phi+\Psi)\sqrt{\Phi\Psi}+\frac{(\Phi+\Psi)}{2}\left(\frac{\Phi\Psi(\Phi-\Psi)}{\sqrt{\Phi\Psi}}\right)\right.$$
$$\left.+6\Phi\Psi(\Phi-\Psi)-4(\Phi+\Psi)^2(\Phi-\Psi)\right]$$
$$=\frac{(\Phi-\Psi)}{4}\left[(\Phi+\Psi)\sqrt{\Phi\Psi}+\sqrt{\Phi\Psi}\frac{(\Phi+\Psi)}{2}+6\Phi\Psi-4(\Phi+\Psi)^2\right]$$
$$=\frac{(\Phi-\Psi)}{4}\left[2AG+AG+6G^2-16A^2\right]$$
$$=\frac{(\Phi-\Psi)}{4}\left[3AG+6G^2-16A^2\right]$$
$$\Phi^2\frac{\partial f}{\partial\Phi}-\Psi^2\frac{\partial f}{\partial\Psi}=\frac{(\Phi-\Psi)}{4}\left[3(AG+2G^2)-16A^2\right]\geq 0$$
$$(\Phi-\Psi)\Phi^2\frac{\partial f}{\partial\Phi}-\Psi^2\frac{\partial f}{\partial\Psi}=\frac{(\Phi-\Psi)^2}{4}\left[3(AG+2G^2)-16A^2\right]\geq 0$$

This proves $f$ is Schur harmonically convex. □

## 5 Conclusion

We conclude that the results listed in Main results are Schur convex for all positive values of $\Phi-\Psi$

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## Author information


Sandeep Kumar, Department of Mathematics, Presidency University, Bangalore-560 064, India.
E-mail: skumars1504@gmail.com

Sangeetha B. K., Department of Mathematics, Presidency University, Bangalore-560 064, India.
E-mail: vinaysangeetha3@gmail.com



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