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NORMAL HOLONOMY ALONG TRANSNORMAL CURVES IN $\mathbb{R}^4$

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Abstract In this paper we study normal holonomy along transnormal curves in $\mathbb{R}^4$. The idea of normal holonomy will be exploited in the form of rotation to construct transnormal curves parallel to a 2-transnormal curve in $\mathbb{R}^4$.

1 Introduction

Let M be a smooth (C^∞) connected m-manifold without boundary and let $f : M \longrightarrow \mathbb{R}^n$ be a smooth embedding of M into $\mathbb{R}^n$. Let $V = f(M)$. Through each point $p \in V$ there passes a unique m-plane T_pV tangent to V at p and a unique $(n-m)$-plane N_pV normal to V at p. Thus, there are maps T and N with $T(p) = T_pV$ and $N(p) = N_pV$. The m-manifold V is *transnormal* in $\mathbb{R}^n$ iff $\forall p, q \in V$, if $q \in N(p)$, then $N(q) = N(p)$ [15]. We define an equivalence relation on V by writing $p \sim q$ to mean $q \in N(p)$. The factor space $V/\sim$ can be identified with the space of normal planes to V, say W. Also $N : V \to W$ is a covering map [15]. The *generating frame* of V at p is $\phi(p) = V \cap N(p)$. It can be shown that, for all $p, q \in V$, $\phi(p)$ is isometric to $\phi(q)$ [16]. If the cardinality of $\phi(p)$ is r, then V is called an r-*transnormal* manifold. The idea of transnormality was introduced and discussed by S.Robertson [15, 16], and then by B.Wegner [18, 19, 20]. For a survey article see [17].

A relatively new start of the work done on transnormality is due to K.Al-Banawi and S.Carter concerning transnormal curves [2] and transnormal partial tubes [3]. Then in [4, 6, 7], K.Al-Banawi had introduced a study of the geometry of transnormal tori in $\mathbb{R}^4$ regarding their focal points, generating polytopes and radii as tori are spherical partial tubes. In [9], A.Al-sariereh and K.Al-Banawi introduced an example of a transnormal partial tube around a non-transnormal manifold. In [8], K.Al-Banawi studied the order of transnormal manifolds in Euclidean spaces. Most recently, H.Al-Aroud and K.Al-Banawi deduced new results regarding transnormal surfaces in Euclidean spaces [1]. While K.Al-Banawi used Morse theory [12] in [5] to study transnormal embeddings of $\mathbf{S}^1$, here we use normal holonomy to build parallel transnormal curves in $\mathbb{R}^4$ of different orders.

2 Normal Holonomy along Parallel Curves

Let $f : \mathbf{R} \longrightarrow \mathbb{R}^{\mathbf{n}}$ be a regular smooth curve in the Euclidean space $\mathbb{R}^n$ with domain $\mathbf{R}$. Let $\{\nu_0, \ldots, \nu_{n-1}\}$ be a frame field along f such that $\tau_f(t) = \nu_0(t)$ is the unit tangent of f at $f(t)$. The connection forms of the frame field of f are

$$\omega_{ij} = < \frac{d\nu_i}{dt}, \nu_j >, \quad i, j = 0, \ldots, n-1 \tag{2.1}$$

Since $\frac{d}{dt} < \nu_i, \nu_j >= 0$, we have $< \frac{d\nu_i}{dt}, \nu_j > + < \nu_i, \frac{d\nu_j}{dt} >= 0$. Thus, $\omega_{ij} + \omega_{ji} = 0$, $i, j = 0, \ldots, n-1$. In particular $\omega_{ii} = 0$, $i = 0, \ldots, n-1$.
The geometry of f is governed by the connection equations of the frame field of f, which are

$$\frac{d\nu_i}{dt} = \sum_{j=0}^{n-1} \omega_{ij}\nu_j, \quad i = 0, \ldots, n-1 \tag{2.2}$$

Let $N_f(t) = f(t) + \nu_f(t)$ be the affine normal plane of f at $f(t)$ where $\nu_f(t)$ is the normal vector space of f at $f(t)$.

Definition 2.1. [11] Let $f, g : \mathbf{R} \longrightarrow \mathbb{R}^{\mathbf{n}}$ be two regular smooth curves in $\mathbb{R}^n$ with domain $\mathbf{R}$ and affine normal planes N_f and N_g. Then f and g are parallel iff for all $t \in \mathbf{R}$, $N_f(t) = N_g(t)$.

A section S of the normal bundle of f is called *parallel*, with respect to the connection of f, if $\frac{dS}{dt}$ is tangential for all $t \in \mathbf{R}$. If $S = \sum_{j=1}^{n-1} \alpha_j \nu_j$, then $\frac{dS}{dt} = A \frac{df}{dt}$ where A and α_j, $j = 1, \ldots, n-1$, are functions of t.

Proposition 2.2. *Let $f, g : \mathbf{R} \longrightarrow \mathbb{R}^{\mathbf{n}}$ be two regular smooth curves in $\mathbb{R}^n$. Then f and g are parallel iff $g - f$ is a parallel section of the normal bundle of f.*

Proof. Let $S = g - f$. If f and g are parallel, then S is a section of the normal bundle of f and $\nu_f(t) = \nu_g(t)$ for all $t \in \mathbf{R}$. Hence $\tau_f(t) = \tau_g(t)$ for all $t \in \mathbf{R}$. Thus,

$$\frac{dS}{dt} = \frac{dg}{dt} - \frac{df}{dt} = \left(\frac{||\frac{dg}{dt}||}{||\frac{df}{dt}||} - 1\right)\frac{df}{dt}.$$

Hence S is parallel. Conversely, if S is a parallel section of the normal bundle of f, then $\frac{dS}{dt} = A \frac{df}{dt}$ where A is a function of t. Thus, $\frac{dg}{dt} = (A+1)\frac{df}{dt}$, i.e. $\tau_f(t) = \tau_g(t)$, for all $t \in \mathbf{R}$. But S is a section of the normal bundle of f. Hence for all $t \in \mathbf{R}$, $N_f(t) = N_g(t)$, and so f and g are parallel . □

A general proof for parallel immersions is in [19]. Proposition 2.2 suggests a method of constructing curves parallel to f using parallel sections of the normal bundle of f in a method usually called *parallel transport* [10].The local parallel sections of the normal bundle of f can be characterized as solutions of a linear system of ordinary differential equations. This is the idea of the next proposition.

Proposition 2.3. *A section $S = \sum_{j=1}^{n-1} \alpha_j \nu_j$ of the normal bundle of f is parallel iff*

$$\frac{d\alpha_i}{dt} = \sum_{j=1}^{n-1} \omega_{ij} \alpha_j, \quad i = 1, \ldots, n-1 \tag{2.3}$$

Proof. If S is parallel, then

$$\sum_{j=1}^{n-1} \alpha_j \frac{d\nu_j}{dt} + \sum_{j=1}^{n-1} \frac{d\alpha_j}{dt} \nu_j = A \frac{df}{dt}.$$

Thus,

$$\sum_{j=1}^{n-1} \alpha_j < \frac{d\nu_j}{dt}, \nu_i > + \sum_{j=1}^{n-1} \frac{d\alpha_j}{dt} < \nu_j, \nu_i >= 0, \quad i = 1, \ldots, n-1.$$

So

$$\sum_{j=1}^{n-1} \omega_{ji} \alpha_j + \frac{d\alpha_i}{dt} = 0, \quad i = 1, \ldots, n-1.$$

That is,

$$\frac{d\alpha_i}{dt} = \sum_{j=1}^{n-1} \omega_{ij} \alpha_j, \quad i = 1, \ldots, n-1.$$

Conversely, if $S = \sum_{j=1}^{n-1} \alpha_j \nu_j$ and $\frac{d\alpha_i}{dt} = \sum_{j=1}^{n-1} \omega_{ij} \alpha_j$, $i = 1, \ldots, n-1$, then for $i = 1, \ldots, n-1$,

$$< \frac{dS}{dt}, \nu_i >= \sum_{j=1}^{n-1} \alpha_j < \frac{d\nu_j}{dt}, \nu_i > + \sum_{j=1}^{n-1} \frac{d\alpha_j}{dt} < \nu_j, \nu_i > \tag{2.4}$$

$$= \sum_{j=1}^{n-1} \alpha_j \omega_{ji} + \frac{d\alpha_i}{dt} \tag{2.5}$$

$$= \sum_{j=1}^{n-1} \omega_{ji} \alpha_j + \sum_{j=1}^{n-1} \omega_{ij} \alpha_j = 0 \tag{2.6}$$

Thus, $\frac{dS}{dt}$ is tangential. Hence S is parallel. □

Recall that the notation $f : [a, b] \longrightarrow \mathbb{R}^4$ means that f is a differentiable map that is one to one on $[a, b)$ with $f(a) = f(b)$. The evaluation of the solution of the above system at $t_0 + b - a$ for given initial conditions at t_0 defines a linear isometry, the *normal holonomy map*

$$hol : N_f(t_0) \longrightarrow N_f(t_0 + b - a) = N_f(t_0),$$

which is orientation preserving. The idea of normal holonomy will be exploited in the form of rotation to construct transnormal curves parallel to a 2-transnormal curve in $\mathbb{R}^4$.

3 Transnormal Parallel Curves in $\mathbb{R}^4$

Let $f : [a, b] \longrightarrow \mathbb{R}^4$ be a smooth simple closed curve in $\mathbb{R}^4$ with $\{\nu_0, \nu_1, \nu_2, \nu_3\}$ as a frame field along f such that $\tau_f(t) = \nu_0(t)$ is the unit tangent of f at $f(t)$. Assume that ν_1 is parallel. Thus, $\nu = \alpha_2\nu_2 + \alpha_3\nu_3$ is a parallel section of the normal bundle of f iff

$$\frac{d\alpha_2}{dt} = \omega_{23}\alpha_3 \quad \text{and} \quad \frac{d\alpha_3}{dt} = \omega_{32}\alpha_2 \tag{3.1}$$

Since hol is an isometry, it is natural to set $\alpha_2 = \cos\theta$, $\alpha_3 = \sin\theta$. Thus,

$$\frac{d\theta}{dt} = \omega_{32},$$

and so

$$\theta(t) = \int_a^t \omega_{32} dt + \theta(a) \tag{3.2}$$

This shows that the normal holonomy map of f is given by a rotation around ν_1 from ν_2 by the angle $\theta(b) - \theta(a)$. The angle $\theta(b) - \theta(a)$ is called the *normal holonomy angle* of f [13]. The angle $\theta(t) - \theta(a)$ is called the *normal holonomy* along f [14]. If f is 2-transnormal in $\mathbb{R}^4$, then normal holonomy allows the construction of transnormal curves parallel to f. The condition that ν_1 is parallel is satisfied when $\nu_1(t_1)$ is chosen as

$$\nu_1(t_1) = \frac{f(t_1^*) - f(t_1)}{||f(t_1^*) - f(t_1)||}$$

where $f(t_1^*)$ is the opposite point of $f(t_1)$. The unit normal ν_1 is parallel since

$$\frac{d\nu_1}{dt}(t_1) = -\frac{2}{||f(t_1^*) - f(t_1)||}\frac{df}{dt}(t_1).$$

Let

$$\mathcal{S} = \{(f(t), \nu) : t \in \mathbb{R}, \nu = \rho(\nu_2\cos\theta(t) + \nu_3\sin\theta(t)), \rho \in \mathbb{R}^+ \cup \{0\}\}.$$

Let

$$\mathcal{S}^\xi f([a, b]) = \{(f(t), \nu) \in \mathcal{S}, ||\nu|| = \xi\}.$$

The next corollary is a special case of Lemma 1 in [3].

Corollary 3.1. *Let $f : [a, b] \longrightarrow \mathbb{R}^4$ be a 2-transnormal curve in $\mathbb{R}^4$ with $p^* = f(t^*)$ being the opposite point of $p = f(t)$, $t, t^* \in [a, b]$. Then for all $\xi > 0$ sufficiently small, $\eta|\mathcal{S}^\xi f([a, b])$ is an immersion and for all $p \in f([a, b])$, for all $(q, \nu) \in \mathcal{S}^\xi f([a, b])$, $\eta(q, \nu) \in N_f(p)$ iff $q \in \{p, p^*\}$.*

The next theorem follows a method suggested by Wegner in [20].

Theorem 3.2. *Let $f : [0, 2\pi] \longrightarrow \mathbb{R}^4$ be a 2-transnormal curve in $\mathbb{R}^4$ with $f(t_1 + \pi)$ being the opposite point of $f(t_1)$, $t_1 \in [0, 2\pi]$. Assume that $\theta(t) = \frac{t}{r}$ is the normal holonomy along f, $r \in \mathbb{R}^+$. If r is rational, then there exists a transnormal embedding g in $\mathbb{R}^4$ of finite order parallel to f. If r is irrational, then there exists an injective transnormal immersion g in $\mathbb{R}^4$ of infinite order parallel to f.*

Proof. By the above argument, the term $\nu = \nu_2 \cos \frac{t}{r} + \nu_3 \sin \frac{t}{r}$ is a parallel section of the normal bundle of f with $\omega_{32} = \frac{1}{r}$. Choose $\xi \in \mathbb{R}^+$ as in Corollary 3.1, then

$$g = f + \xi(\nu_2 \cos \frac{t}{r} + \nu_3 \sin \frac{t}{r}) \tag{3.3}$$

is an immersion in $\mathbb{R}^4$ parallel to f. If r is rational, then $r = \frac{c}{d}$, $c, d \in \mathbf{N}$ with $gcd(c, d) = 1$. Thus, the holonomy angle of f is $\frac{2\pi d}{c}$. Since $gcd(c, d) = 1$, the least natural number k satisfying the equation

$$k \times \frac{2\pi d}{c} \equiv 0 \text{ mod } 2\pi \tag{3.4}$$

is c. Hence the curve g joins up after c periods of f, i.e. the curve g is defined on $[0, 2\pi c]$. The point $g(0)$ lies on the ray containing $\nu_2(0)$ with a distance ξ from $f(0)$. Now for $i = 1, \ldots, c$, the point $g(2\pi i)$ on g is the result of the rotation of $g(0)$ about ν_1 by an angle equal to $\frac{2\pi i}{c}$. Such a point lies in the plane spanned by $\nu_2(0), \nu_3(0)$. The points $g(0)$ and $g(2\pi c)$ coincide. Thus, if $t_1 \in [0, 2\pi]$, then the plane spanned by $\nu_2(t_1), \nu_3(t_1)$ intersects g at c different points, namely $g(t_1 + 2\pi i)$, $i = 0, \ldots, c-1$. The points are the vertices of a regular c-gon, and so they form a transitive set which lies on a circle centred at $f(t_1)$. Since f is 2-transnormal, then $N_f(t_1)$ also contains another set of points on g which are the vertices of another regular c-gon centred at $f(t_1+\pi)$. The points are $g(t_1+\pi+2\pi i)$, $i = 0, \ldots, c-1$. If $g(t) \in N_g(t_1)$, then $g(t) \in N_f(t_1)$. By Corollary 1, $f(t)$ is either $f(t_1)$ or $f(t_1 + \pi)$. Thus, if $Im(g)$ is the image of g, then

$$N_g(t_1) \cap Im(g) = \cup_{i=0}^{c-1}\{g(t_1 + 2\pi i), g(t_1 + \pi + 2\pi i)\}.$$

The affine normal plane of g at all the above $2c$ points is $N_f(t_1) = N_g(t_1)$. Hence g is a $2c$-transnormal embedding in $\mathbb{R}^4$. If r is irrational, then the equation $k \times \frac{2\pi}{r} \equiv 0 \text{ mod } 2\pi$ has no solution for all $k \in \mathbf{Z} - \{0\}$. Hence the curve g will not join up and

$$N_g(t_1) \cap Im(g) = \cup_{i=0}^{\infty}\{g(t_1 + 2\pi i), g(t_1 + \pi + 2\pi i)\}.$$

Again the affine normal plane of g at all the above points is $N_f(t_1) = N_g(t_1)$. Hence g is a transnormal immersion in $\mathbb{R}^4$ of infinite order. The immersion g is injective on $[0, \infty)$. □

It should be mentioned here that the proof of Theorem 3.2 gives a good choice of numbers to build transnormal curves of finite orders parallel to f. Simply, if $r = \frac{c}{d}$, $c, d \in \mathbf{N}$ with $gcd(c, d) = 1$, then the curve g is $2c$-transnormal. Also it is assumed in the proof that $c \geq 3$. If $c = 1$, the generating frame of g at $g(t_1)$ is $\{g(t_1), g(t_1 + \pi)\}$. If $c = 2$, the generating frame of g is the vertices of a tetrahedron. When $c \geq 3$, the generating frame of g at $g(t_1)$ is the vertices of two regular c-gons centred at $f(t_1)$,$f(t_1 + \pi)$. Since the two regular c-gons are contained in two parallel planes, the generating polytope of g is a regular right prism or a twisted regular right prism.

The curve in the next example is due to Wegner [20].

Example 3.3. Consider the embedding f of $\mathbf{S}^1$ in $\mathbb{R}^4$ defined by

$$f(t) = (\sin t, \cos t, R \sin 3t, R \cos 3t) \tag{3.5}$$

where $0 < R < \frac{1}{\sqrt{3}}$ and t is taken mod 2π. The curve f is 2-transnormal [20].
An orthonormal field along f is

$$\tau_f(t) = \frac{1}{\sqrt{1+9R^2}}(\cos t, -\sin t, 3R\cos 3t, -3R \sin 3t), \tag{3.6}$$

$$\nu_1(t) = -\frac{1}{\sqrt{1+R^2}}(\sin t, \cos t, R\sin 3t, R\cos 3t), \tag{3.7}$$

$$\nu_2(t) = \frac{1}{\sqrt{1+9R^2}}(-3R\cos t, 3R\sin t, \cos 3t, -\sin 3t), \tag{3.8}$$

$$\nu_3(t) = \frac{1}{\sqrt{1+R^2}}(-R\sin t, -R\cos t, \sin 3t, \cos 3t) \tag{3.9}$$

with $\tau_f(t)$ being the unit tangent of f at $f(t)$.

The unit normal ν_1 is parallel, and so $\omega_{12} = \omega_{21} = \omega_{13} = \omega_{31} = 0$. Also

$$\frac{d\nu_2}{dt} = \frac{-3\sqrt{1+R^2}}{\sqrt{1+9R^2}}\nu_3 \tag{3.10}$$

$$\frac{d\nu_3}{dt} = \frac{8R}{\sqrt{1+R^2}\sqrt{1+9R^2}}\tau_f + \frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}\nu_2 \tag{3.11}$$

Thus, $\omega_{32} = \frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}$, and so the normal holonomy of f is $\theta_R(t) = \frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t$, and the normal holonomy angle is $\frac{6\pi\sqrt{1+R^2}}{\sqrt{1+9R^2}}$. Such an angle depends on R, and hence is denoted by θ_R.
If $\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3$ is a parallel section of the normal bundle of f, then

$$\frac{d\alpha_1}{dt} = 0 \tag{3.12}$$

$$\frac{d\alpha_2}{dt} = \frac{-3\sqrt{1+R^2}}{\sqrt{1+9R^2}}\alpha_3 \tag{3.13}$$

$$\frac{d\alpha_3}{dt} = \frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}\alpha_2 \tag{3.14}$$

The general solution of the above system is

$$\alpha_1 = \mu \tag{3.15}$$

$$\alpha_2 = \xi\cos\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t + \bar{\xi}\sin\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t \tag{3.16}$$

$$\alpha_3 = \xi\sin\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t - \bar{\xi}\cos\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t \tag{3.17}$$

where ξ,$\bar{\xi}$ and μ are constants.
Let $\bar{\xi} = \mu = 0$. A parallel section of the normal bundle of f is

$$\nu = \xi(\nu_2\cos\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t + \nu_3\sin\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t).$$

A parallel curve to f is defined by

$$g(t) = f(t) + \xi(\nu_2(t)\cos\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t + \nu_3(t)\sin\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t) \tag{3.18}$$

where t is taken mod 2π and ξ as in Corollary 1. The curve g is parallel to f since

$$\frac{dg}{dt} = (1 + \frac{8\xi R}{(1+9R^2)\sqrt{1+R^2}}\sin\frac{3\sqrt{1+R^2}}{\sqrt{1+9R^2}}t)\frac{df}{dt}.$$

It is possible to construct transnormal curves parallel to f of different orders by choosing suitable values of R. To construct a $2r$-transnormal curve, $r \geq 1$, consider the equation

$$\frac{6\pi\sqrt{1+R^2}}{\sqrt{1+9R^2}} = \frac{2\pi k}{r} \tag{3.19}$$

where $k \in \mathbf{N}$ and $\gcd(k,r) = 1$. The last equation reduces to

$$R^2 = \frac{9r^2 - k^2}{9k^2 - 9r^2} \tag{3.20}$$

Since $0 < R^2 < \frac{1}{3}$, k is chosen such that $\sqrt{3}r < k < 3r$. But $\gcd(3r-1,r) = 1$ and for $r \geq 1$, $\sqrt{3}r < 3r - 1 < 3r$. Thus, choose $k = 3r - 1$, and so

$$R = \frac{1}{3}\sqrt{\frac{6r-1}{(4r-1)(2r-1)}} \tag{3.21}$$

If $r = 1$, then $k = 2$, $R = \frac{1}{3}\sqrt{\frac{5}{3}}$, $\theta_{\frac{1}{3}\sqrt{\frac{5}{3}}} = 4\pi$ and g is also 2-transnormal.

A suitable odd multiple of π can serve as a holonomy angle of f, which leads to a 4-transnormal curve parallel to f. In this case $r = 2$, $k = 5$, and hence $R = \frac{1}{3}\sqrt{\frac{11}{21}}$, $\theta_{\frac{1}{3}\sqrt{\frac{11}{21}}} = 5\pi$. The curve is

$$g(t) = f(t) + \xi(\nu_2(t)\cos\frac{5}{2}t + \nu_3(t)\sin\frac{5}{2}t) \tag{3.22}$$

where f, ν_2 and ν_3 are the ones with $R = \frac{1}{3}\sqrt{\frac{11}{21}}$ and $t \in [0, 4\pi]$.
The curve is 4-transnormal with the generating frame

$$\{g(t), g(t+\pi), g(t+2\pi), g(t+3\pi)\}.$$

For a 6-transnormal curve parallel to f, $r = 3$, $k = 8$, and so $R = \frac{1}{3}\sqrt{\frac{17}{55}}$, $\theta_{\frac{1}{3}\sqrt{\frac{17}{55}}} = \frac{16\pi}{3}$. The curve is

$$g(t) = f(t) + \xi(\nu_2(t)\cos\frac{8}{3}t + \nu_3(t)\sin\frac{8}{3}t) \tag{3.23}$$

where f, ν_2 and ν_3 are the ones with $R = \frac{1}{3}\sqrt{\frac{17}{55}}$ and $t \in [0, 6\pi]$.
The curve is 6-transnormal with the generating frame

$$\{g(t), g(t+\pi), \ldots, g(t+5\pi)\}.$$

For an 8-transnormal curve parallel to f, $r = 4$, $k = 11$, and so $R = \frac{1}{3}\sqrt{\frac{23}{105}}$, $\theta_{\frac{1}{3}\sqrt{\frac{23}{105}}} = \frac{11\pi}{2}$. The curve is

$$g(t) = f(t) + \xi(\nu_2(t)\cos\frac{11}{4}t + \nu_3(t)\sin\frac{11}{4}t) \tag{3.24}$$

where f, ν_2 and ν_3 are the ones with $R = \frac{1}{3}\sqrt{\frac{23}{105}}$ and $t \in [0, 8\pi]$.
The curve is 8-transnormal with the generating frame

$$\{g(t), g(t+\pi), \ldots, g(t+7\pi)\}.$$

In general, for any $r \geq 2$, the curve

$$g(t) = f(t) + \xi(\nu_2(t)\cos\frac{3r-1}{r}t + \nu_3(t)\sin\frac{3r-1}{r}t) \tag{3.25}$$

where f, ν_2 and ν_3 are the ones with $R = \frac{1}{3}\sqrt{\frac{6r-1}{(4r-1)(2r-1)}}$ and $t \in [0, 2\pi r]$, is a $2r$-transnormal curve parallel to f with a holonomy angle $\frac{2\pi(3r-1)}{r}$.

If R is chosen such that the holonomy angle θ_R is an irrational multiple of 2π, then the equation

$$k\theta_R \equiv 0 \text{ mod } 2\pi \tag{3.26}$$

has no solution for all $k \in \mathbf{Z} - \{0\}$, and so the result will be an injective immersion of $\mathbb{R}$ into $\mathbb{R}^4$ having an infinite order of transnormalilty. As an example, if $R = \frac{1}{3}$, then $\theta_{\frac{1}{3}} = 2\sqrt{5}\pi$ and the curve which is parallel to f is of infinite order of transnormality.

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