Palestine Journal of Mathematics

Vol 13(2)(2024) , 202–209

A note on the structure of $\mathcal{U}(\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3))$

R. K. Sharma, Yogesh Kumar and D. C. Mishra

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16S34; Secondary 20C05.

Keywords and phrases: Group algebra, Wedderburn decomposition, Unit group.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *A complete characterization of the unit group $\mathcal{U}(\mathbb{F}_qG)$ of the group algebra $\mathbb{F}_qG$ of non–abelian group G of order 27 with exponent 3 over any finite field $\mathbb{F}_q$ is obtained.*

1 Introduction

Let $\mathbb{F}G$ denotes the group algebra of the group G over the finite field $\mathbb{F}$, $\mathcal{U}(\mathbb{F}G)$ denote the unit group of the group algebra $\mathbb{F}G$. It is well known that by using Wedderburn-Malcev theorem ([1], $p.491$), we have

$$\mathcal{U}(\mathbb{F}G) \cong (1 + J(\mathbb{F}G)) \rtimes \mathcal{U}(\frac{\mathbb{F}G}{J(\mathbb{F}G)}).$$

Thus a nice description of Wedderburn decomposition of $\frac{\mathbb{F}G}{J(\mathbb{F}G)}$ is always very helpful to determine unit group of $\mathbb{F}G$. See [2] for further details of the group algebras.

Determination of the unit group of group algebras has been always very fascinating and challenging. A lot of work has been done in finding the algebraic structure of the unit group of the group algebra $\mathbb{F}G$([3]–[21]). In 2010, Gildea[22] determined the unit group of group algebra $\mathbb{F}_{3^k}(C_3 \times D_6)$. Again in 2011, Gildea([23]– [24]) determined unit group of group algebras $\mathbb{F}_{2^k}(C_2 \times D_8)$ and $\mathbb{F}_{3^k}(C_3^2 \rtimes C_2)$. In [25], we have characterized a possible unit group of group algebra $\mathbb{F}_{p^n}S_5$, if $p > 5$. In this paper, we have characterized completely the unit group of the group algebra $\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$ for any finite field $\mathbb{F}_q$, where $\mathbb{Z}_n$ is the multiplicative group of integral modulo n and $H \rtimes K$ denotes the semidirect product with H normal.

There are certain techniques to find the decomposition of group algebra $\mathbb{F}_qG$, when field characteristic does not divide the order of the group. In this paper we use Ferraz[26] techniques. Sandling[27] completely solved the problem to determine the unit group of group algebra $\mathbb{F}_qG$, in case group G is finite abelian p-group. When group G is non-abelian, there are many papers in the literature devoted to particular p-groups G, but the problem is not understood in full generality. Even if $\mathbb{F}$ be finite field, having p elements for some prime p, and G is a p-group, it is certainly not easy to describe the unit group of the modular group algebra $\mathbb{F}G$. The Modular Isomorphism Problem which asks whether non-isomorphic p-groups have always non-isomorphic p-modular group algebras is still open problem.

Units of group rings are of paramount importance from an application point of view. Hurley [28] establishes a relationship between the ring of matrices and group rings. Additionally, Hurley [29] and Dholakia [30] provide a method for constructing convolution codes using units from group rings. To obtain the derivation of rings, refer to the references [31] through [33].

We compute the disjoint conjugacy classes of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. There are 11 of them. For this, we have used the presentation of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ given by

$$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3 = \{a, b, c : a^3 = b^3 = c^3 = e, ab = ba, ac = ca, cb = abc\}.$$

The 11 conjugacy classes of the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ are given as follows:

$$\begin{aligned}
&[e],\ [a],\ [a^2],\\
[b] &= \{b, ab, a^2b\},\\
[b^2] &= \{b^2, ab^2, a^2b^2\},\\
[c] &= \{c, ac, a^2c\},\\
[bc] &= \{bc, cb, acb\},\\
[c^2] &= \{c^2, ac^2, a^2c^2\},\\
[b^2c] &= \{b^2c, ab^2c, cb^2\},\\
[bc^2] &= \{bc^2, abc^2, c^2b\},\\
[b^2c^2] &= \{b^2c^2, c^2b^2, ac^2b^2\}.
\end{aligned}$$

2 Preliminaries

Throughout this paper, we denote a finite field with $q = p^n$ elements as $\mathbb{F} = \mathbb{F}_q$, and G represents a finite group.

We utilize results from Ferraz's work [26].

An element $x \in G$ is termed as p-regular if $p \nmid o(x)$, where $o(x)$ denotes the order of the element x. Let s be the least common multiple (l.c.m.) of the orders of the p-regular elements of G, and θ be a primitive s-th root of unity over $\mathbb{F}$. We define the multiplicative group $T_{G,\mathbb{F}}$ as:

$$T_{G,\mathbb{F}} = \{t \mid \theta \to \theta^t \text{ is an automorphism of } \mathbb{F}(\theta) \text{ over } \mathbb{F}\}.$$

For p-regular elements g, which we denote as γ_g, we consider the sum of all conjugates of g in G. The cyclotomic $\mathbb{F}$-class of γ_g will be denoted by the set:

$$S_{\mathbb{F}}(\gamma_g) = \{\gamma_{g^t} \mid t \in T_{G,\mathbb{F}}\}.$$

The following will be necessary for our discussion.

Proposition 2.1. *[26] The number of simple components of $\frac{\mathbb{F}G}{J(\mathbb{F}G)}$ is equal to the number of cyclotomic $\mathbb{F}$-classes in G.*

Proposition 2.2. *[26] Suppose the Galois group $Gal(\mathbb{F}(\theta) : \mathbb{F})$ is cyclic and t be the number of cyclotomic $\mathbb{F}-$classes in G. If $K_1, K_2, \cdots, K_t$ are the simple components of $\mathcal{Z}(\frac{\mathbb{F}G}{J(\mathbb{F}G)})$ and $S_1, S_2, \cdots, S_t$ are the cyclotomic $\mathbb{F}-$classes of G, then $|S_i| = [K_i : \mathbb{F}]$ with a suitable ordering of the indices.*

Proposition 2.3. *[2] (Perlis Walker) Let G be a finite abelian group of order n, and K be a field such that $char(K) \nmid n$. Then*

$$KG \cong \oplus_{d/n} a_d K(\zeta_d)$$

where ζ_d denotes a primitive root of unity of order d, $a_d = \frac{n_d}{[K(\zeta_d):K]}$, and n_d denotes the number of elements of order d.

Proposition 2.4. *([34], $p.110$) Let N be a normal subgroup of G such that G/N is p-solvable. If $|G/N| = np^a$ where $(p, n) = 1$ then*

$$J(\mathbb{F}G)^{p^a} \subseteq FG \cdot J(FN) \subseteq J(\mathbb{F}G)$$

In particular, if G is p-solvable of order np^a where $(p, n) = 1$, then

$$J(\mathbb{F}G)^{p^a} = 0.$$

Proposition 2.5. *([2], Proposition 3.6.11) Let $\mathbb{F}G$ be a semisimple group algebra. If $G^{'}$ denotes the commutator subgroup of G, then we can write*

$$\mathbb{F}G \cong \mathbb{F}G_{e_{G'}} \oplus \triangle(G, G^{'}),$$

where $\mathbb{F}G_{e_{G'}} \cong \mathbb{F}(\frac{G}{G'})$ is the sum of all commutative simple components of $\mathbb{F}G$ and $\triangle(G, G^{'})$ is the sum of all the others. Here $e_{G'} = \frac{\widehat{G'}}{|G'|}$ where $\widehat{G'}$ is the sum of all elements of $G^{'}$.

3 The Structure of $\mathcal{U}(\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)), p \neq 3.$

Lemma 3.1. *Let $q = p^n$, where $p \neq 3$ is a prime. The Wedderburn decomposition of $\mathbb{F}_q(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is given as follows:*

Table 1.

Condition on n	Wedderburn decomposition of $\mathbb{F}_q(\mathbb{Z}_3 \times \mathbb{Z}_3)$
n is even	$\mathbb{F}_q \oplus 8\mathbb{F}_q$
n is odd with $p \equiv 1 \mod 3$	$\mathbb{F}_q \oplus 8\mathbb{F}_q$
n is odd with $p \equiv -1 \mod 3$	$\mathbb{F}_q \oplus 4\mathbb{F}_{q^2}$

Proof. When n is even, $p^n \equiv 1 \mod 3$, i.e., $3 \mid (p^n - 1)$. This implies that the field $\mathbb{F}_q$ has a primitive third root of unity. Therefore, by Proposition 2.3, we have

$$\mathbb{F}_q(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q$$

Now, let n be odd. We divide this case into two subcases:
(1) $p \equiv 1 \mod 3$ and (2) $p \equiv -1 \mod 3$.

Subcase 1. If $p \equiv 1 \mod 3$ and n is odd, then $p^n \equiv 1 \mod 3$. This implies that the field $\mathbb{F}_q$ has a primitive third root of unity, and hence, by Proposition 2.3, we have

$$\mathbb{F}_q(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q.$$

Subcase 2. If $p \equiv -1 \mod 3$ and n is odd, then $p^n \not\equiv 1 \mod 3$. In this case, the field $\mathbb{F}_q$ doesn't have a primitive third root of unity, and hence, again by Proposition 2.3, we have

$$\mathbb{F}_q(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 4\mathbb{F}_q(\omega)$$

where $\omega \notin \mathbb{F}_q$ is a third root of unity. Now, by using the simple fact that $\mathbb{F}_q(\omega) \cong \mathbb{F}_{q^2}$, we have the lemma. □

Theorem 3.2. *Let $q = p^n, p \neq 3$ be a prime. The Wedderburn decomposition of $\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$ is given as follows:*

Table 2.

condition on n	Wedderburn decomposition of $\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$
n is even	$\mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q)$
n is odd with $p \equiv 1 \mod 3$	$\mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q)$
n is odd with $p \equiv -1 \mod 3$	$\mathbb{F}_q \oplus 4\mathbb{F}_{q^2} \oplus \mathbb{M}_3(\mathbb{F}_{q^2})$

Proof. Observe that $\frac{\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)}{\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)'} \cong \mathbb{F}_q(C_3 \times C_3)$, as $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)' \cong C_3 \times C_3$, where C_3 denotes a cyclic group of order 3. Also, $|(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3| = 27 = 3^3$. Hence, the group algebra $\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$ is semi-simple, as $p \nmid |G|$. Here, $q = p^n$, with $p \neq 3$. By Proposition 2.5, we have

$$\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3) = \mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)e_{((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)'} \oplus \mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)(((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)'-1))$$

where $e_{((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)'} = e_{(C_3\times C_3)} = \frac{\widehat{(C_3\times C_3)'}}{|C_3\times C_3|} = \frac{\sum_{\sigma\in(C_3\times C_3)}\sigma}{9}$, and

$\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)e_{((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)'}$ = sum of all commutative simple components of $\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)$

However,

$$\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)e_{((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)'} \cong \mathbb{F}_q(\frac{((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)}{((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)'}) \cong \mathbb{F}_q(C_3\times C_3) \cong \mathbb{F}_q\oplus 8\mathbb{F}_q \text{ or } \mathbb{F}_q\oplus 4\mathbb{F}_{q^2}.$$

This gives the Wedderburn decomposition

$$\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3) \cong \mathbb{F}_q\oplus 8\mathbb{F}_q\oplus\sum_{i=1}^{2}\mathbb{M}_{n_i}(\mathbb{F}_{q^{k_i}})$$

or

$$\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3) \cong \mathbb{F}_q\oplus 4\mathbb{F}_{q^2}\oplus\sum_{i=1}^{2}\mathbb{M}_{n_i}(\mathbb{F}_{q^{k_i}})$$

for $n_i \geq 2$.

We divide the proof in two cases.

Case 1: When n is even and $p^n \equiv 1 \mod 3$, i.e., the field $\mathbb{F}_q$ contains a primitive third root of unity. In this case, we have

$$S_{\mathbb{F}_q(\gamma_g)} = \{\gamma_g\} \quad \text{for each group element } g,$$

which means

$$|S_{\mathbb{F}_q(\gamma_g)}| = 1$$

for each $g \in (\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3$. This leads to the conclusion

$$\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3) \cong \mathbb{F}_q\oplus 8\mathbb{F}_q\oplus\sum_{i=1}^{2}\mathbb{M}_{n_i}(\mathbb{F}_q),$$

by Lemma 3.1. By dimension constraints, we have

$$\dim_{\mathbb{F}_q}(\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3)) = 1+8+n_1^2+n_2^2,$$

$$27 = 1+8+n_1^2+n_2^2,$$

$$18 = n_1^2+n_2^2.$$

This equation has only one solution, namely, $n_1 = 3$ and $n_2 = 3$. Hence, in this case, we have

$$\mathbb{F}_q((\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_3) \cong \mathbb{F}_q\oplus 8\mathbb{F}_q\oplus\mathbb{M}_3(\mathbb{F}_q)\oplus\mathbb{M}_3(\mathbb{F}_q).$$

Case 2: Let n be odd. We further divide this case into two subcases:

(i) $p \equiv 1 \mod 3$.

(ii) $p \equiv -1 \mod 3$.

Subcase 1: If $p \equiv 1 \mod 3$, and n is odd, observe that

$$S_{\mathbb{F}_q(\gamma_g)} = \{\gamma_g\} \quad \text{for each group element } g,$$

which means

$$|S_{\mathbb{F}_q(\gamma_g)}| = 1$$

for each $g \in (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. Again, as $p \equiv 1 \mod 3$ and n is odd, we have $p^n \equiv 1 \mod 3$, indicating that the field $\mathbb{F}_q$ contains a primitive third root of unity. Thus, as in Case 1 when n is even, by the above lemma, we obtain

$$\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 8\mathbb{F}_q \oplus \mathbb{M}_3(\mathbb{F}_q) \oplus \mathbb{M}_3(\mathbb{F}_q).$$

Subcase 2: When $p \equiv -1 \mod 3$ and n is odd, we have

$$S_{\mathbb{F}_q(\gamma_e)} = \{\gamma_e\} \text{ i.e. } |S_{\mathbb{F}_q(\gamma_e)}| = 1,$$

$$S_{\mathbb{F}_q(\gamma_a)} = \{\gamma_a, \gamma_{a^2}\} \text{ i.e. } |S_{\mathbb{F}_q(\gamma_a)}| = 2,$$

$$S_{\mathbb{F}_q(\gamma_b)} = \{\gamma_b, \gamma_{b^2}\} \text{ i.e. } |S_{\mathbb{F}_q(\gamma_b)}| = 2,$$

$$S_{\mathbb{F}_q(\gamma_c)} = \{\gamma_c, \gamma_{c^2}\} \text{ i.e. } |S_{\mathbb{F}_q(\gamma_c)}| = 2,$$

$$S_{\mathbb{F}_q(\gamma_{bc})} = \{\gamma_{bc}, \gamma_{b^2c^2}\} \text{ i.e. } |S_{\mathbb{F}_q(\gamma_{bc})}| = 2,$$

$$S_{\mathbb{F}_q(\gamma_{b^2c})} = \{\gamma_{b^2c}, \gamma_{bc^2}\} \text{ i.e. } |S_{\mathbb{F}_q(\gamma_{b^2c})}| = 2,$$

That is, in this subcase, we have $2 = |S_i| = |K_i : \mathbb{F}|$ in Proposition 2.2 for $i \neq 1$. Again, since $p \equiv -1 \mod 3$ and n is odd, so $p^n \not\equiv 1 \mod 3$. By using the above lemma, we have

$$\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 4\mathbb{F}_{q^2} \oplus \mathbb{M}_{n_1}(\mathbb{F}_{q^2}).$$

Equating dimensions on both sides, we have

$$\dim_{\mathbb{F}_q}(\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)) = 1 + 4 \cdot 2 + 2n_1^2,$$

$$27 = 9 + 2n_1^2,$$

$$18 = 2n_1^2.$$

The only possible solution to the equation is $n_1 = 3$.
Hence, in this subcase, we have

$$\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \cong \mathbb{F}_q \oplus 4\mathbb{F}_{q^2} \oplus \mathbb{M}_3(\mathbb{F}_{q^2}).$$

□

Corollary 3.3. *Let $q = p^n$, where $p > 3$ be a prime, then $\mathcal{U}(\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3))$ i.e. unit group of the group ring $\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$ is given as follows :*

Table 3.

condition on n	$\mathcal{U}(\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3))$
n is even	$\mathbb{F}_q^* \times 8\mathbb{F}_q^* \times GL_3(\mathbb{F}_q) \times GL_3(\mathbb{F}_q)$
n is odd with $p \equiv 1 \mod 3$	$\mathbb{F}_q^* \times 8\mathbb{F}_q^* \times GL_3(\mathbb{F}_q) \times GL_3(\mathbb{F}_q)$
n is odd with $p \equiv -1 \mod 3$	$\mathbb{F}_q^* \times 4\mathbb{F}_{q^2}^* \times GL_3(\mathbb{F}_{q^2})$

Proof. Proof is simple application of the fact that for any two ring R_1 and R_2, $\mathcal{U}(R_1 \oplus R_2) = \mathcal{U}(R_1) \times \mathcal{U}(R_2)$. □

4 The Structure of $\mathcal{U}(\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)), q = 3^n$.

If $\mathbb{F}_q$ is a finite field of characteristic p, and G is a finite p-group, then the Jacobson radical is the same as the augmentation ideal, and hence the dimension of the Jacobson radical, $dim_{\mathbb{F}_q} J(\mathbb{F}_q G) = |G| - 1$. We have the following results.

Theorem 4.1. *Let $q = 3^n$, then $\frac{\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)}{J(\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3))} \cong \mathbb{F}_q$.*

Proof. An element $x \in (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ is 3-regular if $3 \nmid o(x)$. In our group under consideration, the only 3-regular element is the identity element, denoted as e. Hence by Proposition 2.1, we have $\frac{\mathbb{F}_q((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)}{J((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3))} \cong \mathbb{F}_q$. □

Lemma 4.2. *([34], p.321) If the group $M(p)$ is defined as*

$$M(p) = \langle a, b, c \mid a^p = b^p = c^p = 1, ab = ba, ac = ca, cb = abc \rangle,$$

then the index of nilpotency of $J(\mathbb{F}_{p^m} M(p))$ is $4p - 3$.

Theorem 4.3. *Let $q = 3^n$ and $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. Then $\mathcal{U}(\mathbb{F}_q G) \cong (1 + J(\mathbb{F}_q G)) \rtimes \mathbb{F}_q^*$, where $1 + J(\mathbb{F}_q G)$ is a non-abelian group of exponent 9.*

Proof. By Theorem 4.1, we have $\mathcal{U}(\frac{\mathbb{F}_q G}{J(\mathbb{F}_q G)}) \cong \mathbb{F}_q^*$. As G is non-abelian, $bc \neq cb$ for some $b, c \in G$. Then, $b - 1$ and $c - 1$ are elements of the augmentation ideal $\Delta(G) \subseteq J(\mathbb{F}_q G)$. Therefore, $b = b - 1 + 1$ and $c = c - 1 + 1$ are two non-commutating elements of $1 + J(\mathbb{F}_q G)$. This proves that $1 + J(\mathbb{F}_q G)$ is non-abelian.

Since the group G is 3-solvable, by Proposition 2.4, we have $(1 + J(\mathbb{F}_q G))^{27} = 1$, which means that the exponent of $1 + J(\mathbb{F}_q G)$ is $3, 9$, or 27. Now, using Lemma 4.2, we find that the index of nilpotency of $J(\mathbb{F}_q G)$, $4p - 3 = 9$, and hence the group $1 + J(\mathbb{F}_q G)$ is of exponent 3 or 9.

Observe that $b - 1 \in J(\mathbb{F}_q G)$, so $cb - c \in J(\mathbb{F}_q G)$, and

$$(cb - c)^2 = (cb)^2 - cbc - c^2 b + c^2$$

$$\begin{aligned}(cb - c)^3 &= 1 - (cb)^2 c - cbc^2 b + cbc^2 - c^2 bcb + c^2 bc + b - 1 \\ &= -b^2 - ab^2 + ab - a^2 b^2 + a^2 b + b \neq 0.\end{aligned}$$

This shows that $1 + J(\mathbb{F}_q G)$ is a non-abelian subgroup of $\mathcal{U}(\mathbb{F}_q G)$ with an exponent of 9. □

5 The Structure of $\mathcal{U}(\mathbb{F}_q((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)), q = p^n$.

In this section, we generalize the result from the previous section to the group $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$. This group is a semidirect product of the groups $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p$, where p is an arbitrary odd prime. It has the following presentation:

$$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p = \langle a, b, c \mid a^p = b^p = c^p = e,\ ab = ba,\ ac = ca,\ cb = abc \rangle.$$

Theorem 5.1. *Let $q = p^n$ and $G = (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ for an odd prime p. Then $\frac{\mathbb{F}_q G}{J(\mathbb{F}_q G)} \cong \mathbb{F}_q$.*

Proof. An element $x \in G$ is p-regular if $p \nmid o(x)$. Such an element is the identity element e only, as each non-identity element of G is of order p. Therefore, $\frac{\mathbb{F}_q G}{J(\mathbb{F}_q G)}$ has only one simple component with dimension 1, i.e., $\frac{\mathbb{F}_q G}{J(\mathbb{F}_q G)} \cong \mathbb{F}_q$ as $\dim \mathbb{F}_q G = p^3 - 1$ over $\mathbb{F}_q$. □

Theorem 5.2. *Let $q = p^n$ and $G = (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ for an odd prime p. Then $\mathcal{U}(\mathbb{F}_q G) \cong (1 + J(\mathbb{F}_q G)) \rtimes \mathbb{F}_q^*$, where $1 + J(\mathbb{F}_q G)$ is a non-abelian subgroup of $\mathcal{U}(\mathbb{F}_q(G))$ with exponent p or p^2.*

Proof. By Theorem 5.1, we have $\mathcal{U}(\frac{\mathbb{F}_qG}{J(\mathbb{F}_qG)}) \cong \mathbb{F}_q^*$. As G is non-abelian, $bc \neq cb$ for some $b, c \in G$. Now, $b-1$ and $c-1$ are elements of the augmentation ideal $\Delta(G) \subseteq J(\mathbb{F}_qG)$. Therefore, $b = b-1+1$ and $c = c-1+1$ are two non-commutating elements of $1+J(\mathbb{F}_qG)$. This proves that $1+J(\mathbb{F}_qG)$ is non-abelian.

Since the group G is p-solvable, by Proposition 2.4, we have $(1+J(\mathbb{F}_qG))^{p^3} = 1$. This implies that the exponent of $1+J(\mathbb{F}_qG)$ is p^i, where $1 \leq i \leq 3$. However, Lemma 4.2 states that the index of nilpotency of $J(\mathbb{F}_qG)$ is equal to $4p-3$, and since $p < 4p-3 \leq p^2$ for an odd prime p, the exponent of $1+J(\mathbb{F}_qG)$ is equal to p or p^2. This proves the theorem. □

References

[1] *C. W. Curtis and I. Reiner,* Representation Theory of Finite Groups and Associative Algebras, *Pure and Applied Mathematics, Vol. XI, Interscience Publishers, a division of John Wiley & Sons, New York-London, (1962).*

[2] *C. P. Miles and S. K. Sehgal,* An introduction to group rings*, Kluwer Academic Publishers, Dordrecht, (2002).*

[3] *A. A. Bovdi and A. Szakacs,* Unitary subgroup of the group of units of a modular group algebra of a finite abelian p-group, *Math. Zametki,* **45***, 23–29, (1989).*

[4] *A. A. Bovdi and L. Erdei,* Unitary units in the modular group algebra of the group of order 16, *Technical Reports Debrecen, Dept. of Math., L. Kossuth Univ.,* **96***, 57–72, (1996).*

[5] *A. A. Bovdi and A. L. Rosa,* On the order of the unitary subgroup of a modular group algebra, *Comm. Algebra,* **28***, 1897-1905, (2000).*

[6] *L. Creedon and J. Gildea.* Unitary units of the group a algrbra $\mathbb{F}_{2^k}Q_8$, International Journal of Algebra and Computation, **19***, 283–286, (2009).*

[7] *R. Sandling,* Units in the modular group algebra of a finite abelian p-group, *J. Pure. Algebra,* **33***, 337–347, (1984).*

[8] *J. Gildea, F. Monaghan,* Unit of some group algebras of group of order 12 over any field of characteristic 3*, Algebra Discrete Math.,* **11***, 46–58, (2011).*

[9] *J. Gildea,* The structure of the unitary units of the group algebra $\mathbb{F}_{2^k}D_8$, *Int. Electron. J. Algebra,* **9***, 171–176, (2011).*

[10] *J. Gildea,* The structure of the unit group of the group algebra $\mathbb{F}_{2^k}A_4$, *Czechoslovak. Math. J.,* **61***, 531–539, (2011).*

[11] *J. Gildea and L. Creedon,* The structure of the unit group of the group algebra $\mathbb{F}_{2^k}D_8$, *Canad. Math. Bull.,* **54***, 237–243, (2011).*

[12] *N. Makhijani, R. K. Sharma and J. B. Srivastava,* A note on units in $\mathbb{F}_{p^m}D_{2p^n}$*, Acta Math. Acad. Paedagog. Nyhazi.,* **30***, 17–25, (2014).*

[13] *N. Makhijani, R. K. Sharma and J. B. Srivastava,* The unit group of algebra of circulant matrices, *Int.J.Group Theory,* **3***, 13–16, (2014).*

[14] *N. Makhijani, R. K. Sharma and J. B. Srivastava,* The unit group of finite group algebra of a generalized dihedral group, *Asian European J. Math.,* **7***, 1450034, (2014).*

[15] *N. Makhijani, R. K. Sharma, and J. B. Srivastava,* Units in $\mathbb{F}_{2^k}D_{2n}$, *Int. J. Group Theory,* **3***, 25–34, (2014).*

[16] *N. Makhijani, R. K. Sharma and J. B. Srivastava,* A note on the structure of $\mathbb{F}_{p^k}A_5/J(\mathbb{F}_{p^k}A_5)$, *Acta Sci. Math(Szeged),* **82***, 29–43, (2016).*

[17] *R. K. Sharma and P. Yadav,* Unit group of algebra of circulant matrices*, Int. J. Group Theory,* **2***, 1–6, (2013).*

[18] *R. K. Sharma and P. Yadav,* The unit group of $\mathbb{Z}_pQ_8$, *Algebras Groups Geom.,* **25***, 425–429, (2008).*

[19] *R. K. Sharma, J. B. Srivastava and M. Khan,* The unit group of $\mathbb{F}S_3$, *Acta Math. Acad. Paedagog. Nyhazi.,* **23,** *129–142, (2007).*

[20] *R. K. Sharma, J. B. Srivastava, M. Khan,* The unit group of $\mathbb{F}A_4$, *Publ. Math. Debrecen,* **71***, 21–26, (2007).*

[21] *S. Maheshwari and R. K. Sharma,* The unit group of group algebra $\mathbb{F}_qSL(2,3)$, *J. Algebra Comb. Discrete Appl.,* **3***, 1–6, (2016).*

[22] *J. Gildea,* The structure of the unit group of the group algebra $\mathbb{F}_{3^k}(C_3 \times D_6)$, *Comm. Algebra,* **38***, 3311–3317, (2010).*

[23] *J. Gildea,* Units of the group algebra $\mathbb{F}_{2^k}(C_2 \times D_8)$, *J. Algebra Appl.,* **10***, 643–647, (2011).*

[24] *J. Gildea,* Units of the group algebra $\mathbb{F}_{3^k}(C_3^2 \rtimes C_2)$, *Acta Sci. Math. (Szeged),* **77**, *53–62, (2011).*

[25] *Y. Kumar, R. K. Sharma and J. B. Srivastava,* Unit group of the group algebra $\mathbb{F}S_5$, *Acta Math. Acad. Paedagog. Nyhazi.,* **33**, *187–193, (2017).*

[26] *R. A. Ferraz,* Simple components of center of $FG/J(FG)$ *Comm. Algebra,* **36**, *3191–3199, (2008).*

[27] *R. Sandling,* Units in the modular group algebra of a finite abelian p-group, *J. Pure. Algebra,* **33**, *337–347, (1984).*

[28] *T. Hueley,* Group rings and ring of matrices, *Int.J.Pure Appl. Math.,* **31**, *319–335, (2006).*

[29] *T. Hueley,* Convolutional codes from units in matrix and group rings, *Int. J. Pure Appl. Math.,* **50**, *431–463, (2009).*

[30] *A. Dholakia,* Introduction to Convolutional Codes and Applications, *Kluwer Academic Publishers Norwell, USA, (1994).*

[31] *M. Sahai and S. F. Ansari,* Derivations and Centralizers in Rings, *Palestine Journal of Mathematics,* **11**, *413–419, (2022).*

[32] *M. Sahai, P. Kumari and R. K. Sharma,* PRESENTATIONS OF GENERAL LINEAR GROUPS WITH JORDAN REGULAR GENERATORS, *Palestine Journal of Mathematics,* **9**, *01–14, (2020).*

[33] *S. Ali,* On generalized $*$-derivations in $*$-rings, *Palestine Journal of Mathematics,* **1**, *32–37, (2012).*

[34] *G. Karpilovsky, The Jacobson radical of group algebras, North-Holland, Amsterdam, (1987).*

Author information

R. K. Sharma, Department of Mathematics, Indian Institute of Technology Delhi, New Delhi-110016, India.
E-mail: rksharmaiitd@gmail.com

Yogesh Kumar, Department of Mathematics, Indian Institute of Technology Delhi, New Delhi-110016, India.
E-mail: kumaryogeshiitd@gmail.com

D. C. Mishra, Department of Mathematics, Govt. P. G. College Jaiharikhal, Uttrakhand, India.
E-mail: deepiitdelhi@gmail.com

Received: 2023-03-02
Accepted: 2023-10-23