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Bounds for the Topological Indices of ℧-graph

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Abstract. Topological indices are mathematical measure which correlates to the chemical structures of any simple finite graph. These are used for Quantitative Structure-Activity Relationship (QSAR) and Quantitative Structure-Property Relationship (QSPR). In this paper, we define operator graph namely, ℧-graph and structured properties. Also, establish the limits for few topological indices namely, Inverse sum indeg index, first zagreb index, first reformulated Zagreb index, Sombar index and nirmala index of ℧-graph.

1 Introduction

Harold Wiener formulated topological index in 1947 while he was working on boiling point of paraffin and he named this index as path number, now its is called as Wiener index [13]. Since then many other topological indices have been defined and studied. Topological indices are numerical invariants that are associated with the topological characterization of a compound. The properties of chemical compounds such as toxicological, physicochemical, pharmacological are closely related to topological indices.

Now we recall some well known topological indices.

The first zagreb index was introduced by Gutman.I and Trinajstic.N (1972) [2] and is defined as

$$M_1[G] = \sum_{uv\in E(G)} [d_u + d_v]$$

The first reformulated zagreb index was introduced by A.Milicevic et al. (2004)[9] and is defined as

$$EM_1[G] = \sum_{uv\in E(G)} [d_u + d_v - 2]^2$$

The Sombar index was defined by Gutman in[3] as

$$SO[G] = \sum_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$$

Kulli[4] introduced the Nirmala index of a graph G and it is defined as

$$N[G] = \sum_{uv\in E(G)} \sqrt{d_G(u) + d_G(v)}$$

In 2010, Vukicevic and Gasperow [10,12] introduced bond-additive topological index namely, inverse sum indeg index as a significant predictor of total surface area of octane isomers and is defined as.

$$ISI[G] = \sum_{uv\in E(G)} \left(\frac{d_u d_v}{d_u + d_v}\right)$$

Definition 1.1. Let G and H be two simple connected graphs with n_1,n_2 and m_1,m_2 vertices and edges respectively. Then edge SR-Corona of G and H is a graph which is obtained by taking one copy of $S(G)$ and m_1 copies of $R(H)$ and each i^{th} vertex $[1 \leq i \leq m_1]$ of $I[G]$ [$I(G)$ is a vertex set which are inserted vertices to each edge of G] is adjacent to each vertex of i^{th} copy of $R(H)$.

Theorem 1.2. *Let ℧ be a edge SR-Corona graph, then bounds for the first zagreb index of ℧-graph are given by*

$$M_1[\mho] \leq 2m_1[\Delta_G + 2 + n_2 + m_2] + m_1m_2[8\Delta_H + 15 + n_2 + m_2] + m_1n_2[2\Delta_H + 3 + n_2 + m_2]$$
$$and$$
$$M_1[\mho] \geq 2m_1[\delta_G + 2 + n_2 + m_2] + m_1m_2[8\delta_H + 15 + n_2 + m_2] + m_1n_2[2\delta_H + 3 + n_2 + m_2].$$

Proof.

$$\begin{aligned} M_1[\mho] &= 2m_1[d_G + 2 + n_2 + m_2] + m_1m_2[2d_H + 1 + 2d_H + 1] + 2m_1m_2[2d_H + 1 + 3] \\ &+ m_1n_2[2d_H + 1 + 2 + n_2 + m_2] + m_1m_2[3 + 2 + n_2 + m_2] \\ &= 2m_1[d_G + 2 + n_2 + m_2] + m_1m_2[4d_H + 2] + 2m_1m_2[2d_H + 4] + m_1n_2[2d_H \\ &+ 3 + n_2 + m_2] + m_1m_2[5 + n_2 + m_2] \\ . &= 2m_1[d_G + 2 + n_2 + m_2] + m_1m_2[4d_H + 2 + 4d_H + 8 + 5 + n_2 + m_2] \\ &+ m_1n_2[2d_H + 3 + n_2 + m_2] \\ &= 2m_1[d_G + 2 + n_2 + m_2] + m_1m_2[8d_H + 15 + n_2 + m_2] + m_1n_2[2d_H + 3 + n_2 \\ &+ m_2] \\ M_1[\mho] &\leq 2m_1[\Delta_G + 2 + n_2 + m_2] + m_1m_2[8\Delta_H + 15 + n_2 + m_2] + m_1n_2[2\Delta_H + 3 + n_2 \\ &+ m_2] \\ Similarly, & \\ M_1[\mho] &\geq 2m_1[\delta_G + 2 + n_2 + m_2] + m_1m_2[8\delta_H + 15 + n_2 + m_2] + m_1n_2[2\delta_H + 3 + n_2 \\ &+ m_2] \end{aligned}$$

This proves the result.

□

Example 1.3. Let G and H are two simple connected graphs, then ℧-graph is

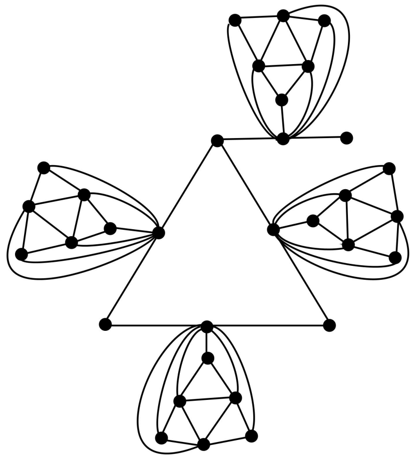

Figure 1: ℧-graph

$$
\begin{aligned}
M_1[\mho] &= \sum_{uv \in E(G)} [d_u + d_v] \\
&= 24(3+5) + 4(8+2) + 15(8+3) + 12(5+5) + 12(8+5) + 1(8+1) \\
&= 24(8) + 4(10) + 15(11) + 12(10) + 12(13) + 9 \\
&= 682 \\
M_1[\mho] &\leq 2m_1[\Delta_G + 2 + n_2 + m_2] + m_1m_2[8\Delta_H + 15 + n_2 + m_2] + m_1n_2[2\Delta_H + 3 + n_2 \\
&+ m_2] \\
&\leq 8[3+2+3+3] + 12[8(2) + 15 + 3 + 3] + 12[2(2) + 3 + 3 + 3] \\
&\leq 8[11] + 12[37] + 12[13] \\
&= 688 \\
M_1[\mho] &\geq 2m_1[\delta_G + 2 + n_2 + m_2] + m_1m_2[8\delta_H + 15 + n_2 + m_2] + m_1n_2[2\delta_H + 3 + n_2 \\
&+ m_2] \\
&\geq 8[1+2+3+3] + 12[8(2) + 15 + 3 + 3] + 12[2(2) + 3 + 3 + 3] \\
&\geq 8[9] + 12[37] + 12[13] \\
&= 672
\end{aligned}
$$

Theorem 1.4. *Let $\mho$ be a edge SR-Corona graph, then bounds for the first reformulated zagreb index of $\mho$-graph are given by*

$$
\begin{aligned}
EM_1[\mho] &\leq 2m_1[\Delta_G + n_2 + m_2]^2 + m_1m_2[24\Delta_H^2 + 17 + 16\Delta_H + n_2^2 + m_2^2 + 6n_2 + 2n_2m_2 \\
&\quad + 6m_2] + m_1n_2[2\Delta_H + 1 + n_2 + m_2]^2 \\
&\text{and} \\
EM_1[\mho] &\geq 2m_1[\delta_G + n_2 + m_2]^2 + m_1m_2[24\delta_H^2 + 17 + 16\delta_H + n_2^2 + m_2^2 + 6n_2 + 2n_2m_2 \\
&\quad + 6m_2] + m_1n_2[2\delta_H + 1 + n_2 + m_2]^2.
\end{aligned}
$$

Proof.

$$\begin{aligned}
EM_1[\mho] &= 2m_1[d_G + 2 + n_2 + m_2 - 2]^2 + m_1m_2[2d_H + 1 + 2d_H + 1 - 2]^2 \\
&+ 2m_1m_2[2d_H + 1 + 3 - 2]^2 + m_1n_2[2d_H + 1 + 2 + n_2 + m_2 - 2]^2 \\
&+ m_1m_2[3 + 2 + n_2 + m_2 - 2]^2 \\
&= 2m_1[d_G + n_2 + m_2]^2 + m_1m_2[4d_H]^2 + 2m_1m_2[2d_H + 2]^2 \\
&+ m_1n_2[2d_H + 1 + n_2 + m_2]^2 + m_1m_2[3 + n_2 + m_2]^2 \\
&= 2m_1[d_G + n_2 + m_2]^2 + m_1m_2[4d_H]^2 + 2m_1m_2[2d_H + 2]^2 + m_1n_2[2d_H + 1 \\
&+ n_2 + m_2]^2 + m_1m_2[3 + n_2 + m_2]^2 \\
&= 2m_1[d_G + n_2 + m_2]^2 + m_1m_2[16d_H^2 + 2(4d_H^2 + 4 + 8d_H) + (3 + n_2 + m_2)^2] \\
&+ m_1n_2[2d_H + 1 + n_2 + m_2]^2 \\
&= 2m_1[d_G + n_2 + m_2]^2 + m_1m_2[24d_H^2 + 8 + 16d_H + (3 + n_2 + m_2)^2] \\
&+ m_1n_2[2d_H + 1 + n_2 + m_2]^2 \\
&= 2m_1[d_G + n_2 + m_2]^2 + m_1m_2[24d_H^2 + 17 + 16d_H + n_2^2 + m_2^2 + 6n_2 + 2n_2m_2 \\
&+ 6m_2] + m_1n_2[2d_H + 1 + n_2 + m_2]^2 \\
EM_1[\mho] &\leq 2m_1[\Delta_G + n_2 + m_2]^2 + m_1m_2[24\Delta_H^2 + 17 + 16\Delta_H + n_2^2 + m_2^2 + 6n_2 + 2n_2m_2 \\
&+ 6m_2] + m_1n_2[2\Delta_H + 1 + n_2 + m_2]^2 \\
Similarly, & \\
EM_1[\mho] &\geq 2m_1[\delta_G + n_2 + m_2]^2 + m_1m_2[24\delta_H^2 + 17 + 16\delta_H + n_2^2 + m_2^2 + 6n_2 + 2n_2m_2 \\
&+ 6m_2] + m_1n_2[2\delta_H + 1 + n_2 + m_2]^2
\end{aligned}$$

This proves the result. □

Example 1.5. Let G and H are two simple connected graphs, then ℧-graph is

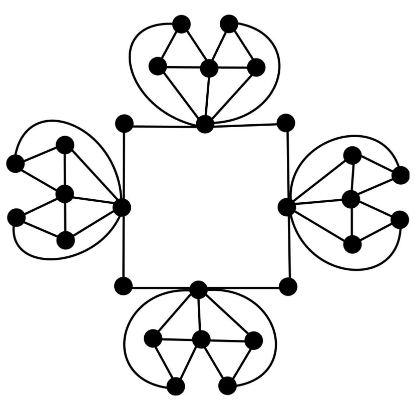

Figure 2: ℧-graph

$$\begin{aligned}
EM_1[\mho] &= \sum_{uv\in E(G)} [d_u + d_v - 2]^2 \\
&= 8(3+3-2)^2 + 8(7+2-2)^2 + 4(7+5-2)^2 + 16(5+3-2)^2 + 16(7+3-2)^2 \\
&= 8(16) + 8(49) + 4(100) + 16(36) + 16(64) \\
&= 2520 \\
EM_1[\mho] &\leq 2m_1[\Delta_G + n_2 + m_2]^2 + m_1m_2[24\Delta_H^2 + 17 + 16\Delta_H + n_2^2 + m_2^2 + 6n_2 + 2n_2m_2 \\
&+ 6m_2] + m_1n_2[2\Delta_H + 1 + n_2 + m_2]^2 \\
&\leq 8[2+3+2]^2 + 8[24(4) + 17 + 16(2) + 9 + 4 + 6(3) + 2(6) + 6(2)] + 12[2(2) \\
&+ 1 + 3 + 2]^2 \\
&\leq 8[49] + 8[96 + 17 + 32 + 13 + 18 + 24] + 12[100] \\
&= 3192 \\
EM_1[\mho] &\geq 2m_1[\delta_G + n_2 + m_2]^2 + m_1m_2[24\delta_H^2 + 17 + 16\delta_H + n_2^2 + m_2^2 + 6n_2 + 2n_2m_2 \\
&+ 6m_2] + m_1n_2[2\delta_H + 1 + n_2 + m_2]^2 \\
&\geq 8[2+3+2]^2 + 8[24(1) + 17 + 16(1) + 9 + 4 + 6(3) + 2(6) + 6(2)] + 12[2(1) \\
&+ 1 + 3 + 2]^2 \\
&\geq 8[49] + 8[24 + 17 + 16 + 13 + 18 + 24] + 12[64] \\
&= 2056
\end{aligned}$$

Theorem 1.6. *Let $\mho$ be a edge SR-Corona graph, then bounds for the Sombar index of $\mho$-graph are given by*

$$\begin{aligned}
SO[\mho] &\leq 2m_1\sqrt{\Delta_G^2 + (2 + n_2 + m_2)^2} + m_1m_2\{\sqrt{2(2\Delta_H + 1)^2} + 2\sqrt{4\Delta_H^2 + 1 + 4\Delta_H + 9} \\
&+ \sqrt{3^2 + (2 + n_2 + m_2)^2}\} + m_1n_2\sqrt{(2\Delta_H + 1)^2 + (2 + n_2 + m_2)^2} \\
&\quad and \\
SO[\mho] &\geq 2m_1\sqrt{\delta_G^2 + (2 + n_2 + m_2)^2} + m_1m_2\{\sqrt{2(2\delta_H + 1)^2} + 2\sqrt{4\delta_H^2 + 1 + 4\delta_H + 9} \\
&+ \sqrt{3^2 + (2 + n_2 + m_2)^2}\} + m_1n_2\sqrt{(2\delta_H + 1)^2 + (2 + n_2 + m_2)^2}
\end{aligned}$$

Proof. Consider,

$$\begin{aligned}
SO[\mho] &= 2m_1\sqrt{d_G^2+(2+n_2+m_2)^2}+m_1m_2\sqrt{(2d_H+1)^2+(2d_H+1)2}\\
&+2m_1m_2\sqrt{(2d_H+1)^2+3^2}+m_1n_2\sqrt{(2d_H+1)^2+(2+n_2+m_2)^2}\\
&+m_1m_2\sqrt{3^2+(2+n_2+m_2)^2}\\
&=2m_1\sqrt{d_G^2+(2+n_2+m_2)^2}+m_1m_2\sqrt{2(2d_H+1)^2}\\
&+2m_1m_2\sqrt{4d_H^2+1+4d_H+9}+m_1n_2\sqrt{(2d_H+1)^2+(2+n_2+m_2)^2}\\
&+m_1m_2\sqrt{3^2+(2+n_2+m_2)^2}\\
&=2m_1\sqrt{d_G^2+(2+n_2+m_2)^2}+m_1m_2\{\sqrt{2(2d_H+1)^2}\\
&+2\sqrt{4d_H^2+1+4d_H+9}+\sqrt{3^2+(2+n_2+m_2)^2}\}\\
&+m_1n_2\sqrt{(2d_H+1)^2+(2+n_2+m_2)^2}\\
SO[\mho] &\le 2m_1\sqrt{\Delta_G^2+(2+n_2+m_2)^2}+m_1m_2\{\sqrt{2(2\Delta_H+1)^2}\\
&+2\sqrt{4\Delta_H^2+1+4\Delta_H+9}+\sqrt{3^2+(2+n_2+m_2)^2}\}\\
&+m_1n_2\sqrt{(2\Delta_H+1)^2+(2+n_2+m_2)^2}
\end{aligned}$$

$Similarly,$

$$\begin{aligned}
SO[\mho] &\ge 2m_1\sqrt{\delta_G^2+(2+n_2+m_2)^2}+m_1m_2\{\sqrt{2(2\delta_H+1)^2}\\
&+2\sqrt{4\delta_H^2+1+4\delta_H+9}+\sqrt{3^2+(2+n_2+m_2)^2}\}\\
&+m_1n_2\sqrt{(2\delta_H+1)^2+(2+n_2+m_2)^2}
\end{aligned}$$

This proves the result. □

Example 1.7. Let G and H are two simple connected graphs, then ℧-graph is

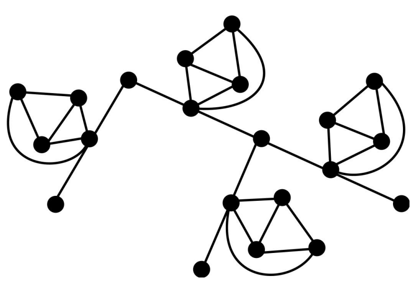

Figure 3: ℧-graph

$$\begin{aligned}
SO[\mho] &= \sum_{uv\in E(G)}\sqrt{d_G(u)^2+d_G(v)^2}\\
&=12\sqrt{9+9}+15\sqrt{9+25}+2\sqrt{25+4}+3\sqrt{25+1}\\
&=12\sqrt{18}+15\sqrt{34}+2\sqrt{29}+3\sqrt{26}\\
&=317.41394
\end{aligned}$$

$$\begin{aligned}
SO[\mho] &\leq 2m_1\sqrt{\Delta_G^2+(2+n_2+m_2)^2}+m_1m_2\{\sqrt{2(2\Delta_H+1)^2}+2\sqrt{4\Delta_H^2+1+4\Delta_H+9}\\
&+\sqrt{3^2+(2+n_2+m_2)^2}\}+m_1n_2\sqrt{(2\Delta_H+1)^2+(2+n_2+m_2)^2}\\
&\leq 8\sqrt{9+(2+2+1)^2}+4\{\sqrt{2(2+1)^2}+2\sqrt{4+1+4+9}+\sqrt{3^2+(2+2+1)^2}\}\\
&+8\sqrt{(2+1)^2+(2+2+1)^2}\\
&\leq 8\sqrt{9+25}+4\{\sqrt{18}+2\sqrt{18}+\sqrt{34}\}+8\sqrt{34}\\
&=344.8831
\end{aligned}$$

$$\begin{aligned}
SO[\mho] &\geq 2m_1\sqrt{\delta_G^2+(2+n_2+m_2)^2}+m_1m_2\{\sqrt{2(2\delta_H+1)^2}+2\sqrt{4\delta_H^2+1+4\delta_H+9}\\
&+\sqrt{3^2+(2+n_2+m_2)^2}\}+m_1n_2\sqrt{(2\delta_H+1)^2+(2+n_2+m_2)^2}\\
&\geq 8\sqrt{1+(2+2+1)^2}+4\{\sqrt{2(2+1)^2}+2\sqrt{4+1+4+9}+\sqrt{3^2+(2+2+1)^2}\}\\
&+8\sqrt{(2+1)^2+(2+2+1)^2}\\
&\geq 8\sqrt{26}+4\{\sqrt{18}+2\sqrt{18}+\sqrt{34}\}+8\sqrt{34}\\
&=161.6752
\end{aligned}$$

Theorem 1.8. *Let $\mho$ be a edge SR-Corona graph, then bounds for the Nirmal index of $\mho$-graph are given by*

$$\begin{aligned}
N[\mho] &\leq 2m_1\sqrt{\Delta_G+2+n_2+m_2}+m_1m_2\left[\sqrt{4\Delta_H+2}+2\sqrt{2\Delta_H+4}+\sqrt{5+n_2+m_2}\right]\\
&+m_1n_2\sqrt{2\Delta_H+3+n_2+m_2}\\
&and\\
N[\mho] &\geq 2m_1\sqrt{\delta_G+2+n_2+m_2}+m_1m_2\{\sqrt{4\delta_H+2}+2\sqrt{2\delta_H+4}+\sqrt{5+n_2+m_2}\}\\
&+m_1n_2\sqrt{2\delta_H+3+n_2+m_2}+m_1m_2\sqrt{3+2+n_2+m_2}
\end{aligned}$$

Proof.

$$\begin{aligned}
N[\mho] &= 2m_1\sqrt{d_G+2+n_2+m_2}+m_1m_2\sqrt{2d_H+1+2d_H+1}+2m_1m_2\sqrt{2d_H+1+3}\\
&+m_1n_2\sqrt{2d_H+1+2+n_2+m_2}+m_1m_2\sqrt{3+2+n_2+m_2}\\
&= 2m_1\sqrt{d_G+2+n_2+m_2}+m_1m_2\left[\sqrt{4d_H+2}+2\sqrt{2d_H+4}+\sqrt{5+n_2+m_2}\right]\\
&+m_1n_2\sqrt{2d_H+3+n_2+m_2}\\
N[\mho] &\leq 2m_1\sqrt{\Delta_G+2+n_2+m_2}+m_1m_2\left[\sqrt{4\Delta_H+2}+2\sqrt{2\Delta_H+4}+\sqrt{5+n_2+m_2}\right]\\
&+m_1n_2\sqrt{2\Delta_H+3+n_2+m_2}
\end{aligned}$$

$Similarly,$

$$\begin{aligned}
N[\mho] &\geq 2m_1\sqrt{\delta_G+2+n_2+m_2}+m_1m_2\left[\sqrt{4\delta_H+2}+2\sqrt{2\delta_H+4}+\sqrt{5+n_2+m_2}\right]\\
&+m_1n_2\sqrt{2\delta_H+3+n_2+m_2}
\end{aligned}$$

This proves the result. □

Example 1.9. Let G and H are two simple connected graphs, then $\mho$-graph is

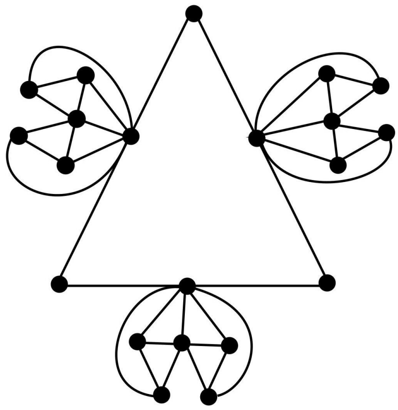

Figure 4: ℧-graph

$$
\begin{aligned}
N[\mho] &= \sum_{uv\in E(G)} \sqrt{d_G(u)+d_G(v)}\\
&= 12\sqrt{7+3}+3\sqrt{7+5}+6\sqrt{2+7}+6\sqrt{3+3}+12\sqrt{5+3}\\
&= 12\sqrt{10}+3\sqrt{12}+6\sqrt{9}+6\sqrt{6}+12\sqrt{8}\\
&= 114.9777
\end{aligned}
$$

$$
\begin{aligned}
N[\mho] &\leq 2m_1\sqrt{\Delta_G+2+n_2+m_2}+m_1m_2\left[\sqrt{4\Delta_H+2}+2\sqrt{2\Delta_H+4}+\sqrt{5+n_2+m_2}\right]\\
&+m_1n_2\sqrt{2\Delta_H+3+n_2+m_2}\\
&\leq 6\sqrt{2+2+2+3}+6\left[\sqrt{8+2}+2\sqrt{4+4}+\sqrt{5+2+3}\right]+9\sqrt{4+3+2+3}\\
&\leq 6\sqrt{9}+6\left[\sqrt{10}+2\sqrt{8}+\sqrt{10}\right]+9\sqrt{12}\\
&= 121.06537
\end{aligned}
$$

$$
\begin{aligned}
N(G) &\geq 2m_1\sqrt{\delta_G+2+n_2+m_2}+m_1m_2\{\sqrt{4\delta_H+2}+2\sqrt{2\delta_H+4}+\sqrt{5+n_2+m_2}\}\\
&+m_1n_2\sqrt{2\delta_H+3+n_2+m_2}\\
&\geq 6\sqrt{2+2+3+2}+6\{\sqrt{4+2}+2\sqrt{2+4}+\sqrt{5+2+3}\}+9\sqrt{2+3+2+3}\\
&\geq 6\sqrt{9}+6\{\sqrt{6}+2\sqrt{6}+\sqrt{10}\}+9\sqrt{10}\\
&= 109.52498
\end{aligned}
$$

Theorem 1.10. *Let ℧ be a edge SR-Corona graph, then bounds for the inverse sum indeg index of ℧-graph are given by*

$$
\begin{aligned}
ISI(\mho) \leq 2m_1&\left[\frac{\Delta_G(2+n_2+m_2)}{\Delta_G+2+n_2+m_2}\right]+m_1m_2\left[\left[\frac{2\Delta_H+1}{2}\right]+\left[\frac{3(2\Delta_H+1)}{\Delta_H+2}\right]+\left[\frac{6+3n_2+3m_2}{5+n_2+m_2}\right]\right]\\
&+m_1n_2\left[\frac{(2\Delta_H+1)(2+n_2+m_2)}{2\Delta_H+3+n_2+m_2}\right].
\end{aligned}
$$

and

$$
\begin{aligned}
ISI(\mho) \geq 2m_1&\left[\frac{\delta_G(2+n_2+m_2)}{\delta_G+2+n_2+m_2}\right]+m_1m_2\left[\left[\frac{2\delta_H+1}{2}\right]+\left[\frac{3(2\delta_H+1)}{\delta_H+2}\right]+\left[\frac{6+3n_2+3m_2}{5+n_2+m_2}\right]\right]\\
&+m_1n_2\left[\frac{(2\delta_H+1)(2+n_2+m_2)}{2\delta_H+3+n_2+m_2}\right].
\end{aligned}
$$

Proof.

$$\begin{aligned}
ISI(\mho) &= 2m_1\left[\frac{d_G(2+n_2+m_2)}{d_G+2+n_2+m_2}\right]+m_1m_2\left[\frac{(2d_H+1)(2d_H+1)}{2d_H+1+2d_H+1}\right]+2m_1m_2\left[\frac{(2d_H+1)3}{2d_H+3+1}\right]\\
&+m_1n_2\left[\frac{(2d_H+1)(2+n_2+m_2)}{2d_H+1+2+n_2+m_2}\right]+m_1m_2\left[\frac{3(2+n_2+m_2)}{3+2+n_2+m_2}\right]\\
&=2m_1\left[\frac{d_G(2+n_2+m_2)}{d_G+2+n_2+m_2}\right]+m_1m_2\left[\frac{4d_H^2+4d_H+1}{4d_H+2}\right]+2m_1m_2\left[\frac{6d_H+3}{2d_H+4}\right]\\
&+m_1n_2\left[\frac{(2d_H+1)(2+n_2+m_2)}{2d_H+3+n_2+m_2}\right]+m_1m_2\left[\frac{6+3n_2+3m_2)}{5+n_2+m_2}\right]\\
&=2m_1\left[\frac{d_G(2+n_2+m_2)}{d_G+2+n_2+m_2}\right]+m_1m_2\left[\frac{(2d_H+1)^2}{2(2d_H+1)}\right]+2m_1m_2\left[\frac{6d_H+3}{2d_H+4}\right]\\
&+m_1n_2\left[\frac{(2d_H+1)(2+n_2+m_2)}{2d_H+3+n_2+m_2}\right]+m_1m_2\left[\frac{6+3n_2+3m_2)}{5+n_2+m_2}\right]\\
ISI(\mho) &\leq 2m_1\left[\frac{\Delta_G(2+n_2+m_2)}{\Delta_G+2+n_2+m_2}\right]+m_1m_2\left[\left[\frac{2\Delta_H+1}{2}\right]+\left[\frac{3(2\Delta_H+1)}{\Delta_H+2}\right]+\left[\frac{6+3n_2+3m_2}{5+n_2+m_2}\right]\right]\\
&+m_1n_2\left[\frac{(2\Delta_H+1)(2+n_2+m_2)}{2\Delta_H+3+n_2+m_2}\right].
\end{aligned}$$

$Similarly$

$$\begin{aligned}
ISI(\mho) &\geq 2m_1\left[\frac{\delta_G(2+n_2+m_2)}{\delta_G+2+n_2+m_2}\right]+m_1m_2\left[\left[\frac{2\delta_H+1}{2}\right]+\left[\frac{3(2\delta_H+1)}{\delta_H+2}\right]+\left[\frac{6+3n_2+3m_2}{5+n_2+m_2}\right]\right]\\
&+m_1n_2\left[\frac{(2\delta_H+1)(2+n_2+m_2)}{2\delta_H+3+n_2+m_2}\right].
\end{aligned}$$

This proves the result. □

Example 1.11. Let G and H are two simple connected graphs, then $\mho$-graph is

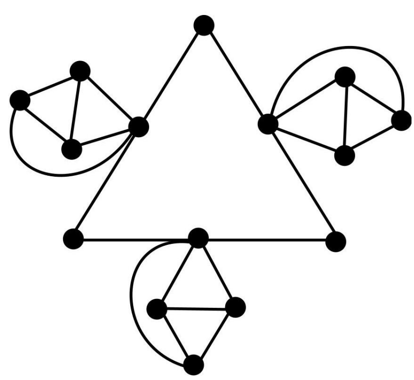

Figure 5: $\mho$-graph

$$
\begin{aligned}
ISI(\mho) &= \sum_{uv\in E(G)} \left(\frac{d_u d_v}{d_u + d_v}\right)\\
&= 9\frac{9}{6} + 9\frac{15}{8} + 6\frac{10}{7}\\
&= 9(1.5) + 9(1.875) + 6(1.4285)\\
&= 38.946
\end{aligned}
$$

$$
\begin{aligned}
ISI(\mho) &\le 2m_1\left[\frac{\Delta_G(2+n_2+m_2)}{\Delta_G+2+n_2+m_2}\right] + m_1m_2\left[\left[\frac{2\Delta_H+1}{2}\right] + \left[\frac{3(2\Delta_H+1)}{\Delta_H+2}\right] + \left[\frac{6+3n_2+3m_2}{5+n_2+m_2}\right]\right]\\
&+ m_1n_2\left[\frac{(2\Delta_H+1)(2+n_2+m_2)}{2\Delta_H+3+n_2+m_2}\right].\\
&\le 6\left[\frac{2(2+2+1)}{2+2+2+1}\right] + 3\left[\left[\frac{2+1}{2}\right] + \left[\frac{6+3}{1+2}\right] + \left[\frac{6+6+3}{5+2+1}\right]\right] + 6\left[\frac{(2+1)(2+2+1)}{2+3+2+1}\right]\\
&\le 6\left[\frac{10}{7}\right] + 3\left[\left[\frac{3}{2}\right] + \left[\frac{9}{3}\right] + \left[\frac{15}{8}\right]\right] + 6\left[\frac{15}{8}\right]\\
&\le 6(1.42857) + 31.5 + 3 + 1.875 + 6(1.875)\\
&= 38.946
\end{aligned}
$$

$$
\begin{aligned}
ISI(\mho) &\ge 2m_1\left[\frac{\delta_G(2+n_2+m_2)}{\delta_G+2+n_2+m_2}\right] + m_1m_2\left[\left[\frac{2\delta_H+1}{2}\right] + \left[\frac{3(2\delta_H+1)}{\delta_H+2}\right] + \left[\frac{6+3n_2+3m_2}{5+n_2+m_2}\right]\right]\\
&+ m_1n_2\left[\frac{(2\delta_H+1)(2+n_2+m_2)}{2\delta_H+3+n_2+m_2}\right].\\
&\ge 6\left[\frac{2(2+2+1)}{2+2+2+1}\right] + 3\left[\left[\frac{2+1}{2}\right] + \left[\frac{6+3}{1+2}\right] + \left[\frac{6+6+3}{5+2+1}\right]\right] + 6\left[\frac{(2+1)(2+2+1)}{2+3+2+1}\right]\\
&\ge 6\left[\frac{10}{7}\right] + 3\left[\left[\frac{3}{2}\right] + \left[\frac{9}{3}\right] + \left[\frac{15}{8}\right]\right] + 6\left[\frac{15}{8}\right]\\
&\ge 6(1.42857) + 31.5 + 3 + 1.875 + 6(1.875)\\
&= 38.946
\end{aligned}
$$

2 Conclusion

In this work, we considered ℧-graph and concentrated five important topological indices and determine their bounds. Similar way, researchers can considering different class of topological indices and determine their corresponding bounds for ℧-graph.

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