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Solution of Certain Pell Equations and The Period Length of Continued Fraction

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Abstract Let d be a positive integer, which is not a perfect square. Then, we determine the form of d in which the period length of the continued fraction of $\sqrt{d}$ is three. Using these results, we express all positive integer solutions of the Pell equations $x^2 - dy^2 = \pm 1, \pm 4$ in the generalized Fibonacci and Lucas sequences.

1 Introduction

Let d be a positive integer, which is not a perfect square. The quadratic Diophantine equation of the form

$$x^2 - dy^2 = \pm 1$$

and

$$x^2 - dy^2 = \pm 4$$

are called a Pell equation (see [1, 2, 3, 4]). For $f, g \in \mathbb{Z}$ $(f \neq 0, g \neq 0, f^2 + 4g^2 > 0)$, we define the generalized Fibonacci sequences $U_n(f,g)$ and the generalized Lucas sequences $V_n(f,g)$ as follows (see[6, 7]):

$$U_0(f,g) = 0, \ \ U_1(f,g) = 1, \ \ U_{n+1}(f,g) = fU_n(f,g) + gU_{n-1}(f,g),$$

$$V_0(f,g) = 2, \ \ V_1(f,g) = f, \ \ V_{n+1}(f,g) = fV_n(f,g) + gV_{n-1}(f,g).$$

Jones [2] considered positive integer solutions of the Pell equations in the case of $d = k^2 \pm 1$ and $k^2 \pm 4$ and showed that all positive integer solutions of the equations $x^2 - dy^2 = \pm 1, \pm 4$ can be expressed by the generalized Fibonacci sequences and the generalized Lucas sequences. Keskin-Güney [5] gave the continued fractions of $\sqrt{d}$ for these integers d and gave another proof of the results of Jones [2].
In this paper, we consider positive integer solutions of the Pell equations $x^2 - dy^2 = \pm 1, x^2 - dy^2 = \pm 4$ when the period length of the continued fraction of $\sqrt{d}$ is three. The following are the main results of this paper.

Theorem 1.1. *Let d be a positive integer. Assume that there exist positive integers k and a satisfying*

$$a < k, \ \ d = k^2 + \frac{4ak+1}{4a^2+1}.$$

Let $(f,g) = (2k(4a^2+1) + 4a, 1)$.

(i) All positive integer solutions of the equation $x^2 - dy^2 = 1$ *are given by*

$$(x_n, y_n) = \left(\frac{1}{2}V_{2n}(f,g), (4a^2+1)U_{2n}(f,g)\right)$$

with $n \geq 1$.

(ii) All positive integer solutions of the equation $x^2 - dy^2 = -1$ *are given by*

$$(x_n, y_n) = \left(\frac{1}{2}V_{2n-1}(f,g), (4a^2+1)U_{2n-1}(f,g)\right)$$

with $n \geq 1$.

Theorem 1.2. *Let* d *be a positive integer. Assume that there exist positive integers* k *and* a *satisfying*

$$a < k, \quad d = k^2 + \frac{4ak+1}{4a^2+1}.$$

Let $(f,g) = (2k(4a^2+1)+4a, 1)$.

(i) All positive integer solutions of the equation $x^2 - dy^2 = 4$ *are given by*

$$(x_n, y_n) = (V_{2n}(f,g), 2(4a^2+1)U_{2n}(f,g))$$

with $n \geq 1$.

(ii) All positive integer solutions of the equation $x^2 - dy^2 = -4$ *are given by*

$$(x_n, y_n) = (V_{2n-1}(f,g), 2(4a^2+1)U_{2n-1}(f,g))$$

with $n \geq 1$.

This thesis is organized as follows: Section 2 review the primary results of continued fractions and Pell equations. In section 3, we determine a positive integer d such that the period length of the continued fraction of $\sqrt{d}$ is three. In section 4, we prove Theorem 1.1 and Theorem 1.2.

2 PRELIMINARIES

Definition 2.1. Let α be a real number. A continued fraction of α is an expression of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}. \quad (a_0 \in \mathbb{Z},\ a_1, a_2, \dots \in \mathbb{N})$$

This fraction is denoted by

$$\alpha = [a_0, a_1, a_2, \dots].$$

Then $\lfloor \alpha \rfloor = a_0$, where $\lfloor x \rfloor$ is a floor function of x.

Lemma 2.2. *Let* d *be a positive integer, which is not a perfect square. Then, the continued fraction of* $\sqrt{d}$ *is expressed by*

$$\sqrt{d} = [a_0, a_1, \dots, a_n, \dots].$$

Then there exists a positive integer l *satisfying*

$$\sqrt{d} = [a_0, a_1, \dots, a_l, a_1, \dots, a_l, a_1, \dots].$$

We use the notation

$$\sqrt{d} = [a_0, \overline{a_1, \ldots, a_l}]$$

as a convenient abbreviation. The natural number $l = l(\sqrt{d})$ is called the period length of $\sqrt{d}$.

Lemma 2.3. *Let d be a positive integer which is not perfect square.*

(1) The continued fraction of $\sqrt{d}$ can be expressed in the form

$$\sqrt{d} = [a_0, \overline{a_1, a_2, \ldots, a_{l-1}, 2a_0}].$$

(2) The periodic terms of the continued fraction of $\sqrt{d}$ are palindrome. That means

$$[a_0, \overline{a_1, a_2, \ldots, a_{l-1}, 2a_0}] = [a_0, \overline{a_{l-1}, a_{l-2}, \ldots, a_1, 2a_0}].$$

Definition 2.4. Let $\sqrt{d} = [a_0, a_1, a_2, \ldots]$, then the n^{th} convergent of $\sqrt{d}$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

with $n \geq 0$.

Next, we explain how to solve Pell equations using continued fractions.

Definition 2.5. Let d be a positive integer, which is not a perfect square. Then the equations

$$x^2 - dy^2 = \pm 1, \pm 4$$

are called Pell equations.

Definition 2.6. Let the pair of integers (a, b) be a solution of the Pell equation. If both a and b are positive, (a, b) is called a positive integer solution of the Pell equation $x^2 - dy^2 = \pm 1, \pm 4$. Also, if $a + b\sqrt{d}$ is the least possible value in solutions of Pell equations, then (a, b) is called the fundamental solution.

First, we consider the fundamental solution of the equations $x^2 - dy^2 = \pm 1$.

Lemma 2.7. *Let $l(\sqrt{d}) = l$.*

(1) If l is even, then

(1-i) The fundamental solution of the equation $x^2 - dy^2 = 1$ is given by (p_{l-1}, q_{l-1}).

(1-ii) The equation $x^2 - dy^2 = -1$ has no integer solution.

(2) If l is odd, then

(2-i) The fundamental solution of the equation $x^2 - dy^2 = 1$ *is given by* (p_{2l-1}, q_{2l-1}).

(2-ii) The fundamental solution of the equation $x^2 - dy^2 = -1$ *is given by* (p_{l-1}, q_{l-1}).

Next, we consider the fundamental solution of the equations $x^2 - dy^2 = \pm 4$.

Lemma 2.8. *Let* $d \equiv 2, 3 \pmod 4$ *or* $d \equiv 1 \pmod 8$. *The positive integer solutions of the equations* $x^2 - dy^2 = \pm 1$ *and* $x^2 - dy^2 = \pm 4$ *have a one-to-one correspondence by* $(a, b) \mapsto (2a, 2b)$. *In this correspondence, the fundamental solution of the equations* $x^2 - dy^2 = \pm 1$ *corresponds to the fundamental solution of the equations* $x^2 - dy^2 = \pm 4$.

Lemma 2.9. *Let* $d \equiv 5 \pmod 8$. *Let* ϵ *be the fundamental unit of the ring of integer* $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$. *If* $\epsilon \in \mathbb{Z}[\sqrt{d}]$, *then* ϵ *is the fundamental unit of* $\mathbb{Z}[\sqrt{d}]$. *If* $\epsilon \notin \mathbb{Z}[\sqrt{d}]$, *then* ϵ^3 *is the fundamental unit of* $\mathbb{Z}[\sqrt{d}]$.

It is known that there is a one-to-one correspondence between the unit of $\mathbb{Z}[\sqrt{d}]$ and the solution of the equations $x^2 - dy^2 = \pm 1$. In particular, the fundamental unit of $\mathbb{Z}[\sqrt{d}]$ corresponds to the fundamental solution of the equations $x^2 - dy^2 = \pm 1$.

Remark 2.10. Let ϵ in Lemma 2.9 be denoted by

$$\epsilon = a + b\frac{1+\sqrt{d}}{2} = \frac{(2a+b) + b\sqrt{d}}{2}$$

where $a, b \in \mathbb{Z}$. If $2a + b$ is even, then b is even. Also, If $2a + b$ is odd, then b is odd. Thus, if both $2a + b$ and b are odd, then

$$s + t\sqrt{d} = \epsilon^3 \in \mathbb{Z}[\sqrt{d}]$$

where (s, t) is the fundamental solution of the equations $x^2 - dy^2 = \pm 1$. Also, if both $2a + b$ and b are even, then

$$s + t\sqrt{d} = \epsilon \in \mathbb{Z}[\sqrt{d}]$$

where (s, t) is the fundamental solution of the equations $x^2 - dy^2 = \pm 1$.

If the fundamental solutions of the Pell equations exist, then all positive integer solutions are given by the fundamental solution.

Theorem 2.11. *Let* (x_1, y_1) *be the fundamental solution of the equation* $x^2 - dy^2 = 1$. *Then all positive integer solutions to the equation* $x^2 - dy^2 = 1$ *are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

with $n \geq 1$.

Theorem 2.12. *Let* (x_1, y_1) *be the fundamental solution of the equation* $x^2 - dy^2 = -1$. *Then all positive integer solutions to the equation* $x^2 - dy^2 = -1$ *are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^{2n-1}$$

with $n \geq 1$. *Also, the fundamental solution* $(x_1^{'}, y_1^{'})$ *of the equation* $x^2 - dy^2 = 1$ *is given by*

$$x_1^{'} + y_1^{'}\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^2.$$

We consider the positive integer solutions of the equations $x^2 - dy^2 = \pm 4$.

Theorem 2.13. *Let (x_1, y_1) be the fundamental solution of the equation $x^2 - dy^2 = 4$. Then all positive integer solutions to the equation $x^2 - dy^2 = 4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^{n-1}}$$

with $n \geq 1$.

Theorem 2.14. *Let (x_1, y_1) be the fundamental solution of the equation $x^2 - dy^2 = -4$. Then all positive integer solutions to the equation $x^2 - dy^2 = -4$ are given by*

$$x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^{2n-1}}{4^{n-1}}$$

with $n \geq 1$. Also, the fundamental solution $(x_1^{'}, y_1^{'})$ of the equation $x^2 - dy^2 = 4$ is given by

$$\frac{x_1^{'} + y_1^{'}\sqrt{d}}{2} = \left(\frac{x_1 + y_1\sqrt{d}}{2}\right)^2.$$

Next, we review the properties of the generalized Fibonacci sequence and the generalized Lucas sequence.

Definition 2.15. Let $f, g \in \mathbb{Z}$ $(f \neq 0, g \neq 0, f^2 + 4g > 0)$. The generalized Fibonacci sequence $\{U_n(f, g)\}$ is defined by

$$U_0(f, g) = 0,\ \ U_1(f, g) = 1,\ \ U_{n+1}(f, g) = fU_n(f, g) + gU_{n-1}(f, g)$$

for $n \geq 1$, and the generalized Lucas sequence $\{V_n(f, g)\}$ is defined by

$$V_0(f, g) = 2,\ \ V_1(f, g) = f,\ \ V_{n+1}(f, g) = fV_n(f, g) + gV_{n-1}(f, g)$$

for $n \geq 1$, respectively.

Lemma 2.16. *Let*

$$\alpha = \frac{f + \sqrt{f^2 + 4g}}{2},\ \ \beta = \frac{f - \sqrt{f^2 + 4g}}{2}$$

in Definition 2.15. Then, we obtain

$$U_n(f, g) = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ \ V_n(f, g) = \alpha^n + \beta^n.$$

For more information about continued fractions, Pell equations, and generalized Fibonacci and Lucas sequence, one can consult [1], [3], and [4].

3 The period length of the continued fraction is three

In this section, we consider the period length of the continued fraction is 3.

Lemma 3.1. *If $l(\sqrt{d}) = 3$, then the continued fraction of $\sqrt{d}$ is $[k, \overline{2a, 2a, 2k}]$ with $k, a \in \mathbb{N}$.*

Proof. The continued fraction of $\sqrt{d}$ is $[k, \overline{s, t, 2k}]$ with $s, t \in \mathbb{N}$. Thus, we get

$$\sqrt{d} = k + \cfrac{1}{s + \cfrac{1}{t + \cfrac{1}{2k + (\sqrt{d} - k)}}}.$$

Therefore, we obtain

$$(std - stk^2 + d - sk - k^2 - tk - 1) + (s - t)\sqrt{d} = 0. \tag{3.1}$$

Hence, by $\{1, \sqrt{d}\}$ is linearly independent, $s = t$ holds. Thus, we get

$$d = k^2 + \frac{2sk + 1}{s^2 + 1},$$

since equation (3.1). Also, by the fraction part is an integer, s is even. Hence, Lemma 3.1 holds. □

We determine the positive integer d such that $l(\sqrt{d}) = 3$ since Lemma 3.1.

Theorem 3.2. *$l(\sqrt{d}) = 3$ if and only if there exist positive integers k and a satisfying*

$$a < k, \quad d = k^2 + \frac{4ak + 1}{4a^2 + 1}.$$

Proof. Assume $l(\sqrt{d}) = 3$. Then, $\sqrt{d} = [k, \overline{2a, 2a, 2k}]$ follows from Lemma 3.1. Thus, we get

$$\sqrt{d} = k + \cfrac{1}{2a + \cfrac{1}{2a + \cfrac{1}{2k + \sqrt{d} - k}}}.$$

By the above equation, we get $d = k^2 + \frac{4ak+1}{4a^2+1}$. Also, $4ak + 1 \geq 4a^2 + 1$ follows from d is a positive integer. Therefore, we obtain $k \geq a$. If $k = a$, $l(\sqrt{d}) = 1$, which is a contradiction. Hence, $k > a$.
Now, assume that positive integers k and a satisfying $a < k,\ d = k^2 + \frac{4ak+1}{4a^2+1}$. Then, by $k^2 + \frac{4ak+1}{4a^2+1}$ is a positive integer, we get $k \equiv a \pmod{4a^2 + 1}$. So, we obtain $k = (4a^2 + 1)n + a$ with $n \in \mathbb{N}$. Thus,

$$\sqrt{d} = k + \cfrac{1}{\cfrac{\sqrt{d} + k}{d - k^2}} = k + \cfrac{1}{\cfrac{\sqrt{d} + k}{4an + 1}} \tag{3.2}$$

follows from $\lfloor \sqrt{d} \rfloor = k$. Then, we obtain

$$\lfloor \sqrt{d} + k \rfloor = 2k = 8a^2n + 2n + 2a = 2a(4an + 1) + 2n.$$

By $2n < 4an + 1$, we obtain

$$\left\lfloor \frac{\sqrt{d} + k}{d - k^2} \right\rfloor = \left\lfloor \frac{\sqrt{d} + k}{4an + 1} \right\rfloor = 2a.$$

Next, we get

$$\frac{\sqrt{d} + k}{d - k^2} = 2a + \left(\frac{\sqrt{d} + k}{4an + 1} - 2a \right) = 2a + \cfrac{1}{\cfrac{4an + 1}{\sqrt{d} - (4a^2n - n + a)}}.$$

Also,

$$\frac{4an+1}{\sqrt{d}-(4a^2n-n+a)} = \frac{(4an+1)(\sqrt{d}+4a^2n-n+a)}{d-(4a^2n-n+a)^2} = \frac{\sqrt{d}+4a^2n-n+a}{4an+1} \tag{3.3}$$

holds. Hence, by

$$\lfloor \sqrt{d}+4a^2n-n+a \rfloor = k+4a^2n-n+a = 2a(4an+1),$$

we get

$$\left\lfloor \frac{4an+1}{\sqrt{d}-(4a^2n-n+a)} \right\rfloor = \left\lfloor \frac{\sqrt{d}+4a^2n-n+a}{4an+1} \right\rfloor = 2a.$$

Since equation (3.3), we obtain

$$\frac{4an+1}{\sqrt{d}-(4a^2n-n+a)} = 2a + \left(\frac{\sqrt{d}+4a^2n-n+a}{4an+1} - 2a \right) = 2a + \frac{1}{\dfrac{4an+1}{\sqrt{d}-k}}.$$

Moreover, we get

$$\frac{4an+1}{\sqrt{d}-k} = \frac{(4an+1)(\sqrt{d}+k)}{\dfrac{4ak+1}{4a^2+1}} = \sqrt{d}+k.$$

By $\lfloor \sqrt{d}+k \rfloor = 2k$, we obtain

$$\sqrt{d}+k = 2k + (\sqrt{d}+k-2k) = 2k + \frac{1}{\dfrac{\sqrt{d}+k}{d-k^2}}.$$

So, we obtain $\sqrt{d} = [k, \overline{2a, 2a, 2k}]$ by the above equation and equation (3.2). From the assumption, $a \neq k$ holds. Therefore, the proof follows. □

Theorem 3.3. *The equivalent holds.*

$$l(\sqrt{d}) = 3 \iff d = ((4a^2+1)x+a)^2 + 4ax + 1 \ (\exists a, \exists x \in \mathbb{N}).$$

Proof. It was proven by Theorem 3.2. □

Lemma 3.4. *If $l(\sqrt{d}) = 3$, then $4 \nmid d$.*

Proof. Positive integers k and a exist from Theorem 3.2 satisfying $a < k$, $d = k^2 + \frac{4ak+1}{4a^2+1}$. Assume $4 \mid d$. We obtain

$$d = k^2 + \frac{4ak+1}{4a^2+1} \equiv 0 \pmod 4.$$

So, we get

$$k^2(4a^2+1) + (4ak+1) \equiv 0 \pmod 4.$$

Thus, $k^2 \equiv 3 \pmod 4$. However, the congruence is a contradictory since $x^2 \equiv 0, 1 \pmod 4$ with $x \in \mathbb{N}$. Hence, the proof follows. □

4 Solutions of the Pell equations

First, we consider positive integer solutions of Pell equations $x^2 - dy^2 = \pm 1$ at $l(\sqrt{d}) = 3$. In this section, we assume $l(\sqrt{d}) = 3$. Then, positive integers k and a satisfying

$$a < k, \quad d = k^2 + \frac{4ak+1}{4a^2+1}$$

exist. Note that $l(\sqrt{d})$ is odd; $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ have positive integer solutions.

Theorem 4.1. *The fundamental solution (x_1, y_1) of the equation $x^2 - dy^2 = -1$ is given by*

$$x_1 + y_1\sqrt{d} = (k(4a^2+1) + 2a) + (4a^2+1)\sqrt{d}.$$

Proof. Let $\frac{p_n}{q_n}$ be the n^{th} convergent of $\sqrt{d}$. By $l(\sqrt{d}) = 3$, $x_1 + y_1\sqrt{d} = p_2 + q_2\sqrt{d}$ follows from Lemma 2.7. Therefore, we get a conclusion from

$$\frac{p_2}{q_2} = k + \cfrac{1}{2a + \cfrac{1}{2a}} = \frac{k(4a^2+1) + 2a}{4a^2+1}.$$

□

Theorem 4.2. *The fundamental solution (x_1, y_1) to the equation $x^2 - dy^2 = 1$ is given by*

$$x_1 + y_1\sqrt{d} = ((k(4a^2+1) + 2a) + (4a^2+1)\sqrt{d})^2.$$

Proof. This Theorem holds by Theorem 2.12, 4.1. □

We show that the generalized Fibonacci and Lucas sequences can express all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$.

Theorem 4.3. *Let $(f, g) = (2k(4a^2+1) + 4a, 1)$.*

(i) All positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$(x_n, y_n) = \left(\frac{1}{2}V_{2n}(f, g), (4a^2+1)U_{2n}(f, g)\right)$$

with $n \geq 1$.

(ii) All positive integer solutions of the equation $x^2 - dy^2 = -1$ are given by

$$(x_n, y_n) = \left(\frac{1}{2}V_{2n-1}(f, g), (4a^2+1)U_{2n-1}(f, g)\right)$$

with $n \geq 1$.

Proof. By Theorem 2.11, 2.12, 4.1, 4.2, all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ are given by

$$x_n + y_n\sqrt{d} = ((k(4a^2+1) + 2a) + (4a^2+1)\sqrt{d})^n$$

with $n \geq 1$. Note that if n is even, then the equation becomes the solution of the equation $x^2 - dy^2 = 1$. If n is odd, then the equation becomes the solution of the equation $x^2 - dy^2 = -1$. Now, we set

$$\alpha = (k(4a^2+1) + 2a) + (4a^2+1)\sqrt{d},$$

$$\beta = (k(4a^2+1) + 2a) - (4a^2+1)\sqrt{d}.$$

Then, we get $\alpha^n = x_n + y_n\sqrt{d},\ \beta^n = x_n - y_n\sqrt{d}$. Therefore, we obtain

$$x_n = \frac{\alpha^n + \beta^n}{2},\ \ y_n = \frac{\alpha^n - \beta^n}{2\sqrt{d}}.$$

Now, let

$$(f, g) = (2k(4a^2 + 1) + 4a, 1).$$

Then, we obtain

$$x_n = \frac{\alpha^n + \beta^n}{2} = \frac{1}{2}V_n(f, g),$$

$$y_n = (4a^2 + 1)\frac{\alpha^n - \beta^n}{\alpha - \beta} = (4a^2 + 1)U_n(f, g).$$

□

Next, we consider positive integer solutions of the Pell equations $x^2 - dy^2 = \pm 4$ at $l(\sqrt{d}) = 3$. Note that both $x^2 - dy^2 = 4$ and $x^2 - dy^2 = -4$ have positive integer solutions where $l(\sqrt{d})$ is odd.

Lemma 4.4. *If* $d \equiv 5 \pmod 8$ *and* $l(\sqrt{d}) = 3$*, then both* x *and* y *of the fundamental solution of the equation* $x^2 - dy^2 = -4$ *are even.*

Proof. Let

$$\epsilon = p + q\frac{1 + \sqrt{d}}{2} = \frac{(2p + q) + q\sqrt{d}}{2} \quad (p, q \in \mathbb{Z})$$

be the fundamental unit of the ring of integer $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$. Then, $(2p + q, q)$ is the fundamental solution of the equation $x^2 - dy^2 = -4$. Now, we assume both $2p + q$ and q are odd. By Lemma 2.9, $\epsilon^3 \in \mathbb{Z}[\sqrt{d}]$ is the fundamental solution of the equation $x^2 - dy^2 = -1$. Therefore, we obtain

$$\epsilon^3 = \frac{(2p + q)^3 + 3(2p + q)q^2 d + (3(2p + q)^2 q + q^3 d)\sqrt{d}}{8}. \tag{4.1}$$

By $\epsilon^3 = (k(4a^2 + 1) + 2a) + (4a^2 + 1)\sqrt{d}$, we get $m = \frac{dq^2 - 1}{4}$ with $m \in \mathbb{N}$. Since equation (4.1) and $(2p + q)^2 - dq^2 = -4$, we get

$$2p + q = \frac{k(4a^2 + 1) + 2a}{2m},\ \ q = \frac{4a^2 + 1}{2m - 1}.$$

Particularly,

$$2m - 1 \mid 4a^2 + 1. \tag{4.2}$$

These equations are substituted for the equation $x^2 - dy^2 = -4$. We obtain

$$(4a^2 + 1)^2(4m - 1)k^2 + 4a(4a^2 + 1)(4m - 1)k$$
$$+4(4a^2 m - a^2 + m^2 - 4m^2(2m - 1)^2) = 0.$$

By the quadratic formula, we get

$$k = \frac{-4a(4a^2 + 1)(4m - 1) \pm \sqrt{D}}{2(4a^2 + 1)^2(4m - 1)}.$$

Hence, we obtain

$$D = 16m^2(4a^2 + 1)^2(4m - 1)^2(4m - 3).$$

Since D is a square number, $4m-3$ becomes a square number. $4m-3$ is odd; we set $4m-3=(2s-1)^2$ with $s\in\mathbb{N}$. Now, $m=s^2-s+1$. These equations are substituted for the equation of k; we obtain

$$k=\frac{2(-a\pm m(2s-1))}{4a^2+1}.$$

By $k>0$, the equation becomes

$$k=\frac{2(-a+m(2s-1))}{4a^2+1}. \tag{4.3}$$

(i) Let $a=1$, we get $m=1,3$. Assume $m=1$, then $k=0$ since $s=1$, which is unsuitable. Assume $m=3$, then $k=\frac{16}{5}$ since $s=2$ which is unsuitable.

(ii) Let $a>1$ and $m=1$, we get $s=1$. Then, the equation becomes $-a+m(2s-1)=-a+1<0$ which is unsuitable.

(iii) Let $a>1$ and $m>1$, we get $s>1$. By equation (4.2), (4.3), we get

$$2m-1\mid -a+m(2s-1).$$

Now, since

$$(2m-1)=(2s^2-2s+1),$$
$$-a+m(2s-1)=2s^3-3s^2+3s-1-a,$$

we obtain $2s^2-2s+1\mid 2s^3-3s^2+3s-1-a$. Also,

$$\frac{2s^3-3s^2+3s-1-a}{2s^2-2s+1}=s-\frac{s^2-2s+1+a}{2s^2-2s+1}$$

is a positive integer. So,

$$s^2-2s+1+a\geq 2s^2-2s+1$$

holds by $2s^2-2s+1\mid s^2-2s+1+a$. Thus, we get $a\geq s^2=m+(s-1)$. Also, we found $a>m$ and $a>s-1$. Therefore, we obtain

$$-a+m(2s-1)<-a+a(2s-1)=2a(s-1)<2a^2<4a^2+1$$

and $k<2$. However, if $k=1$, i.e., $a=1$, this result is unsuitable by (i).

Thus, there is a contradiction in all of the cases (i), (ii), (iii). Therefore, both $2p+q$ and q are even. □

Condition $l(\sqrt{d})=3$ is vital in Lemma 4.4. For instance, if $d=5$, then $\sqrt{5}=[2,\overline{4}]$ and $l(\sqrt{5})=1$. The fundamental solution (x,y) of the equation $x^2-5y^2=-4$ becomes $(x,y)=(1,1)$.

Theorem 4.5. *The fundamental solution (x_1,y_1) of the equation $x^2-dy^2=-4$ is given by*

$$x_1+y_1\sqrt{d}=2(k(4a^2+1)+2a)+2(4a^2+1)\sqrt{d}.$$

Proof. Since $l(\sqrt{d})=3$ and Lemma 3.4, we get $d\equiv 1,2,3 \pmod 4$. Also, by Theorem 4.1, the fundamental solution of the equation $x^2-dy^2=-1$ is given by $(k(4a^2+1)+2a)+(4a^2+1)\sqrt{d}$. So, from Lemma 2.8, 4.4 and Remark 2.10, the fundamental solution of the equation $x^2-dy^2=-4$ is given by

$$x_1+y_1\sqrt{d}=2(k(4a^2+1)+2a)+2(4a^2+1)\sqrt{d}.$$

□

Theorem 4.6. *The fundamental solution* (x_1, y_1) *of the equation* $x^2 - dy^2 = 4$ *is given by*

$$x_1 + y_1\sqrt{d} = 2((k(4a^2+1)+2a)+(4a^2+1)\sqrt{d})^2.$$

Proof. This Theorem holds by Theorem 2.14, 4.5. □

Finally, we express all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ by the generalized Fibonacci and Lucas sequences when $l(\sqrt{d}) = 3$.

Theorem 4.7. *Let* $(f, g) = (2k(4a^2+1)+4a, 1)$.

(i) All positive integer solutions of the equation $x^2 - dy^2 = 4$ *are given by*

$$(x_n, y_n) = (V_{2n}(f,g), 2(4a^2+1)U_{2n}(f,g))$$

with $n \geq 1$.

(ii) All positive integer solutions of the equation $x^2 - dy^2 = -4$ *are given by*

$$(x_n, y_n) = (V_{2n-1}(f,g), 2(4a^2+1)U_{2n-1}(f,g))$$

with $n \geq 1$.

Proof. By Theorem 2.13, 2.14, 4.5, 4.6, all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ are given by

$$x_n + y_n\sqrt{d} = 2((k(4a^2+1)+2a)+(4a^2+1)\sqrt{d})^n$$

with $n \geq 1$. Note that if n is even, then the equation becomes the solution of the equation $x^2 - dy^2 = 4$. If n is odd, then the equation becomes the solution of the equation $x^2 - dy^2 = -4$. Now, we set

$$\alpha = (k(4a^2+1)+2a)+(4a^2+1)\sqrt{d},$$

$$\beta = (k(4a^2+1)+2a)-(4a^2+1)\sqrt{d}.$$

Then, we get $2\alpha^n = x_n + y_n\sqrt{d}$, $2\beta^n = x_n - y_n\sqrt{d}$. Therefore, we obtain

$$x_n = \alpha^n + \beta^n, \quad y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}}.$$

Now, let

$$(f, g) = (2k(4a^2+1)+4a, 1).$$

Then, we obtain

$$x_n = \alpha^n + \beta^n = V_n(f,g),$$

$$y_n = 2(4a^2+1)\frac{\alpha^n - \beta^n}{\alpha - \beta} = 2(4a^2+1)U_n(f,g).$$

□

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