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A VARIANT OF S-1-ABSORBING PRIMENESS

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Abstract Let R be a commutative ring with nonzero identity. Let $S \subseteq R$ be a multiplicative closed subset of R and M be a unital module. A submodule N of M with $(N :_R M) \cap S = \phi$ is said to be S-1-absorbing primary, if $abm \in N$ for some non-unit elements $a, b \in R$ and $m \in M$, then there exists an $s \in S$ such that $sab \in \sqrt{N :_R M}$ or $sm \in N$. In the present work, we discuss some properties, characterizations and applications of S-1-absorbing primary submodules of a module.

1 Introduction

In commutative algebra, prime submodules (prime ideals) have very significant role to characterize different classes of modules (rings). In 2020, Badawi et al. introduced the idea of 1-absorbing primary ideal as a generalization of primary ideals. In 2021, Farzalipour et al. [3] generalized the notion of prime submodules to S-1-absorbing prime submodules and discussed some applications. They called that a submodule N of M with $(N :_R M) \cap S = \phi$ is an *S-1-absorbing prime*, if there exists an $s \in S$ whenever $abm \in N$ for some non-unit elements $a, b \in R$ and $m \in M$, then $sab \in (N :_R M)$ or $sm \in N$.

We know that a prime submodule (prime ideal) is always a primary submodule (primary ideal). Therefore, it motivates us to study the notion of S-1-absorbing primary submodules as a natural generalization of S-1-absorbing prime submodules.

Definition 1.1. Let R be a commutative ring with nonzero identity. Let $S \subseteq R$ be a multiplicative closed subset of R and M be a unital module. A submodule N of M with $(N :_R M) \cap S = \phi$ is said to be S-1-absorbing primary, if $abm \in N$ for some non-unit elements $a, b \in R$ and $m \in M$, then there exists an $s \in S$ such that $sab \in \sqrt{N :_R M}$ or $sm \in N$.

During investigation, we find that many properties of S-1-absorbing prime submodules do not hold in case of S-1-absorbing primary submodules of a module. In support, we provide some examples in Section 2. In Theorem 2.4, we give some characterizations of an S-1-absorbing primary submodule in terms of its radicals. In Theorem 2.6, we characterize an S-1-absorbing primary submodule N of a module M in terms of all S-1-absorbing primary submodules of M of the form $(N :_M s^2)$ for some $s \in S$.

In Remark 3.5(ii), we show that a result analogous to [3, Theorem 3.1] does not hold for S-1-absorbing primary ideal. In Theorem 3.6, we generalize it in case of S-1-absorbing primary ideal: Let M be a multiplication faithful R-module and S be a multiplicative closed subset of R. Let I be an S-1-absorbing primary ideal of R. Let a and b be two non-unit elements in R and $m \in M$. If $abm \in IM$, then there exists an s in S such that $sab \in \sqrt{I}$ or $sm \in IM$.

In Section 4, we study some properties of S-1-absorbing primary submodule of a finitely generated module. In Proposition 4.2, we prove that $(N :_R M)$ is an S-1-absorbing primary submodule of M. The converse is true if M is a multiplication module (In general it is not true see Remark 4.3). In Proposition 4.6, we provide a characterization of an S-1-absorbing primary submodule of a finitely generated von-Neumann regular module. Finally, in Section 5, we discuss properties of S-1-absorbing primary submodule over a singleton multiplicative closed subset of the ring.

Throughout R is a commutative ring with nonzero identity and M is a unital module. Let N and L be two submodules of an R-module M and K be an ideal of a ring R. Then the residual N by L is $(N :_R L) = \{x \in R : xL \subseteq N\}$ and residual N by K is $(N :_M K) = \{m \in M : Km \subseteq N\}$. We denote $(0 :_R M)$ by $ann(M)$ and $(N :_M Rs)$ by $(N :_M s)$ where Rs is the principal ideal generated by an element $s \in R$. According to [6], a prime (resp. primary) submodule is a proper submodule N of M with the property that for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M)$ (resp. $a^k \in (N :_R M)$ for some positive integer k). In [10], a nonempty subset S of R is said to be a *multiplicatively closed subset* of R if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $ss^{'} \in S$ for all $s, s^{'} \in S$.

2 Examples and characterizations

Example 2.1. (i). An S-1-absorbing prime submodule is always an S-1-absorbing primary but the converse need not be true. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{16}$ and $N = \{\bar{0}, \bar{8}\}$. Then $(N :_R M) = \{a \in \mathbb{Z} \mid a\mathbb{Z}_{16} \subseteq N\} = 8\mathbb{Z}$ and $\sqrt{N :_R M} = \{a \in \mathbb{Z} \mid a^n\mathbb{Z}_{16} \subseteq N, for\ some\ n \in \mathbb{N}\} = 2\mathbb{Z}$. Let $S = \{1\}$ be a multiplicative closed subset of R. Then $(N :_R M) \cap S = \phi$. Consider $a = 2, b = 2$ and $m = \bar{2}$ and so $2.2.\bar{2} \in N$. This implies that $1.2.2 \in \sqrt{N :_R M}$. Then N is an S-1-absorbing primary submodule of M. But $1.2.2 \notin (N :_R M)$ and $1.\bar{2} \notin N$. So N is not an S-1-absorbing prime submodule of M.

(ii). An S-1-absorbing primary submodule of a module need not be a primary submodule. Consider an example from [3, Example 2.1(iii)]. Let the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}_4$ and the zero submodule $N = 0 \times 0$. Now $(N :_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}_4)) = 0$ and $\sqrt{N :_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}_4)} = 0$. Let $S = \mathbb{Z} - 0$ and put $s = 4$. Then N is an S-1-absorbing prime submodule of M. Therefore N is an S-1-absorbing primary submodule of M. Now, let $a = 4$ and $m = (0, \bar{1})$. Then $am = 4(0, \bar{1}) \in N$, but $4 \notin \sqrt{N :_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}_4)}$ and $(0, \bar{1}) \notin N$. Hence N is not a primary submodule of M.

(iii). Let N be a proper submodule of an R-module M such that $(N :_R M)$ is a prime ideal of R. Then the notions of S-1-absorbing primary and S-1-absorbing prime submodules are same for N.

Further, we discuss sufficient conditions for a submodule to be an S-1-absorbing primary submodule.

Proposition 2.2. *Let $S \subseteq R$ be a multiplicative closed subset of R and M be an R-module.*

(i) Let P be a primary submodule of M with $(P :_R M) \cap S = \phi$. Then P is an S-1-absorbing primary submodule of M.

(ii) Let S_1 and S_2 are multiplicative closed subset of R such that $S_1 \subseteq S_2$ and P be an S_1-1-absorbing primary submodule of M with $(P :_R M) \cap S_2 = \phi$. Then P is an S_2-1-absorbing primary submodule of M.

(iii) P is an S-1-absorbing primary submodule of M if and only if P is an S^*-1-absorbing primary submodule of M.

(iv) Let P be an S-1-absorbing primary submodule of M with $(S^{-1}P : S^{-1}M) \cap S = \phi$. Then $S^{-1}P$ is an S-1-absorbing primary submodule of $S^{-1}M$.

Proof. (1). Clear.

(2). Let $abm \in P$, for some non-units $a, b \in R$ and $m \in M$. Since P is an S_1-1-absorbing primary submodule of M with $(P :_R M) \cap S_2 = \phi$, therefore there exists $s_1 \in S_1$ such that $s_1ab \in \sqrt{P :_R M}$ or $s_1m \in P$. Since $S_1 \subseteq S_2$, therefore $s_1 \in S_2$. Hence P is an S_2-1-absorbing primary submodule of M.

(3). Let P be an S-1-absorbing primary submodule of M. First, we show that $(P :_R M) \cap S^* = \phi$. Let $x \in (P :_R M) \cap S^* = \phi$. Then $x \in S^*$. This implies that $\frac{x}{1}$ is a unit of $S^{-1}R$ and so $\frac{x}{1}.\frac{a}{s} = 1$, for some $a \in R$. Thus $xa = s$ or $s^{'}s = s^{'}xa$ for some $s^{'} \in S$. Also, $s^{''} = s^{'}s = s^{'}xa$. It follows that $s^{''} \in (P :_R M) \cap S$, which is a contradiction as $(P :_R M) \cap S = \phi$. Thus

$(P :_R M) \cap S^* = \phi$. Since $S \subseteq S^*$, P is an S^*-1-absorbing primary submodule of M by (2). Conversely, let $abm \in P$ for some non-units $a, b \in R$ and $m \in M$. Then there exists $s^{'} \in S^*$ such that $s^{'}ab \in \sqrt{P :_R M}$ or $s^{'}m \in P$ as P is an S^*-1-absorbing primary submodule of M. Since $\frac{s^{'}}{1}$ is a unit of $S^{-1}R$, $\frac{s^{'}}{1}\frac{a^{'}}{s} = 1$. It gives $s^{'}a^{'} = s$. It follows that $s_1 s = s_1 s^{'} a^{'}$ for some $s_1 \in S$. Also, $s^{''} = s_1 s = s_1 s^{'} a^{'}$. Now, $s^{''}ab = s_1 a^{'} s^{'} ab \in \sqrt{P :_R M}$ or $s^{''}m = s_1 a^{'} s^{'} m \in P$. Hence P is an S-1-absorbing primary submodule of M.

(4). Let P be an S-1-absorbing primary submodule of M. Let $\frac{r_1}{s_1}\frac{r_2}{s_2}\frac{m}{t} \in S^{-1}P$ for some non-units $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$. Then $xr_1r_2m \in P$ for some $x \in S$ and also r_1 and r_2 are non-units. Suppose that r_1 is a unit, so there exists $x_1 \in R$ such that $r_1x_1 = 1 = x_1r_1$. Then we can write $\frac{r_1}{s_1}\frac{s_1x_1}{1} = 1 = \frac{s_1x_1}{1}\frac{r_1}{s_1}$. So $\frac{r_1}{s_1}$ is a unit, which is a contradiction. Since P is an S-1-absorbing primary submodule of M, there exists an $s \in S$ such that $sxr_1r_2 \in \sqrt{P :_R M}$ or $sm \in P$. Then $s\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{sxr_1r_2}{xs_1s_2} \in S^{-1}\sqrt{P :_R M} \subseteq \sqrt{S^{-1}P : S^{-1}M}$. So $s\frac{r_1}{s_1}\frac{r_2}{s_2} \in \sqrt{S^{-1}P : S^{-1}M}$ or $s\frac{m}{t} = \frac{sm}{t} \in S^{-1}P$. Hence $S^{-1}P$ is an S-1-absorbing primary submodule of $S^{-1}M$. □

Remark 2.3. The converse of Proposition 2.2(4) need not be true. For example, consider the setup from [3, Example 2.3]. Let $\mathbb{Q} \times \mathbb{Q}$ be a $\mathbb{Z}$-module where $\mathbb{Q}$ is the field of rational numbers. Let $S = \mathbb{Z} - \{0\}$ be a multiplicatively closed subset of $\mathbb{Z}$ and $\mathbb{N} = \mathbb{Z} \times 0$ be a submodule of $\mathbb{Q} \times \mathbb{Q}$. Now $(N :_{\mathbb{Z}} (\mathbb{Q} \times \mathbb{Q})) = 0$ and $\sqrt{\mathbb{N} :_{\mathbb{Z}} (\mathbb{Q} \times \mathbb{Q})} = 0$. Let s be an element of S. Let p be a prime number with gcd $(p, s) = 1$. Then, we observe that for $a = p$, $b = p$ and $m = (\frac{1}{p}, 0)$, $abm = (p, 0) \in N$. But $sp^2 \notin \sqrt{\mathbb{N} :_{\mathbb{Z}} (\mathbb{Q} \times \mathbb{Q})}$ and $s(\frac{1}{p}, 0) \notin N$. Thus N is not an S-1-absorbing primary submodule of M. Also, by [3, Example 2.3], $S^{-1}N$ is a 1-absorbing prime submodule of $S^{-1}(\mathbb{Q} \times \mathbb{Q})$. Hence $S^{-1}N$ is an S-1-absorbing primary submodule of $S^{-1}(\mathbb{Q} \times \mathbb{Q})$.

Now we provide some characterizations of an S-1-absorbing primary submodule of a module.

Theorem 2.4. *Let S be a multiplicative closed subset of R and M be an R-module. Let $N \subseteq M$ with $(N :_R M) \cap S = \phi$. Then the following are equivalent:*

(i) N is an S-1-absorbing primary submodule of M.

(ii) There exists an $s \in S$ such that if $abN_1 \subseteq N$ for some non-units $a, b \in R$ and a submodule N_1 of M, then either $sab \in \sqrt{N :_R M}$ or $sN_1 \subseteq N$.

(iii) There exists an $s \in S$ such that if $IJN_1 \subseteq N$ for some ideals $I, J \in R$ and a submodule N_1 of M, then either $sIJ \subseteq \sqrt{N :_R M}$ or $sN_1 \subseteq N$.

Proof. (1) $\Longrightarrow$ (2). Let $abN_1 \subseteq N$ for some non-units $a, b \in R$ and a submodule N_1 of M. Suppose that $sN_1 \nsubseteq N$. Then there exists some non-unit $n_1 \in N$ such that $sn_1 \notin N$. But $abn_1 \in N$ and N is an S-1-absorbing primary submodule of M. Then $sab \in \sqrt{N :_R M}$.

(2) $\Longrightarrow$ (3). Let $IJN_1 \subseteq N$ for some ideals $I, J \in R$ and a submodule N_1 of M. Suppose $sIJ \nsubseteq \sqrt{N :_R M}$. Then there exist non-units $a \in I$ and $b \in J$ such that $sab \notin \sqrt{N :_R M}$. But $abN_1 \subseteq N$. Then $sN_1 \subseteq N$ by (2).

(3) $\Longrightarrow$ (1). Let $abm \in N$ for some non-units $a, b \in R$ and $m \in M$. Let $N_1 = Rm$, $I = Ra$ and $J = Rb$. Then $IJN_1 = RaRbRm = Rabm \subseteq N$. Thus there exists an $s \in S$ such that $sIJ \subseteq \sqrt{N :_R M}$ or $sN_1 \subseteq N$. So $sRaRb \subseteq \sqrt{N :_R M}$ or $sRm \subseteq N$. It follows that $sab \in \sqrt{N :_R M}$ or $sm \in N$. Hence N is an S-1-absorbing primary submodule of M.

□

Corollary 2.5. *Let S be a multiplicative closed subset of R and N be an ideal of R with $N \cap S = \phi$. Then the following are equivalent:*

(i) N is an S-1-absorbing primary ideal of R.

(ii) There exists an $s \in S$ such that if $abI \subseteq N$ for some non-units $a, b \in R$ and an ideal I of R, then either $sab \in \sqrt{N}$ or $sI \subseteq N$.

(iii) There exists an $s \in S$ such that if $IJK \subseteq N$ for some ideals I, J and K of R, then either $sIJ \subseteq \sqrt{N}$ or $sK \subseteq N$.

In the following, we generalize [3, Theorem 2.3].

Theorem 2.6. *Let S be a multiplicative closed subset of R and M be an R-module. Let N be a submodule of M with $(N :_R M) \cap S = \phi$. Then the following are equivalent:*

(i) N is an S-1-absorbing primary submodule of M.

(ii) $(N :_M s^2)$ is an S-1-absorbing primary submodule of M for some $s \in S$.

Proof. $(1) \implies (2)$. Let $abm \in (N :_M s^2)$ for some non-units $a, b \in R$ and $m \in M$. Then $s^2abm \in N$. Since N is an S-1-absorbing primary submodule of M, there exists an $s_1 \in S$ such that $s_1s^2ab \in \sqrt{N :_R M}$ or $s_1m \in N$. Also, $s_2sab \in \sqrt{N :_R M}$ or $s_1m \in N$ where $s_1s = s_2$. This implies that $sab \in \sqrt{(N :_R M) :_R s_2} \subseteq \sqrt{N :_R M} :_R s_2^2$. Now, we show that $\sqrt{N :_R M} :_R s_2^2 \subseteq \sqrt{(N :_R M) :_R s^2}$. Let $x \in \sqrt{N :_R M} :_R s_2^2$. So $s_2^2x \in \sqrt{N :_R M}$. Then $(s_2^n)^2x^n \in (N :_R M)$ for some $n \in \mathbb{N}$, so $x \in \sqrt{(N :_R M) : (s_2^n)^2}$. Also, $x \in \sqrt{(N :_R M) :_R (s_2^n)^2} \subseteq \sqrt{(N :_R M) : s}$ by Lemma 3.3(ii). This implies that $x \in \sqrt{(N :_R M) : s} \subseteq \sqrt{(N :_R M) :_R s^2}$. Thus $(\sqrt{(N :_R M)} :_R s^2) \subseteq \sqrt{(N :_R M) : s^2}$. So $sab \in \sqrt{(N :_R M) :_R s^2} \subseteq \sqrt{(N :_R s^2) :_R M}$. Now $s_1m \in N$ implies that $m \in (N :_M s_1) \subseteq (N :_M s_1^2)$ and $(N :_M s_1^2) \subseteq (N :_M s) \subseteq (N :_M s^3)$ by Lemma 3.3(ii). Then $sm \in (N :_M s^2)$. It follows that $(N :_M s^2)$ is an S-1-absorbing primary submodule of M.

$(2) \implies (1)$. Let $abm \in N$. Then $abm \in (N :_M s^2)$. Since $(N :_M s^2)$ is an S-1-absorbing primary submodule of M, there exists $s \in S$ such that $sab \in \sqrt{(N :_R s^2) :_R M}$ or $sm \in (N :_M s^2)$. So $s^n(ab)^n \in (N :_R M) :_R s^2 \subseteq (N :_R M) :_R (s^2)^n$ for some $n \in \mathbb{N}$. Then $(s^2)^n s^n (ab)^n \in (N :_R M)$. Thus $s_1ab \in \sqrt{(N :_R M)}$ where $s_1 = s^2s$. Now $sm \in (N :_M s^2)$. Then $s^2sm \in N$. Hence N is an S-1-absorbing primary submodule of M. □

3 General properties

Proposition 3.1. *Let M and $M^{'}$ be R-modules and $f : M \to M^{'}$ be an R-homomorphism.*

(i) If $N^{'}$ is an S-1-absorbing primary submodule of $M^{'}$ with $(f^{-1}(N^{'}) :_R M) \cap S = \phi$, then $f^{-1}(N^{'})$ is an S-1-absorbing primary submodule of M.

(ii) If f is an epimorphism and N is an S-1-absorbing primary submodule of M with $Kerf \subseteq N$, then $f(N)$ is an S-1-absorbing primary submodule of $M^{'}$.

Proof. (1). Let $abm \in f^{-1}(N^{'})$ for some non-units $a, b \in R$ and $m \in M$. Then $f(abm) = abf(m) \in N^{'}$. Since $N^{'}$ is an S-1-absorbing primary submodule of $M^{'}$ there exists $s \in S$ such that $sab \in \sqrt{N^{'} :_R M^{'}}$ or $sf(m) \in N^{'}$. Now, we show that $\sqrt{N^{'} :_R M^{'}} \subseteq \sqrt{f^{-1}(N^{'}) :_R M}$. Let $x \in \sqrt{N^{'} :_R M^{'}}$. Then $x^nM^{'} \subseteq N^{'}$, $n \in \mathbb{N}$. Now, $f(x^nM) = x^nf(M) \subseteq x^nM^{'} \subseteq N^{'}$, so $f(x^nM) \subseteq N^{'}$. This implies that $x^nM \subseteq x^nM + Kerf = f^{-1}(f(x^nM)) \subseteq f^{-1}(N^{'})$. So, $x^nM \subseteq f^{-1}(N^{'})$. It follows that $x \in \sqrt{f^{-1}(N^{'}) :_R M}$. Then $\sqrt{N^{'} :_R M^{'}} \subseteq \sqrt{f^{-1}(N^{'}) :_R M}$. Thus either $sab \in \sqrt{f^{-1}(N^{'}) :_R M}$ or $sm \in f^{-1}(N^{'})$. Hence $f^{-1}(N^{'})$ is an S-1-absorbing primary submodule of M.

(2). First we show that $(f(N) :_R M^{'}) \cap S = \phi$. Let $s \in (f(N) :_R M^{'}) \cap S$. Then $s \in (f(N) :_R M^{'})$. So, $sM^{'} \subseteq f(N)$. This implies that $f(sM) = sf(M) \subseteq sM^{'} \subseteq f(N)$. So $sM \subseteq sM + Kerf \subseteq N + Kerf = N$. Then $sM \subseteq N$. It follows that $s \in (N :_R M)$, which is a contradiction as N is an S-1-absorbing primary submodule of M. Hence $(f(N) :_R M^{'}) \cap S = \phi$. Let $abm^{'} \in f(N)$ for some non-units $a, b \in R$ and $m^{'} \in M^{'}$. Since f is epimorphism, there exists an $m \in M$ such that $f(m) = m^{'}$. Then $f(abm) = abf(m) = abm^{'} \in f(N)$. This implies that $abm \in N + Kerf \subseteq N$, so $abm \in N$. Then there exists $s \in S$ such that $sab \in \sqrt{N :_R M}$ or $sm \in N$ as N is an S-1-absorbing primary submodule of M. Since $\sqrt{N :_R M} \subseteq \sqrt{f(N) :_R M'}$, therefore $sab \in \sqrt{f(N) :_R M'}$ or $sm^{'} = sf(m) = f(sm) \in f(N)$. Hence $f(N)$ is an S-1-absorbing primary submodule of $M^{'}$. □

Proposition 3.2. *Let S be a multiplicative closed subset of R and L be a submodule of an R-module M. Let N be a submodule of M and $L \subseteq N$. Then N is an S-1-absorbing primary submodule of M if and only if N/L is an S-1-absorbing primary submodule of M/L.*

Proof. Let N be a submodule of M with $L \subseteq N$. Consider a canonical homomorphism $\pi : M \to M/L$ by $\pi(m) = m + L$ for all $m \in M$. Then N/L is an S-1-absorbing primary submodule of M/L by Proposition 3.1(ii). Conversely, suppose that N/L is an S-1-absorbing primary submodule of M/L. Let $abm \in N$ for some non-units $a, b \in R$ and $m \in M$. Then $ab(m + L) \in N/L$. Since N/L is an S-1-absorbing primary submodule of M/L, there exists $s \in S$ such that $sab \in \sqrt{N/L :_R M/L} \subseteq \sqrt{N :_R M}$ or $s(m+L) \in N/L$. Then $sab \in \sqrt{N :_R M}$ or $sm \in N$. Hence N is an S-1-absorbing primary submodule of M. □

As a consequence of the following result, we have [3, Lemma 2.2].

Lemma 3.3. *Let S be a multiplicative closed subset of R and M be an R-module. If N is an S-1-absorbing primary submodule of M, then the following conditions hold for some $s \in S$:*

(i) $(N :_M s_1^2) \subseteq (N :_M s)$ *for all* $s_1 \in S$.

(ii) $((N :_R M) :_R s_1^2) \subseteq ((N :_R M) :_R s)$ *for all* $s_1 \in S$.

Proof. (1). Let $m \in (N :_M s_1^2)$. Then $s_1^2 m \in N$. If s_1 is a unit, then $m \in N$ and we are done. Suppose s_1 is non-unit. Since N is an S-1-absorbing primary submodule of M, there exists an $s \in S$ such that $ss_1^2 \in \sqrt{N :_R M}$ or $sm \in N$. If $ss_1^2 \in \sqrt{N :_R M}$, then $(ss_1^2)^n \in (N :_R M)$ for some $n \in \mathbb{N}$. So $(ss_1^2)^n \in (N :_R M) \cap S$ which is not possible as N is an S-1-absorbing primary submodule of M. Then $sm \in N$ implies that $m \in (N :_M s)$. Thus $(N :_M s_1^2) \subseteq (N :_M s)$ for all $s_1 \in S$.

(2). Let $x \in ((N :_R M) :_R s_1^2)$. Then $xM \subseteq (N :_M s_1^2) \subseteq (N :_M s)$ by (1). This implies that $x \in (N :_R M) :_R s$. Hence $((N :_R M) :_R s_1^2) \subseteq ((N :_R M) :_R s)$ for all $s_1 \in S$. □

Corollary 3.4. *[3, Lemma 2.2] Let S be a multiplicative closed subset of R and N be an S-1-absorbing prime submodule of an R-module M. The following statements hold for some $s \in S$:*

(i) $(N :_M s'^2) \subseteq (N :_M s)$ *for all* $s' \in S$.

(ii) $((N :_R M) :_R s'^2) \subseteq ((N :_R M) :_R s)$ *for all* $s' \in S$.

Remark 3.5. (i) The converse of Lemma 3.3 need not be true. For example, consider the case from Remark 2.3. Then N is not an S-1-absorbing primary submodule of M. Let s_1 be any element of S and s be some element of S. Then $(N :_M s_1^2) = 0$ and $(N :_M s) = 0$. Thus $(N :_M s_1^2) \subseteq (N :_M s)$.

(ii) We find that a result analogous to [3, Theorem 3.1] does not hold for S-1-absorbing primary ideal. For example, consider $M = \mathbb{Z}$ as $\mathbb{Z}$-module. We know that M is a faithful multiplication $\mathbb{Z}$-module. Let $N = 8\mathbb{Z}$ be a submodule of M and $S = \{1\}$ be a multiplicatively closed subset of $\mathbb{Z}$. Then $(N :_R M) = 8\mathbb{Z}$ and $\sqrt{N :_R M} = 2\mathbb{Z}$. Suppose $a = 2$, $b = 2$ and $m = 2$, then $abm \in N$. Consider $I = (N :_R M)$ which is an ideal of $\mathbb{Z}$. It is easy to see that $(N :_R M)$ is an S-1-absorbing primary ideal. Here $IM = (N :_R M)M = N$ as M is a multiplicattion module. But $1.a.b \notin (N :_R M)$ and $1.m \notin N$. In the following, we generalize [3, Theorem 3.1] and get it's proof interesting by taking S-1-absorbing primary ideals in place of S-1-absorbing prime ideals.

Recall from [3], let M be an R-module. If P is a maximal ideal of R then $T_P(M) = \{m \in M : (1-p)m = 0 \ for\ some\ p \in P\}$. Clearly $T_P(M)$ is a submodule of M. Also, M is P-cyclic provided there exist $q \in P$ and $m \in M$ such that $(1 - q)M \subseteq Rm$.

Theorem 3.6. *Let M be a multiplication faithful R-module and S be a multiplicative closed subset of R. Let I be an S-1-absorbing primary ideal of R. Let a and b be two non-unit elements in R and $m \in M$. If $abm \in IM$, then there exists an s in S such that $sab \in \sqrt{I}$ or $sm \in IM$.*

Proof. Let a, b, c be any non-unit elements of R. Then $abc \in I$ implies that there exists an $s \in S$ such that $sab \in \sqrt{I}$ or $sc \in I$ as I is an S-1-absorbing primary ideal of R. Let $abm \in IM$, $m \in M$ and $sab \notin \sqrt{I}$. Consider $K = \{r \in R : rsm \in IM\}$. If $K = R$, then we are done. If $K \neq R$, then there exists a maximal ideal P of R such that $K \subseteq P$. We show that $m \notin T_P(M)$. Let $m \in T_P(M)$, so there exists an element $p \in P$ such that $(1 - p)m = 0$. Then

$(1-p) \in K \subseteq P$ which is a contradiction. Hence $T_P(M) \neq M$. By [7, Theorem 1.2], M is a P-cyclic module as M is a multiplication module. Then there are $p^{'} \in P$ and $m^{'} \in P$ such that $(1-p^{'})M \subseteq Rm^{'}$. So $(1-p^{'})sm \in Rm^{'}$. Then, there is $r_1 \in R$ such that $(1-p^{'})sm = r_1m^{'}$. It gives $(1-p^{'})sabm = r_1abm^{'} \in IM$ and $(1-p^{'})sabm \in Rm^{'}$. Then, there exists $a_1 \in I$ such that $(1-p^{'})sabm = a_1m^{'}$. Now, $r_1abm^{'} = a_1m^{'}$. This implies that $r_1ab - a_1 \in ann(m^{'})$. Then, $(1-p^{'})M \subseteq Rm^{'}$ implies that $(1-p^{'})ann(m^{'})M \subseteq Rann(m^{'})m^{'} = 0$. It follows that $(1-p^{'})ann(m^{'}) \subseteq ann(M)$. Since M is faithful, therefore $(1-p^{'})ann(m^{'}) = 0$. Thus $(1-p^{'})(r_1ab - a_1) = 0$. So $r_1ab(1-p^{'}) = a_1(1-p^{'}) \in I$. Hence $r_1ab(1-p^{'}) \in I$. Now, we have two cases for r_1. First suppose that r_1 is a unit. Then $ab(1-p^{'}) \in I$. Now, if $(1-p^{'})$ is a unit, then $ab \in I$. So $sab \in \sqrt{I}$, a contradiction. Suppose that $(1-p^{'})$ is non-unit. Then $sab \in \sqrt{I}$ or $s(1-p^{'}) \in I$ as I is an S-1-absorbing primary ideal of R. If $sab \in \sqrt{I}$, we have a contradiction. Let $s(1-p^{'}) \in I$, so $(1-p^{'})sm \in IM$. Then $(1-p^{'}) \in K \subseteq P$ which is a contradiction. Now, suppose that r_1 is a non-unit. If $(1-p^{'})$ is a unit, then $r_1ab \in I$. So, $sab \in \sqrt{I}$ or $sr_1 \in I$ because I is an S-1-absorbing primary ideal of R. If $sab \in \sqrt{I}$, then a contradiction. Suppose $sr_1 \in I$, then $sr_1m^{'} \in IM$. Also, $sr_1m^{'} = (1-p^{'})sm$ implies that $(1-p^{'})sm \in IM$. So, $(1-p^{'}) \in K \subseteq P$ which is a contradiction. If $(1-p^{'})$ is a non-unit, then $r_1ab(1-p^{'}) \in I$ implies that $sab \in \sqrt{I}$ or $sr_1(1-p^{'}) \in I$. Since I is an S-1-absorbing primary, $sab \in \sqrt{I}$ or $s^2r_1 \in \sqrt{I}$ or $s(1-p^{'}) \in I$. If $sab \in \sqrt{I}$, we have a contradiction. If $s^2r_1 \in \sqrt{I}$, then $(s^n)^2r_1^n \in I$ for some $n \in \mathbb{N}$. This implies that $r_1^n \in I :_R (s^n)^2 \subseteq I :_R s$, by Lemma 3.3. So, $sr_1^n \in I$. Also, $sr_1^nm^{'} \in IM$. Thus $r_1^n \in K \subseteq P$. It follows that $r_1 \in \sqrt{P}$. Since R is a commutative ring with identity, therefore P is a prime ideal. So, $r_1 \in P$ which is a contradiction as P is a maximal ideal of R. Let $s(1-p^{'}) \in I$. This implies that $(1-p^{'})sm \in IM$. Then, $(1-p^{'}) \in K \subseteq P$ which is again a contradiction. So, $K = R$. Hence $sm \in IM$. □

Proposition 3.7. *If N is a proper submodule of a Noetherian module M and S is a multiplicative closed subset of R with $(N_i :_R M) \cap S = \phi$ for each i, then N has an S-1-absorbing primary decomposition.*

Proof. If N is a proper submodule of a Noetherian module M, N has a primary decomposition. Since every primary submodule with $(N_i :_R M) \cap S = \phi$ is an S-1-absorbing primary submodule, N has an S-1-absorbing primary decomposition. □

4 Properties over a finitely generated module

Proposition 4.1. *Let S be a multiplicative closed subset of R and M be a finitely generated R-module. Let N be a submodule of M with $(S^{-1}N :_R S^{-1}M) \cap S = \phi$. Then the following are equivalent:*

(i) N is an S-1-absorbing primary submodule of M.

(ii) $S^{-1}N$ is an S-1-absorbing primary submodule of $S^{-1}M$ and there exists an $s \in S$ with $(N :_M s_1^2) \subseteq (N :_M s)$ for all $s_1 \in S$.

Proof. (1) $\implies$ (2). It follows from Proposition 2.2(iv) and Lemma 3.3.

(2) $\implies$ (1). Let $abm \in N$ for some non-units $a, b \in R$ and $m \in M$. Then $\frac{abm}{1} \in S^{-1}N$. Since $S^{-1}N$ is an S-1-absorbing primary submodule of $S^{-1}M$ and M is finitely generated, therefore there exists an $s \in S$ such that $(sab)^nS^{-1}M \subseteq S^{-1}N$ or $sm \in S^{-1}N$. So $(sab)^n \in (S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ or $sm \in S^{-1}N$. Then $s_1(sab)^n \in (N :_R M)$ for some $s_1 \in S$. Also, $s_1^n(sab)^n \in (N :_R M)$. Then $s_3ab \in \sqrt{N :_R M}$, where $s_3 = s_1s$. Thus $s_2s_3ab \in \sqrt{N :_R M}$, $s_2 \in S$. Let $s_2sm \in N$ for some $s_2 \in S$. This implies that $s_1s_2sm \in N$. Hence N is an S-1-absorbing primary submodule of M. □

In [7], an R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Since $I \subseteq (N :_R M)$, therefore $N = IM \subseteq (N :_R M)M \subseteq N$, so $N = (N :_R M)M$. Let $N = I_1M$ and $L = I_2M$ are submodules of a multiplication R-module M. Then the product NL of N and L is defined by $NL = I_1I_2M$. Now, we discuss some properties over a multiplication and finitely generated multiplication modules.

Proposition 4.2. *Let S be a multiplicative closed subset of R and M be an R-module. If N is an S-1-absorbing primary submodule of M, then $(N :_R M)$ is an S-1-absorbing primary ideal of R. The converse is true, whenever M is a multiplication module.*

Proof. Let N be an S-1-absorbing primary submodule of M. Let $abc \in N :_R M$ for some non-units $a, b, c \in R$. So $ab(cM) \subseteq N$ or $RaRb(cM) \subseteq N$. Then by Theorem 2.4, there exists $s \in S$ such that $sRaRb \subseteq \sqrt{N :_R M}$ or $s(cM) \subseteq N$. Thus either $sab \in \sqrt{N :_R M}$ or $sc \in (N :_R M)$. Hence $(N :_R M)$ is an S-1-absorbing primary ideal of R. Conversely, let M be a multiplication module and $(N :_R M)$ is an S-1-absorbing primary ideal of R. Let $IJN_1 \subseteq N$ for some ideals I, J of R and some $N_1 \subseteq M$. First, we show that $IJ(N_1 :_R M) \subseteq (IJN_1 :_R M)$. Let $x \in IJ(N_1 :_R M)$. So $x = IJa$ where $a \in N_1 :_R M$. Thus $aM \subseteq N_1$. Then $xM = IJaM \subseteq IJN_1$, so $xM \subseteq IJN_1$. This implies that $x \in (IJN_1 :_R M)$. Therefore $IJ(N_1 :_R M) \subseteq (IJN_1 :_R M)$. Let $x \in (IJN_1 :_R M)$, so $xM \subseteq IJN_1$. Then, $xM \subseteq N$, this implies that $x \in (N :_R M)$. So $(IJN_1 :_R M) \subseteq (N :_R M)$. It follows that $IJ(N_1 :_R M) \subseteq (IJN_1 :_R M) \subseteq (N :_R M)$. Since $(N :_R M)$ be an S-1-absorbing primary ideal of R, therefore there exists $s \in S$ such that $sIJ \subseteq \sqrt{N :_R M}$ or $s(N_1 :_R M) \subseteq (N :_R M)$ by Corollary 2.5. Also, either $sIJ \subseteq \sqrt{N :_R M}$ or $sN_1 = s(N_1 :_R M)M \subseteq N$ as M is a multiplication module. Hence N is an S-1-absorbing primary submodule of M. □

Remark 4.3. The converse of Proposition 4.2 need not be true in general. For example, consider the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$. Let $N = 6\mathbb{Z} \times 0$ be a submodule of M. Then $(6\mathbb{Z} \times 0) :_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) = 0$. So $(N :_{\mathbb{Z}} M)$ is an S-absorbing primary ideal of R. Now we have $\sqrt{N :_{\mathbb{Z}} M} = 0$. Let $a = 2$, $b = 3$ and $m = (1, 0)$. Then $abm = 2.3.(1, 0) \in N$. Let $S = \mathbb{Z} - 6\mathbb{Z}$. Then for any $s \in S$, $sab \notin \sqrt{N :_{\mathbb{Z}} M}$ and $s(1, 0) = (s, 0) \notin N$. Hence N is not an S-1-absorbing primary submodule of M.

Proposition 4.4. *Let S be a multiplicative closed subset of R and M be an R-module. Let N be a submodule of M with $(N :_R M) \cap S = \phi$. If M is a finitely generated multiplication module, then the following are equivalent:*

(i) N is an S-1-absorbing primary submodule of M.

(ii) $KLP \subseteq N$ for some submodules K, L and P of M implies that there exists an $s \in S$ such that $sKL \subseteq rad(N)$ or $sP \subseteq N$.

Proof. (1) $\implies$ (2). Let $KLP \subseteq N$ for some submodules K, L and P of M. Then $(K : M)(L : M)(P : M)M \subseteq N$. So $(K : M)(L : M)P \subseteq N$ as M is a multiplication module. By Theorem 2.4, there exists $s \in S$ such that $s(K : M)(L : M) \subseteq \sqrt{N :_R M}$ or $sP \subseteq N$. This implies that $sKL \subseteq \sqrt{N :_R M}M$ or $sP \subseteq N$. Since M is a multiplication module, $\sqrt{N :_R M}M = rad(N)$ by [7, Theorem 2.12]. Then $sKL \subseteq rad(N)$ or $sP \subseteq N$.

(2) $\implies$ (1). Let $IJK \subseteq N : M$ for some ideals I, J and K of R. So $IJKM \subseteq N$. Then $(IM)(JM)(KM) \subseteq N$ as M is a multiplication module. Since N is an S-1-absorbing primary submodule of M, therefore there exists $s \in S$ such that $s(IM)(JM) \subseteq rad(N)$ or $s(KM) \subseteq N$. Also $sIJ \subseteq rad(N) : M$ or $sK \subseteq N : M$. Since M is finitely generated, $(rad(N) : M) = \sqrt{N :_R M}$ by [5, Theorem 4.4]. Then $sIJ \subseteq \sqrt{N :_R M}$ or $sK \subseteq (N :_R M)$. Thus by Corollary 2.5, $(N :_R M)$ is an S-1-absorbing primary ideal of R. Hence by Proposition 4.2, N is an S-1-absorbing primary submodule of M. □

In the following, we generalize [3, Theorem 3.3].

Theorem 4.5. *Let S be a multiplicative closed subset of R and M be an R-module. Let N be a submodule of M with $(N :_R M) \cap S = \phi$ and M be a finitely generated multiplication module. Then the following are equivalent:*

(i) N is an S-1-absorbing primary submodule of M.

(ii) $(N :_R M)$ is an S-1-absorbing primary ideal of R.

(iii) $N = IM$ for some S-1-absorbing primary ideal I of R with $ann(M) \subseteq I$.

Proof. (1) $\iff$ (2). It follows from Proposition 4.2.

(2) $\implies$ (3). Let $(N :_R M)$ is an S-1-absorbing primary ideal of R. Then N is an S-1-absorbing primary submodule of a multiplication module M. So $N = IM$ for some S-1-absorbing primary ideal I of R. Let $x \in ann(M) = (0 :_R M)$, so $xM = 0 \in N = IM$. This implies that $x \in I$. Thus $ann(M) \subseteq I$.

(3) $\implies$ (1). Let $JKL \subseteq N$ for some ideals J, K of R and some submodule $L \subseteq M$. Then $JK(L :_R M)M \subseteq N = IM$ as N is a submodule of a multiplication module M, so $JK(L :_R M)M \subseteq IM$. By [8, Theorem 9], $JK(L :_R M) \subseteq I + annM = I$ as M is a finitely generated multiplication module. This implies that $JK(L :_R M) \subseteq I$ as $ann(M) \subseteq I$. Then by Corollary 2.5, there exists $s \in S$ such that $sJK \subseteq \sqrt{I}$ or $s(L :_R M) \subseteq I$. Now, let $x \in \sqrt{I}$. Then $x^n M \subseteq IM = N$, so $x \in \sqrt{N :_R M}$. Thus $sJK \subseteq \sqrt{N :_R M}$ or $sL = s(L :_R M)M \subseteq IM = N$. Hence N is an S-1-absorbing primary submodule of M. □

In [4], an R-module M is said to be *von-Neumann regular* module if for each $m \in M$, there exists $a \in R$ such that $Rm = aM = a^2M$. It is easy to see that von-Neumann regular modules are multiplication (see [9]). Also, if M is a finitely generated von-Neumann regular module, then $IM \cap JM = IJM$ for every ideal I and J of R by [4, Lemma 6 and Theorem 1]. Now, we discuss some properties over a finitely generated von-Neumann regular module.

Proposition 4.6. *Let S be a multiplicative closed subset of R and M be a finitely generated von-Neumann regular R-module. Let P be a submodule of M with $(P :_R M) \cap S = \phi$. Then P is an S-1-absorbing primary submodule of M if and only if there exists $s \in S$ such that $K \cap L \cap N \subseteq P$ for some submodules K, L and N of M implies that either $s(K \cap L) \subseteq rad(P)$ or $sN \subseteq P$.*

Proof. Let $K \cap L \cap N \subseteq P$ for some submodules K, L and N of M. Now $KLN = (K : M)(L : M)(N : M)M \subseteq K \cap L \cap N \subseteq P$. Since P is an S-1-absorbing primary submodule of M, there exists $s \in S$ such that $sKL \subseteq rad(P)$ or $sN \subseteq P$ by Proposition 4.4. Since M is a finitely generated von-Neumann regular module, for any N, N' of M we have $NN' = (N : M)(N' : M)M = (N : M)M \cap (N' : M)M = N \cap N'$ by [4, Lemma 6 and Theorem 1]. Then $s(K \cap L) \subseteq rad(P)$ or $sN \subseteq P$. Conversely, let $KLN \subseteq P$ for some submodules K, L and N of M. Now $K \cap L \cap N = (K : M)M \cap (L : M)M \cap (N : M)M = (K : M)(L : M)(N : M)M \subseteq KLN \subseteq P$. It follows by the assumption that there exists an $s \in S$ such that $s(K \cap L) \subseteq rad(P)$ or $sN \subseteq P$. Thus by [4, Lemma 6 and Theorem 1], it follows that $sKL \subseteq rad(P)$ or $sN \subseteq P$. Hence by Proposition 4.4, P is an S-1-absorbing primary submodule of M. □

5 Properties over a singleton multiplicative closed subest of a ring

Recall from [2], a submodule N of M is said to be *irreducible* if it cannot be expressed as the intersection of two submodules of M.

Proposition 5.1. *Let N be a proper submodule of an R-module M and $S = \{1\}$ be a multiplicative closed subset of R with $(N :_R M) \cap S = \phi$. If N is an irreducible submodule of M. Then the following are equivalent:*

(i) N is an S-1-absorbing primary submodule of M.

(ii) $(N :_M r) = (N :_M r^2)$ for some non-unit $r \in R \backslash \sqrt{N :_R M}$.

Proof. (1) $\implies$ (2). Let N be an S-1-absorbing primary submodule of M. Since $(N :_M r) \subseteq (N :_M r^2)$, first we show that $(N :_M r^2) \subseteq (N :_M r)$. Let $m \in (N :_M r^2)$, so $r^2m \in N$. Then $r^2 \in \sqrt{N :_R M}$ or $m \in N$. If $r^2 \in \sqrt{N :_R M}$, then $r \in \sqrt{N :_R M}$ which is a contradiction as $r \in R \backslash \sqrt{N :_R M}$. Suppose $m \in N$, so $rm \in N$. Then $m \in (N :_M r)$, so $(N :_M r^2) \subseteq (N :_M r)$. Hence $(N :_M r) = (N :_M r^2)$ for some non-unit $r \in R \backslash \sqrt{N :_R M}$.

(2) $\implies$ (1). Let $abm \in N$ for some non-units $a, b \in R$ and $m \in M$. Suppose $ab \notin \sqrt{N :_R M}$ and $m \notin N$. Then $a \notin \sqrt{N :_R M}$ and $b \notin \sqrt{N :_R M}$. If $a \in \sqrt{N :_R M}$ and $b \in \sqrt{N :_R M}$, then $ab \in (\sqrt{N :_R M})^2 \subseteq \sqrt{N :_R M}$, which is a contradiction. So, we assume that $(N : a) = (N : a^2)$ or $(N : b) = (N : b^2)$ by hypothesis. Suppose $(N : a) = (N : a^2)$. Now $N \subseteq (N+Ram) \cap (N+Rbm)$, then we have to show that $(N+Ram) \cap (N+Rbm) \subseteq N$. Let $x \in (N+Ram) \cap (N+Rbm)$. Then $x = p_1 + r_1am = p_2 + r_2bm$. So, $ax = ap_1 + r_1a^2m = ap_2r_2abm$. Since $ap_1 \in N$, $ap_2 \in N$ and $r_2abm \in N$, $r_1a^2m \in N$ implies that $r_1m \in (N : a^2) = (N : a)$.

So $r_1am \in N$, also $x \in N$. Thus $(N + Ram) \cap (N + Rbm) = N$, which is a contradiction. By the similar argument for $(N : b) = (N : b^2)$, we have again a contradiction. Therefore N is an S-1-absorbing primary submodule of M. □

Recall from [2], a module M is *cancellative* if whenever $am = an$ for elements $m, n \in M$ and $a \in R$, then $m = n$. Recall from [2], a submodule N of M is said to be *pure* if $rN = N \cap rM$ for every $r \in R$.

Proposition 5.2. *Let N be a proper submodule of a cancellative R-module M and $S = \{1\}$ be a multiplicative closed subset of R with $(N :_R M) \cap S = \phi$. Then N is a pure submodule of M if and only if N is an S-1-absorbing primary submodule of M with $\sqrt{N :_R M} = \{0\}$.*

Proof. Let N be a pure submodule of an R-module M with $abm \in N$ for some non-units $a, b \in R$ and $m \in M$. Suppose $1.ab \notin \sqrt{N :_R M}$. Then $abm \in abM \cap N = abN$ as N is a pure submodule of M. This implies that $abm = abn$ for some $n \in N$. So $m = n \in N$. It gives $1.m \in N$ as M is cancellative. Therefore N is an S-1-absorbing primary submodule of M. Now, let $0 \neq a \in \sqrt{N :_R M}$. Since $N \neq M$, there exists $x_1 \in M \backslash N$ such that $a^n x_1 \in a^n M \cap N = a^n N$ for some $n \in \mathbb{N}$. So, there exists $x_2 \in \mathbb{N}$ such that $a^n x_1 = a^n x_2$. This implies that $x_1 = x_2$ which is a contradiction. Hence $\sqrt{N :_R M} = \{0\}$. Conversely suppose that $abm \in abM \cap N$, so $abm \in N$. Since N is an S-1-absorbing primary submodule of M, $1.ab \in \sqrt{N :_R M}$ or $1.m \in N$. Suppose that $ab \in \sqrt{N :_R M}$. Then $ab = 0$ as $\sqrt{N :_R M} = \{0\}$. So we have $abm = 0 \in abN$. Suppose that $1.m \in N$. So $abm \in abN$, then $abM \cap N \subseteq abN$. Also, $abN \subseteq abM \cap N$. Thus $abM \cap N = abN$. Hence N is a pure submodule of M. □

In the following, we find a sufficient condition for a ring R to be quasilocal. First we recall a result.

Lemma 5.3. *[1, Lemma 1] Let R be a ring. Suppose that for every non-unit element w of R and for every unit element u of R, we have $w + u$ is a unit element of R. Then R is a quasilocal ring.*

Proposition 5.4. *Let $S = \{1\}$ be a multiplicative closed subset of a ring R. Let I be an S-1-absorbing primary ideal of R which is not a primary ideal of R. Then R is a quasilocal ring.*

Proof. Let I be an S-1-absorbing primary ideal of R and I is not a primary ideal of R. Then there exist non-unit elements a and b with $ab \in I$ such that $a \notin \sqrt{I}$ and $b \notin I$. Let c be a non-unit element of R. Then $cab \in I$ implies that $ca \in \sqrt{I}$ or $b \in I$ as I is an S-1-absorbing primary ideal of R. So, $ca \in \sqrt{I}$ as $b \notin I$. Now, consider a unit element u of R. Suppose that $c + u$ is not a unit element of R. Then $(c + u)ab \in I$ implies that $(c + u)a \in \sqrt{I}$ or $b \in I$ as I is an S-1-absorbing primary ideal of R. Then $(c + u)a \in \sqrt{I}$ as $b \notin I$. So $ca + ua \in \sqrt{I}$. Since $ca \in \sqrt{I}$, therefore $a \in \sqrt{I}$. This is a contradiction. Therefore, $c + u$ is a unit element of R. Hence R is a quasilocal ring. □

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