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A NEW LOOK AT THE EULER-RODRIGUES FORMULA FOR THREE-DIMENSIONAL ROTATION

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Abstract In this short paper, we review the Euler-Rodrigues formula for three-dimensional rotation via fractional powers of matrices. We derive the rotations by any angle through the spectral behavior of the fractional powers of the rotation matrix by $\frac{\pi}{2}$ in $\mathbb{R}^3$ about some axis.

1 Introduction

The Euler-Rodrigues formula describes the rotation of a vector in three dimensions, it was first discovered by Euler [4] and later rediscovered independently by Rodrigues [8] and it is related to several interesting problems in computer graphics, dynamics, kinematics, mathematics, and robotics, see Cheng and Gupta [2] and references therein.

Reviews of the Euler–Rodrigues formula in different mathematical forms can be found in the literature, see e.g., Dai [3], Kahvecí, Yayli and Gök [5] and Mebius [6]. Here, we explored the geometric aspect of the classical Balakrishnan formula in [1] to obtain a new algorithm for the generation of a three-dimensional rotation matrix.

Fractional powers have been extensively studied in various branches of mathematics, playing a significant role in the understanding of complex phenomena. They find applications in diverse areas, including differential equations and fractional calculus, see e.g. [9], [7].

To our best knowledge, this treatment on the Euler–Rodrigues formula has not yet been explored in the literature.

2 Three-dimensional rotations

Firstly, we present some facts of the theory of fractional powers of matrices. Secondly, we establish the main results of this paper; namely, we review the Euler-Rodrigues formula via the Balakrishnan formula on fractional powers of matrices.

2.1 Fractional powers of operators

In this subsection, we recall some definitions and summarize without proof the results of the theory of fractional powers of matrices, in the sense of Balakrishnan [1].

Definition 2.1. For $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $(-\infty, 0)$ and $\alpha \in \mathbb{R}$, $A^\alpha = e^{\alpha \log A}$, where $\log A$ is the principal logarithm.

Thanks to Balakrishnan [1] we following results are well-known.

Proposition 2.2. *Let* $0 < \alpha < 1$. *We have*

(i)

$$A^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda I + A)^{-1} d\lambda; \tag{2.1}$$

(ii) *Let β be a real number, then*

$$(A^\alpha)^\beta = A^{\alpha\beta}.$$

2.2 Main results

In this subsection, we present the main results of this paper. We explored the geometric aspect of the classical Balakrishnan formula (2.1) (see, e.g., Balakrishnan [1]) to obtain a new algorithm for the generation of three-dimensional rotation matrices. Here, the matrix representations of linear operators on $\mathbb{R}^3$ are considered using the standard basis of $\mathbb{R}^3$, and $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ denotes a vector in $\mathbb{R}^3$ with $n_1^2 + n_2^2 + n_3^2 = 1$.

Lemma 2.3. *The matrix which represents the rotation by an angle $\frac{\pi}{2}$ about the axis $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ is given by*

$$A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big) = \begin{bmatrix} n_1^2 & n_1n_2 - n_3 & n_1n_3 + n_2 \\ n_1n_2 + n_3 & n_2^2 & n_2n_3 - n_1 \\ n_1n_3 - n_2 & n_2n_3 + n_1 & n_3^2 \end{bmatrix}. \tag{2.2}$$

Proof. Choose two vectors, $\hat{\mathbf{l}}$ and $\hat{\mathbf{m}}$, such that $\{\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}\}$ is a right-handed orthonormal basis. Let $u = a\hat{\mathbf{l}} + b\hat{\mathbf{m}} + c\hat{\mathbf{n}}$, with $a, b, c \in \mathbb{R}$, be any vector to be rotated by an angle $\frac{\pi}{2}$ counterclockwise about the axis $\hat{\mathbf{n}}$. The resulting vector u' is the vector u with its component in the $\hat{\mathbf{l}}, \hat{\mathbf{m}}$ plane rotated by $\frac{\pi}{2}$

$$\begin{aligned} u' &= -b\hat{\mathbf{l}} + a\hat{\mathbf{m}} + c\hat{\mathbf{n}} \\ &= \hat{\mathbf{n}} \times u + \langle u, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}}. \end{aligned}$$

Consider the standard basis $\{\hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}, \hat{\mathbf{e_3}}\}$ of $\mathbb{R}^3$. If u is written as

$$u = u_1\hat{\mathbf{e_1}} + u_2\hat{\mathbf{e_2}} + u_3\hat{\mathbf{e_3}},$$

then

$$\begin{aligned} u' &= \hat{\mathbf{n}} \times u + \langle u, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}} \\ &= (n_2u_3 - n_3u_2 + u_1n_1^2 + u_2n_1n_2 + u_3n_1n_3)\hat{\mathbf{e_1}} \\ &+ (n_3u_1 - n_1u_3 + u_1n_1n_2 + u_2n_2^2 + u_3n_2n_3)\hat{\mathbf{e_2}} \\ &+ (n_1u_2 - n_2u_1 + u_1n_1n_3 + u_2n_2n_3 + u_3n_3^2)\hat{\mathbf{e_3}}. \end{aligned}$$

Therefore, the matrix representation of this rotation is

$$A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big) = \begin{bmatrix} n_1^2 & n_1n_2 - n_3 & n_1n_3 + n_2 \\ n_1n_2 + n_3 & n_2^2 & n_2n_3 - n_1 \\ n_1n_3 - n_2 & n_2n_3 + n_1 & n_3^2 \end{bmatrix}. \square$$

□

Remark 2.4. Thanks to the characterization in (2.2) of the matrix which represents the rotation by an angle $\frac{\pi}{2}$ about the axis $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ we can obtain a matrix characterization of the linear semigroup generated by $A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)$, namely the uniformly continuous semigroup of bounded linear operators generated by $A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)$, denoted by $T(\cdot)$, has the following explicit representation

$$T(t) = e^{tA(\hat{\mathbf{n}}, \frac{\pi}{2})} = \sum_{n=0}^{\infty} \frac{(tA(\hat{\mathbf{n}}, \frac{\pi}{2}))^n}{n!} =$$

$$\begin{bmatrix} n_1^2(e^t - \cos t) + \cos t & n_1n_2(e^t - \cos t) - n_3 \sin t & n_1n_3(e^t - \cos t) + n_2 \sin t \\ n_1n_2(e^t - \cos t) + n_3 \sin t & n_2^2(e^t - \cos t) + \cos t & n_2n_3(e^t - \cos t) - n_1 \sin t \\ n_1n_3(e^t - \cos t) - n_2 \sin t & n_2n_3(e^t - \cos t) + n_1 \sin t & n_3^2(e^t - \cos t) + \cos t \end{bmatrix}$$

for any $t \geqslant 0$.

Remark 2.5. An explicit formula for the matrix elements of a general 3×3 rotation matrix can be find in Rodrigues [8]; namely, if $R(\hat{\mathbf{n}}, \theta)$ denotes the a rotation by an angle θ about an axis $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ $(n_1^2 + n_2^2 + n_3^2 = 1)$, whose elements are denoted by $R_{ij}(\hat{\mathbf{n}}, \theta)$, then we have the Rodrigues formula

$$R_{ij}(\hat{\mathbf{n}}, \theta) = \cos(\theta)\delta_{ij} + (1 - \cos(\theta))n_i n_j - \sin(\theta)\epsilon_{ijk} n_k, \tag{2.3}$$

where δ_{ij} denotes the Kronecker delta, i.e.,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and ϵ_{ijk} denotes the Levi-Civita tensor, i.e.,

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}, \\ -1, & \text{if } (i, j, k) \in \{(3, 2, 1), (1, 3, 2), (2, 1, 3)\}, \\ 0, & \text{if } i = j, \text{ or } j = k, \text{ or } k = i, \end{cases}$$

which is called the angle-and-axis parameterization of the three-dimensional rotation matrix.

We wish to derive all the rotations by any angle $\theta \in \mathbb{R}$ through the rotation by $\frac{\pi}{2}$ and its fractional powers. To get this result we first explicit, in the following theorem, the fractional power, for $0 \leq \alpha \leq 1$, of the rotation $A(\hat{\mathbf{n}}, \frac{\pi}{2})$ in Lemma 2.3. It is one of the main results of this work.

Theorem 2.6. *Let $A(\hat{\mathbf{n}}, \frac{\pi}{2})$ be the matrix that represents the rotation by an angle $\frac{\pi}{2}$ about the axis $\hat{\mathbf{n}} = (n_1, n_2, n_3)$. For $0 \leqslant \alpha \leqslant 1$, the fractional power of the rotation $A(\hat{\mathbf{n}}, \frac{\pi}{2})$ is given by*

$$A^{\alpha}\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big) =$$
$$\begin{bmatrix} n_1^2(1 - \cos(\frac{\alpha\pi}{2})) + \cos(\frac{\alpha\pi}{2}) & n_1 n_2(1 - \cos(\frac{\alpha\pi}{2})) - n_3 \sin(\frac{\alpha\pi}{2}) & n_1 n_3(1 - \cos(\frac{\alpha\pi}{2}))) + n_2 \sin(\frac{\alpha\pi}{2}) \\ n_1 n_2(1 - \cos(\frac{\alpha\pi}{2})) + n_3 \sin(\frac{\alpha\pi}{2}) & n_2^2(1 - cos\frac{\alpha\pi}{2}) + cos\frac{\alpha\pi}{2} & n_2 n_3(1 - cos\frac{\alpha\pi}{2}) - n_1 \sin(\frac{\alpha\pi}{2}) \\ n_1 n_3(1 - cos\frac{\alpha\pi}{2}) - n_2 \sin(\frac{\alpha\pi}{2}) & n_2 n_3(1 - \cos(\frac{\alpha\pi}{2})) + n_1 \sin(\frac{\alpha\pi}{2}) & n_3^2(1 - cos\frac{\alpha\pi}{2}) + \cos(\frac{\alpha\pi}{2}) \end{bmatrix}.$$

Proof. The proof consists of the explicit calculation of the fractional power of the operator $A(\hat{\mathbf{n}}, \frac{\pi}{2})$ through the formula (2.1) for $0 < \alpha < 1$.

$$A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)^{\alpha} = \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} \lambda^{\alpha-1} A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)\Big(\lambda I + A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)\Big)^{-1} d\lambda, \ 0 < \alpha < 1. \tag{2.4}$$

Note that

$$\Big(\lambda I + A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)\Big)^{-1} =$$
$$\frac{1}{(\lambda+1)(\lambda^2+1)} \begin{bmatrix} a^2(1-\lambda) + \lambda(1+\lambda) & ab(1-\lambda) + c(1+\lambda) & ac(1-\lambda) - b(1+\lambda) \\ ab(1-\lambda) - c(1+\lambda) & b^2(1-\lambda) + \lambda(1+\lambda) & bc(1-\lambda) + a(1+\lambda) \\ ac(1-\lambda) + b(1+\lambda) & bc(1-\lambda) - a(1+\lambda) & c^2(1-\lambda) + \lambda(1+\lambda) \end{bmatrix}$$

and

$$A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)\Big(\lambda I + A\Big(\hat{\mathbf{n}}, \frac{\pi}{2}\Big)\Big)^{-1} =$$
$$\frac{1}{(\lambda+1)(\lambda^2+1)} \begin{bmatrix} a^2\lambda(\lambda-1) + 1 + \lambda & ab\lambda(\lambda-1) - c\lambda(1+\lambda) & ac\lambda(\lambda-1) + b\lambda(1+\lambda) \\ ab\lambda(\lambda-1) + c\lambda(1+\lambda) & b^2\lambda(\lambda-1) + 1 + \lambda & bc\lambda(\lambda-1) - a\lambda(1+\lambda) \\ ac\lambda(\lambda-1) - b\lambda(1+\lambda) & bc\lambda(\lambda-1) + a\lambda(1+\lambda) & c^2\lambda(\lambda-1) + 1 + \lambda \end{bmatrix}.$$

Since

$$\begin{aligned}\frac{\lambda(\lambda-1)}{(\lambda+1)(\lambda^2+1)} &= \frac{1}{\lambda+1}-\frac{1}{\lambda^2+1}\\ \frac{\lambda+1}{(\lambda+1)(\lambda^2+1)} &= \frac{1}{\lambda^2+1}\\ \frac{\lambda(\lambda+1)}{(\lambda+1)(\lambda^2+1)} &= \frac{\lambda}{\lambda^2+1}\end{aligned}$$

from right side of the equation (2.4) and $A\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big)\Big(\lambda I + A\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big)\Big)^{-1}$, and by (2.1) we obtain

$$A^{\alpha}\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big)=$$

$$\begin{bmatrix} n_1^2(1-\cos(\frac{\alpha\pi}{2}))+\cos(\frac{\alpha\pi}{2}) & n_1n_2(1-\cos(\frac{\alpha\pi}{2}))-n_3\sin(\frac{\alpha\pi}{2}) & n_1n_3(1-\cos(\frac{\alpha\pi}{2})))+n_2\sin(\frac{\alpha\pi}{2}) \\ n_1n_2(1-\cos(\frac{\alpha\pi}{2}))+n_3\sin(\frac{\alpha\pi}{2}) & n_2^2(1-cos\frac{\alpha\pi}{2})+cos\frac{\alpha\pi}{2} & n_2n_3(1-cos\frac{\alpha\pi}{2})-n_1\sin(\frac{\alpha\pi}{2}) \\ n_1n_3(1-cos\frac{\alpha\pi}{2})-n_2\sin(\frac{\alpha\pi}{2}) & n_2n_3(1-\cos(\frac{\alpha\pi}{2}))+n_1\sin(\frac{\alpha\pi}{2}) & n_3^2(1-cos\frac{\alpha\pi}{2})+\cos(\frac{\alpha\pi}{2}) \end{bmatrix}.$$

Finally, cases $\alpha = 0$ and $\alpha = 1$ are immediate, and the proof is complete. □ □

Corollary 2.7. *The fractional power $A^{\alpha}(\hat{\mathbf{n}},\frac{\pi}{2})$ coincides with matrix $R(\hat{\mathbf{n}},\frac{\alpha\pi}{2}) = [R_{ij}(\hat{\mathbf{n}},\frac{\alpha\pi}{2})]$, where $R_{ij}(\hat{\mathbf{n}},\frac{\alpha\pi}{2})$ is given by (2.3), for $0\leqslant\alpha\leqslant 1$.*

We are now in a position to give our definition for the rotation matrix by an angle θ through fractional powers of the rotation by $\frac{\pi}{2}$.

Definition 2.8. The rotation by $\theta\in\mathbb{R}$, denoted by $A(\hat{\mathbf{n}},\theta)$, is defined to be

$$A(\hat{\mathbf{n}},\theta) := A^{\frac{2\theta}{\pi}}\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big). \tag{2.5}$$

Note that $A(\hat{\mathbf{n}},\frac{\pi}{2})$ is such that the fractional power $A^{\alpha}(\hat{\mathbf{n}},\frac{\pi}{2})$ is well-defined for $\alpha\in\mathbb{R}$. Theorem 2.6 states that the definition in (2.5) agrees with the classical one given by *Rodrigues formula* in (2.3) for $0\le\theta\le\frac{\pi}{2}$. The following theorem extends this result for $\theta\in\mathbb{R}$.

Theorem 2.9. *Let $A(\hat{\mathbf{n}},\theta)$ be the rotation defined in (2.5). Then*

$$A(\hat{\mathbf{n}},\theta) = R(\hat{\mathbf{n}},\theta) \tag{2.6}$$

for any $\theta\in\mathbb{R}$.

Proof. Firstly for $\theta\geqslant 0$, it is sufficient to show that (2.6) is satisfied for each

$$\frac{(n-1)\pi}{2}\le\theta\le\frac{n\pi}{2},$$

for $n\in\mathbb{N}$. We proceed by induction. The case $n=1$ follows from Theorem 2.6. If we assume (2.6) for n, we can prove the result for $n+1$. Set

$$\frac{n\pi}{2}\le\theta\le\frac{(n+1)\pi}{2}$$

so that

$$\frac{(n-1)\pi}{2}\le\theta-\frac{\pi}{2}\le\frac{n\pi}{2}$$

Hence

$$A(\hat{\mathbf{n}},\theta) = A^{\frac{2\theta}{\pi}}\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big) = A^{\frac{2\theta}{\pi}-1}\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big)A\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big) = A\Big(\hat{\mathbf{n}},\theta-\frac{\pi}{2}\Big)A\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big) \tag{2.7}$$

and by induction hypothesis

$$A\Big(\hat{\mathbf{n}},\theta-\frac{\pi}{2}\Big) = R\Big(\hat{\mathbf{n}},\theta-\frac{\pi}{2}\Big) \tag{2.8}$$

combining (2.7) with (2.8) we obtain

$$\begin{aligned} A(\hat{\mathbf{n}},\theta) &= R\Big(\hat{\mathbf{n}},\theta-\frac{\pi}{2}\Big)A\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big) \\ &= R(\hat{\mathbf{n}},\theta)R\Big(\hat{\mathbf{n}},-\frac{\pi}{2}\Big)R\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big) \\ &= R(\hat{\mathbf{n}},\theta) \end{aligned}$$

above we use some basic properties of the Euler-Rodrigues formula.

Secondly, for $-\frac{\pi}{2} \leqslant \theta \leqslant 0$, and proceeding analogously to the proof of Theorem (2.6) we can obtain the expression

$$A^{-\alpha}\Big(\hat{\mathbf{n}},\frac{\pi}{2}\Big) = \begin{bmatrix} n_1^2(1-\cos(\frac{\alpha\pi}{2}))+\cos(\frac{\alpha\pi}{2}) & n_1n_2(1-\cos(\frac{\alpha\pi}{2}))+n_3\sin(\frac{\alpha\pi}{2}) & n_1n_3(1-\cos(\frac{\alpha\pi}{2})))-n_2\sin(\frac{\alpha\pi}{2}) \\ n_1n_2(1-\cos(\frac{\alpha\pi}{2}))-n_3\sin(\frac{\alpha\pi}{2}) & n_2^2(1-\cos(\frac{\alpha\pi}{2}))+\cos(\frac{\alpha\pi}{2}) & n_2n_3(1-\cos(\frac{\alpha\pi}{2}))+n_1\sin(\frac{\alpha\pi}{2}) \\ n_1n_3(1-\cos(\frac{\alpha\pi}{2}))+n_2\sin(\frac{\alpha\pi}{2}) & n_2n_3(1-\cos(\frac{\alpha\pi}{2}))-n_1\sin(\frac{\alpha\pi}{2}) & n_3^2(1-\cos(\frac{\alpha\pi}{2}))+\cos(\frac{\alpha\pi}{2}) \end{bmatrix}$$

and so the definition in (2.5) agrees with the classical one given by the Euler-Rodrigues formula in (2.3) for $-\frac{\pi}{2} \leqslant \theta \leqslant 0$. Finally, an analogous argument of induction as in the first part of this proof shows that (2.5) agrees with the Euler-Rodrigues formula in (2.3) for $\theta \leqslant 0$. □ □

Corollary 2.10. *The family* $\{A(\hat{\mathbf{n}},\theta);\theta\in\mathbb{R}\}$, *where*

$$A(\hat{\mathbf{n}},\theta) = \begin{bmatrix} n_1^2(1-\cos(\theta))+\cos(\theta) & n_1n_2(1-\cos(\theta))-n_3\sin(\theta) & n_1n_3(1-\cos(\theta))+n_2\sin(\theta) \\ n_1n_2(1-\cos(\theta))+n_3\sin(\theta) & n_2^2(1-\cos(\theta))+\cos(\theta) & n_2n_3(1-\cos(\theta))-n_1\sin(\theta) \\ n_1n_3(1-\cos(\theta))-n_2\sin(\theta) & n_2n_3(1-\cos(\theta))+n_1\sin(\theta) & n_3^2(1-\cos(\theta))+\cos(\theta) \end{bmatrix}$$

is a uniformly continuous group on $\mathbb{R}^3$ *with infinitesimal generator* $G:\mathbb{R}^3\to\mathbb{R}^3$ *given by*

$$G = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}.$$

Proof. That family $\{A(\hat{\mathbf{n}},\theta);\theta\in\mathbb{R}\}$ is a group is an immediate consequence of the definition of $A(\hat{\mathbf{n}},\theta)$ in (2.5). We obtain G easily from the definition of infinitesimal generator of a group

$$D(G) = \Big\{u\in\mathbb{R}^3; \lim_{\theta\to 0}\frac{A(\hat{\mathbf{n}},\theta)u-u}{\theta} \text{ exists}\Big\}$$

and

$$Gu = \lim_{\theta\to 0}\frac{A(\hat{\mathbf{n}},\theta)u-u}{\theta}, \text{ for any } u\in D(G).$$

Since G is a bounded linear operator, we conclude that $\{A(\hat{\mathbf{n}},\theta);\theta\in\mathbb{R}\}$ is a uniformly continuous group on $\mathbb{R}^3$. □ □

Remark 2.11. In particular, we can obtain the explicit expression of the logarithm of rotations $A(\hat{\mathbf{n}},\theta)$ thanks to the fact that the logarithm is the infinitesimal generator of the uniformly continuous group $\{A^{\alpha}(\hat{\mathbf{n}},\theta);\alpha\in\mathbb{R}\}$ on $\mathbb{R}^3$; namely, we have

$$\log A(\hat{\mathbf{n}},\theta) = \begin{bmatrix} 0 & -\theta n_3 & \theta n_2 \\ \theta n_3 & 0 & -\theta n_1 \\ -\theta n_2 & \theta n_1 & 0 \end{bmatrix}.$$

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