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ON THE HAUSDORFF SATURATION OF SOME TRIGONOMETRIC-KIES FAMILIES

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Abstract The purpose of this paper is to examine some trigonometric transformations of the Kies probability distribution. We consider the saturation in the Hausdorff sense which measures the speed of occurrence of a random event. Semi-closed form formulas for this saturation are derived. Also, some duality relations are obtained. Several numerical experiments are provided to validate the theoretical conclusions. The derived results can be applicable in many scientific and practical fields such as Biostatistics, Survival theory, Population dynamics, Lifetime analysis, Growth theory, Debugging and Test theory, Computer viruses propagation, Insurance mathematics, etc.

1 Introduction

The statistical literature contains several trigonometric extensions of some probability distributions. This way their applicability and flexibility are significantly increased – we refer to [1, 4, 8, 10, 13, 20, 21, 22]. These results will be of interest to specialists in many modern scientific branches – for some details see [11].

In particular, the sin-G family is considered in [20] – the corresponding cumulative distribution function (CDF, hereafter) is

$$F(t) = \sin\left(\frac{\pi}{2}G(t)\right), \tag{1.1}$$

where $G(t)$ is the CDF of the underlying distribution. A cos-G family is studied in [21] – the CDF is

$$F(t) = 1 - \cos\left(\frac{\pi}{2}G(t)\right). \tag{1.2}$$

A tan-style modification can be found in [22] (see, also [8]) – the CDF is

$$F(t) = \tan\left(\frac{\pi}{4}G(t)\right). \tag{1.3}$$

In addition to these trigonometric transformations, we consider a cotangent-style one. Its cumulative distribution function is defined as

$$F(t) = 1 - \cot\left(\frac{\pi}{4}\left(1 + G(t)\right)\right). \tag{1.4}$$

In [1] the authors propose new Log-Logistic-Tan generalized families of CDFs. Various modifications of this ”powerful” class of functions have been proposed and studied by several researchers (for some details, see [9])

$$F(t) = 1 - \left(1 + s^{-c}\left(\tan\left(\frac{\pi}{2}\left(1 - e^{-t^b}\right)^{\alpha}\right)\right)^{c}\right)^{-1}$$
$$F(t) = 1 - \left(1 + s^{-c}\left(\tan\left(\frac{\pi}{2}\left(1 - e^{-at^2}\right)^{\alpha}\right)\right)^{c}\right)^{-1}$$
$$F(t) = 1 - \left(1 + s^{-c}\left(\tan\left(\frac{\pi}{2}\left(1 - e^{-bt}\right)^{\alpha}\right)\right)^{c}\right)^{-1}.$$

Some questions related to the synthesis and analysis of transfer functions, radiation diagrams, and filter characteristics are elaborated in detail in [9]. Properties and applications of a TAN–G family of ”adaptive functions”

$$F(t) = \tan\left(\frac{\pi}{4}\left(1 - e^{-bt}\right)\right)$$

is studied in [10]. A new class of probability distributions via cosine and sine functions with applications is proposed in [4]:

$$F(t) = \frac{(\alpha+\gamma)\sin(\frac{\pi}{2}G(t))}{\alpha + \beta\cos(\frac{\pi}{2}G(t)) + \gamma\sin(\frac{\pi}{2}G(t)) + \theta\sin(\frac{\pi}{2}G(t))\cos(\frac{\pi}{2}G(t))}.$$

For some other models, we refer to [13, 23].

The Kies probability model was proposed in [6] as an alternative to the extended Weibull models as it provides a more efficient fit to some real-life data sets in comparison to the aforementioned models. Also, differently from the Weibull distribution, the Kies one is defined on a finite domain. For some additional details see [3, 7, 15, 16, 17]. The CDF of the two-parameter Kies distribution is given by

$$F(t) = 1 - e^{-k\left(\frac{t}{1-t}\right)^{b}}. \tag{1.5}$$

where $0 < t < 1$. Note that the distribution's domain can be set to an arbitrary interval $a < t < b$ as the CDF

$$F(t) = 1 - \exp\left(-\lambda\left(\frac{t-a}{b-t}\right)^{\beta}\right),$$

see [15] together with [25]. A three parameter Kies distribution of the form

$$F(t) = 1 - e^{-k\left(\frac{t}{1-k_1 t}\right)^{a}}$$

is applied in [12] to a specific framework of chemical reaction networks. Several extensions of the exponential distribution are available in the scientific literature. One example is the modified Kies-exponential one introduced by [2] with the CDF

$$F(t) = 1 - e^{-\left(e^{kt}-1\right)^{a}},$$

where $t > 0$. In [26] is provided a discussion on some properties of a new power-modified-exponential family with an original Kies-correction with CDF for $0 < t < 1$

$$F(t) = 1 - e^{-\left(e^{k\left(\frac{t}{1-t}\right)^{b}}-1\right)^{a}}.$$

The distributional properties of the minimum and maximum of several Kies distributions are discussed in [27]. In the present article we consider the saturation in the Hausdorff sense, see [18], of the trigonometric families (1.1), (1.2), (1.3), and (1.4) considering the Kies underlying distribution (1.5). This way the CDF generates the following four families

$$
\begin{aligned}
H_{sin}(t) &= \sin\left(\frac{\pi}{2}\left(1-e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right)\\
H_{cos}(t) &= 1-\cos\left(\frac{\pi}{2}\left(1-e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right)\\
H_{tan}(t) &= \tan\left(\frac{\pi}{4}\left(1-e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right)\\
H_{cot}(t) &= 1-\cot\left(\frac{\pi}{4}\left(2-e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right),
\end{aligned}
\tag{1.6}
$$

where $0 < t < 1$, $k > 0$, and $b > 0$. The Hausdorff saturation is defined for distributions with connected domains and finite left endpoints. It is given as the Hausdorff distance between the CDF and a Γ-shaped curve connecting its endpoints. In fact, the saturation measures the speed of occurrence of a random event. This way it is an indicator of the mass location of the distribution – the lower saturation, the more left-skewed distribution, and vice versa. We obtain semi-closed form formulas for the Hausdorff saturation using some duality arguments. A discussion of this measure for a different family can be found in [24]. For some studies devoted to related topics we refer to [5, 14, 19].

The paper is organized as follows. In Section 2 we present our results for the Hausdorff saturation in the light of a distributional duality. Some probability properties of the considered families can be found in Section 3. The sin- and cos-transformations and their relations are discussed in Section 4, whereas the tangent and cotangent ones are considered in Section 5. Some numerical results are presented in Section 6.

2 Distributional duality and Hausdorff saturation

We begin defining a duality between distributions.

Definition 2.1. Let $G(t)$ be a CDF of a continuous distribution on the domain $[0,1]$. We define its dual one as the distribution with CDF $\widetilde{G}(t) = 1 - G(1-t)$.

We shall consider hereafter only distributions on the interval $[0,1]$. Note that the results can be easily extended to an arbitrary connected finite domain. Below we present an immediate corollary of Definition 2.1.

Corollary 2.2. *The following relations between the trigonometric-G distributions stand*

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}\widetilde{G}(t)\right) &= \cos\left(\frac{\pi}{2}G(1-t)\right)\\
1-\cos\left(\frac{\pi}{2}\widetilde{G}(t)\right) &= 1-\sin\left(\frac{\pi}{2}G(1-t)\right).
\end{aligned}
$$

Now we define the Hausdorff distance in the sense of [18] and the related saturation.

Definition 2.3. The Hausdorff distance (the H–distance), $\rho(f,g)$, between two interval functions $f(\cdot)$ and $g(\cdot)$ on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$
\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||,\ \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},
$$

wherein $||.||$ is any norm in $\mathbb{R}^2$. We use the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$ and hence the distance between the points $A = (t_A, x_A)$ and $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

Definition 2.4. The saturation of a distribution defined on the interval $[0,1]$ is the Hausdorff distance between its CDF and the Γ-shaped curve consisting of two lines – one between the points $\{(0,0),(0,1)\}$ and another between $\{(0,1),(1,1)\}$.

We can easily obtain the following proposition for the saturation d.

Proposition 2.5. *The saturation d is the solution of the following equation*

$$H(d) = 1 - d \tag{2.1}$$

in the interval $[0, 1]$, *where* $H(d)$ *is the CDF.*

Remark 2.6. Note that equation (2.1) has a unique solution because $H(d)+d-1$ is an increasing function with an initial value of -1 and a final value of 1.

The next proposition presents the relation between the saturations of the dual distributions.

Theorem 2.7. *The sum of the saturations of the following pairs of distributions is one:* $\left\{\widetilde{G} - sin; G - cos\right\}$, $\left\{\widetilde{G} - cos; G - sin\right\}$, $\left\{\widetilde{G} - tan; G - cot\right\}$, *and* $\left\{\widetilde{G} - cot; G - tan\right\}$.

Proof. Let us consider $\widetilde{G} - sin$ and $G - cos$ distributions and x and y be their Hausdorff saturations. Using Proposition 2.5 we see that x is the solution of

$$\sin\left(\frac{\pi}{2}\widetilde{G}(x)\right) = 1 - x,$$

which is equivalent to

$$\widetilde{G}(x) = \frac{2}{\pi}\arcsin(1 - x). \tag{2.2}$$

Using $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ we transform equation (2.2) to

$$G(1 - x) = \frac{2}{\pi}\arccos(1 - x).$$

The change $y = 1 - x$ shows

$$G(y) = \frac{2}{\pi}\arccos(y),$$

which leads to

$$1 - \cos\left(\frac{\pi}{2}G(y)\right) = 1 - y.$$

It is left to use again Proposition 2.5 to finish the proof for the pair $\left\{\widetilde{G} - sin; G - cos\right\}$. Some symmetrical arguments prove the theorem for the $\widetilde{G} - cos$ and $G - sin$ distributions.

Next, we consider the pair $\left\{\widetilde{G} - tan; G - cot\right\}$. Suppose again that the saturations are x and y, respectively. Using Proposition 2.5 we derive

$$\begin{aligned}
1 - (1 - y) = y &= \cot\left(\frac{\pi}{4}(1 + G(y))\right) \\
&= \tan\left(\frac{\pi}{2} - \frac{\pi}{4}(1 + G(y))\right) \\
&= \tan\left(\frac{\pi}{4}(1 - G(y))\right) \\
&= \tan\left(\frac{\pi}{4}(1 - G(1 - (1 - y)))\right) \\
&= \tan\left(\frac{\pi}{4}\left(\widetilde{G}(1 - y)\right)\right).
\end{aligned}$$

Proposition 2.5 leads that $x = 1 - y$. The last part of the theorem can be proven by symmetric arguments. $\square$

3 Some distributional properties

Let us assume hereafter that the underlying distribution is the Kies one, which CDF is given by formula (1.5). This way the CDFs of the resulting distributions take form (1.6). We can write the probability density function (PDF) of the original Kies distribution as

$$f(t) = \frac{kbt^{b-1}\exp\left(-k\left(\frac{t}{1-t}\right)^{b}\right)}{(1-t)^{b+1}}. \tag{3.1}$$

This PDF can exhibit a very varied structure. Its right endpoint is zero, but the left one may be zero, an arbitrary finite point, or infinity. In fact, the PDF behavior near the zero (the left endpoint) is related to the position of the parameter b w.r.t. one. More precisely, $f(0) = \infty$ for $b < 1$; $f(0) = k$ for $b = 1$; and $f(0) = 0$ for $b > 1$. Also, PDF (3.1) may have a peak, but it can be a decreasing function too. For the whole behavior and its proof see [26], proposition 2.1.

Having in mind the trigonometric-Kies CDFs given by formulas (1.6) we derive for the PDFs

$$\begin{aligned}
h_{sin}(t) &= \frac{\pi}{2}\cos\left(\frac{\pi}{2}\left(1-e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)\right)f(t)\\
&= \frac{\pi}{2}\sin\left(\frac{\pi}{2}e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)f(t)\\
h_{cos}(t) &= \frac{\pi}{2}\sin\left(\frac{\pi}{2}\left(1-e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)\right)f(t)\\
&= \frac{\pi}{2}\cos\left(\frac{\pi}{2}e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)f(t)\\
h_{tan}(t) &= \frac{\pi}{4}\left(1+\left(\tan\left(\frac{\pi}{4}\left(1-e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)\right)\right)^{2}\right)f(t)\\
&= \frac{\pi}{4}\left(1+\left(\cot\left(\frac{\pi}{4}\left(1+e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)\right)\right)^{2}\right)f(t)\\
h_{cot}(t) &= \frac{\pi}{4}\left(1+\left(\cot\left(\frac{\pi}{4}\left(2-e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)\right)\right)^{2}\right)f(t)\\
&= \frac{\pi}{4}\left(1+\left(\tan\left(\frac{\pi}{4}e^{-k\left(\frac{t}{1-t}\right)^{b}}\right)\right)^{2}\right)f(t).
\end{aligned}$$

The PDF behavior of these compositions has to follow the original distribution together with the correction function. Obviously, the value at the right endpoint is zero. Since the Hausdorff saturation is closely related to the left tail of the distribution we consider the PDF limit when $t \to 0$:

Proposition 3.1. *The following statements for the left endpoint values of the PDFs hold.*

(i) We have for the sin-Kies PDF $h_{sin}(0) = \infty$ for $b < 1$, $h_{sin}(0) = \frac{\pi}{2}k$ for $b = 1$, and $h_{sin}(0) = 0$ for $b > 1$.

(ii) We have for the cos-Kies PDF $h_{cos}(0) = \infty$ for $b < \frac{1}{2}$, $h_{cos}(0) = \frac{\pi^2 k^2}{8}$ for $b = \frac{1}{2}$, and $h_{cos}(0) = 0$ for $b > \frac{1}{2}$.

(iii) We have for the tan-Kies PDF $h_{tan}(0) = \infty$ for $b < 1$, $h_{tan}(0) = \frac{\pi}{4}k$ for $b = 1$, and $h_{tan}(0) = 0$ for $b > 1$.

(iv) We have for the cot-Kies PDF $h_{cot}(0) = \infty$ for $b < 1$, $h_{cot}(0) = \frac{\pi}{2}k$ for $b = 1$, and $h_{cot}(0) = 0$ for $b > 1$.

Proof. The proof of the first, third, and fourth statements is a consequence of the mentioned above behavior of the Kies PDF, namely $f(0) = \infty$ for $b < 1$, $f(0) = k$ for $b = 1$, and $f(0) = 0$ for $b > 1$.

Let us consider the cos-Kies family. Obviously, $h_{cos}(0) = 0$ for $b \geq 1$. Suppose now $b < 1$. Using $\lim_{t\to 0} \frac{\sin(x)}{x} = 1$, Kies PDF expression (3.1), and the Taylor expansion of the exponent we derive

$$\begin{aligned}
h_{cos}(0) &= \frac{\pi}{2} \lim_{t\to 0} \left\{ \sin\left(\frac{\pi}{2}\left(1 - e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right) f(t) \right\} \\
&= \frac{\pi}{2} \lim_{t\to 0} \left\{ \frac{\sin\left(\frac{\pi}{2}\left(1 - e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right)}{\frac{\pi}{2}\left(1 - e^{-k\left(\frac{t}{1-t}\right)^b}\right)} \frac{\pi}{2}\left(1 - e^{-k\left(\frac{t}{1-t}\right)^b}\right) f(t) \right\} \\
&= \frac{\pi^2}{4} \lim_{t\to 0} \left\{ \left(1 - e^{-kt^b}\right) f(t) \right\} \\
&= \frac{\pi^2}{4} kb \lim_{t\to 0} \left\{ \sum_{n=1}^{\infty} \left((-1)^{n+1} \frac{k^n t^{nb}}{n!} \right) t^{b-1} \right\} \\
&= \frac{\pi^2}{4} b \lim_{t\to 0} \left\{ \sum_{n=2}^{\infty} \left((-1)^n \frac{k^n t^{nb-1}}{(n-1)!} \right) \right\}.
\end{aligned}$$

We can see that if $b > \frac{1}{2}$, then the whole sum tends to zero. Also, if $b < \frac{1}{2}$, then the first term tends to infinity and thus the whole sum is infinitely large. Finally, if $b = \frac{1}{2}$, then only the first term is non-zero and its value is k^2. These conclusions finish the proof. □

4 Formulas for the sin- cos-transformations

The dual-Kies CDF is

$$\widetilde{G}(t) = e^{-k\left(\frac{1-t}{t}\right)^a}.$$

The following propositions present semi-closed form formulas for the saturation of the sin-G and cos-G families.

Proposition 4.1. *For a positive parameter y we define the function $\gamma(y)$ as*

$$\gamma(y) = y \left(\frac{\cos\left(\frac{\pi}{2} e^{-y}\right)}{1 - \cos\left(\frac{\pi}{2} e^{-y}\right)} \right)^b \tag{4.1}$$

and suppose that $k = \gamma(y)$ for some value of y. Then the saturation of the sin-G distribution is

$$d(y) = 1 - \cos\left(\frac{\pi}{2} e^{-y}\right).$$

This presentation is equivalent to

$$d(k; b) = 1 - \cos\left(\frac{\pi}{2} e^{-\gamma^{-1}(k;b)}\right)$$

since function (4.1) is continuous and increasing and thus it is invertible.

Proof. We know from Proposition 2.5 that the saturation is the solution of the equation

$$\sin\left(\frac{\pi}{2}\left(1 - e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right) = 1 - t$$

equivalent of

$$\frac{\pi}{2} e^{-k\left(\frac{t}{1-t}\right)^b} = \arccos(1 - t). \tag{4.2}$$

Let us change the variables as

$$z = \frac{1}{k} e^{k\left(\frac{t}{1-t}\right)^b} \tag{4.3}$$

or

$$t = \frac{\left(\ln\left(kz\right)\right)^{\frac{1}{b}}}{\left(\ln\left(kz\right)\right)^{\frac{1}{b}} + k^{\frac{1}{b}}}. \tag{4.4}$$

This way equation (4.2) can be written as

$$\frac{\pi}{2kz} = \arccos\left(1 - t\right). \tag{4.5}$$

Having in mind equations (4.4) and (4.5) and changing the variables as $y = \ln\left(kz\right)$ (equivalently to $z = \frac{e^y}{k}$) we derive

$$\frac{\pi}{2}e^{-y} = \arccos\left(\frac{k^{\frac{1}{b}}}{y^{\frac{1}{b}} + k^{\frac{1}{b}}}\right), \tag{4.6}$$

which is equivalent to

$$k = y\left(\frac{\cos\left(\frac{\pi}{2}e^{-y}\right)}{1 - \cos\left(\frac{\pi}{2}e^{-y}\right)}\right)^b. \tag{4.7}$$

It left to put equation (4.7) together with $y = \ln\left(kz\right)$ into (4.4) to finish the proof. □

Proposition 4.2. *Let the function $\gamma\left(y\right)$ be defined as*

$$\gamma\left(y\right) = y\left(\frac{1 - \sin\left(\frac{\pi}{2}e^{-y}\right)}{\sin\left(\frac{\pi}{2}e^{-y}\right)}\right)^b \tag{4.8}$$

for a positive parameter y and suppose that $k = \gamma\left(y\right)$ for some value of y. Then the cos-G distribution's saturation is

$$d\left(y\right) = \sin\left(\frac{\pi}{2}e^{-y}\right).$$

This presentation is equivalent to

$$d\left(k;b\right) = \sin\left(\frac{\pi}{2}e^{-\gamma^{-1}\left(k;b\right)}\right).$$

Proof. Using again Proposition 2.5 we derive the saturation as the solution to the equation

$$\cos\left(\frac{\pi}{2}\left(1 - e^{-k\left(\frac{t}{1-t}\right)^b}\right)\right) = t$$

or to

$$\frac{\pi}{2}e^{-k\left(\frac{t}{1-t}\right)^b} = \arcsin\left(t\right). \tag{4.9}$$

Let us change the variables accordingly to formulas (4.3) and (4.4). This way equation (4.9) turns to

$$\frac{\pi}{2kz} = \arcsin\left(t\right).$$

Thus using again the notation $y = \ln\left(kz\right)$ we derive

$$\frac{\pi}{2}e^{-y} = \arcsin\left(\frac{y^{\frac{1}{b}}}{y^{\frac{1}{b}} + k^{\frac{1}{b}}}\right),$$

which is equivalent to $k = \gamma\left(y\right)$. It is left to put this value of k into equation (4.4) to finish the proof. □

The next proposition is devoted to the saturation of the trigonometric dual Kies distributions – we shall consider first the sin-$\widetilde{G}$ case. Its CDF can be written as

$$\widetilde{G}(t) = \sin\left(\frac{\pi}{2}\left(e^{-k\left(\frac{1-t}{t}\right)^b}\right)\right). \tag{4.10}$$

The following proposition stands.

Proposition 4.3. *Let the function $\gamma(y)$ be defined by equation (4.8) and $k = \gamma(y)$. Then the sin-$\widetilde{G}$ saturation is*

$$d(y) = 1 - \sin\left(\frac{\pi}{2}e^{-y}\right),$$

which is equivalent to

$$d(k;b) = 1 - \sin\left(\frac{\pi}{2}e^{-\gamma^{-1}(k;b)}\right).$$

Proof. First, using CDF (4.10) we present the saturation as the solution of

$$\sin\left(\frac{\pi}{2}\left(e^{-k\left(\frac{1-t}{t}\right)^b}\right)\right) = 1 - t$$

equivalent to

$$\frac{\pi}{2}e^{-k\left(\frac{1-t}{t}\right)^b} = \arcsin(1-t).$$

Changing the variables as

$$z = \frac{1}{k}e^{k\left(\frac{1-t}{t}\right)^b} \Leftrightarrow t = \frac{k^{\frac{1}{b}}}{(\ln(kz))^{\frac{1}{b}} + k^{\frac{1}{b}}} \tag{4.11}$$

together with $y = \ln(kz)$ we derive

$$\frac{\pi}{2}e^{-y} = \arcsin\left(\frac{y^{\frac{1}{b}}}{y^{\frac{1}{b}} + k^{\frac{1}{b}}}\right),$$

which leads to $k = \gamma(y)$. We finish the proof estimating k in the presentation of t in (4.11). □

Remark 4.4. Combining the results of Propositions 4.2 and 4.3 we see a confirmation of Proposition 2.7, namely the sum of both saturations is one.

Analogously we can derive for the saturation of the cos-dual-Kies distribution

$$d(k;b) = \cos\left(\frac{\pi}{2}e^{-\gamma^{-1}(k;b)}\right),$$

where the function $\gamma(\cdot)$ is given by equation (4.1). This is true because the saturation can be derived as the solution of the equation

$$\cos\left(\frac{\pi}{2}\left(e^{-k\left(\frac{1-t}{t}\right)^b}\right)\right) = t$$

and after the change (4.11) we get equation (4.6).

5 Tangent and cotangent transformations

We shall discuss now briefly the tangent and cotangent Kies corrected distributions. We shall only sketch the proofs of the following theorems, because of their similarity to the sin- and cos-transformations.

Proposition 5.1. *Let the function $\gamma(y)$ be defined as*

$$\gamma(y) = y\left(\frac{\tan\left(\frac{\pi}{4}\left(1-e^{-y}\right)\right)}{1-\tan\left(\frac{\pi}{4}\left(1-e^{-y}\right)\right)}\right)^b \tag{5.1}$$

and suppose that $k = \gamma(y)$ for some value of y. Then the saturation of the tan-Kies distribution is

$$d(y) = 1 - \tan\left(\frac{\pi}{4}\left(1 - e^{-y}\right)\right).$$

Proof. Proposition 2.5 now leads to the equation

$$\frac{\pi}{4}\left(1 - e^{-k\left(\frac{t}{1-t}\right)^b}\right) = \arctan(1-t). \tag{5.2}$$

Using the change of variables (4.3) and (4.4) we rewrite equation (5.2) as

$$\frac{\pi}{4}\left(1 - e^{-y}\right) = \arctan\left(\frac{k^{\frac{1}{b}}}{y^{\frac{1}{b}} + k^{\frac{1}{b}}}\right),$$

which is equivalent to $k = \gamma(y)$. □

Proposition 5.2. *If the function $\gamma(y)$ is*

$$\gamma(y) = y\left(\frac{1 - \tan\left(\frac{\pi}{4}e^{-y}\right)}{\tan\left(\frac{\pi}{4}e^{-y}\right)}\right)^b \tag{5.3}$$

and $k = \gamma(y)$ for some y, then the saturation of the cot-Kies distribution is

$$d(y) = \tan\left(\frac{\pi}{4}e^{-y}\right).$$

Proof. Having in mind $\cot\left(\frac{\pi}{2} - x\right) = \tan(x)$ and Proposition 2.5 we reach the equation

$$\frac{\pi}{4}e^{-k\left(\frac{t}{1-t}\right)^b} = \arctan(t).$$

Changing the variables as (4.3) and (4.4) we derive

$$\frac{\pi}{4}e^{-y} = \arctan\left(\frac{y^{\frac{1}{b}}}{y^{\frac{1}{b}} + k^{\frac{1}{b}}}\right),$$

which is equivalent to $k = \gamma(y)$. □

Remark 5.3. Let us discuss the Hausdorff distance in light of the duality presented in Theorem 2.7. The tan-dual-Kies distribution's CDF is

$$H(t) = \tan\left(\frac{\pi}{4}e^{-k\left(\frac{1-t}{t}\right)^b}\right),$$

which after the change of variables (4.11) leads to the equivalent equations

$$\arctan(1-t) = \frac{\pi}{4}e^{-k\left(\frac{1-t}{t}\right)^b}$$

$$\arctan\left(\frac{y^{\frac{1}{b}}}{y^{\frac{1}{b}} + k^{\frac{1}{b}}}\right) = \frac{\pi}{4}e^{-y}.$$

This means that k has to be given by formula (5.3). The difference between changes (4.4) and (4.11) shows that the sum of the saturations of the tan-dual-Kies and cot-Kies distributions is one.

Finally, let us consider the cot-dual-Kies and tan-Kies distributions. The CDF of the first one can be written as

$$H(t) = \tan\left(\frac{\pi}{4}\left(1 - e^{-k\left(\frac{1-t}{t}\right)^b}\right)\right).$$

Proposition 2.5 together with the changes of variables (4.11) and $y = \ln(kz)$ shows that $k = \gamma(y)$ where the function $\gamma(\cdot)$ is given in formula (5.1).

Finally, we obtain the corresponding statement for the original Kies distribution.

Proposition 5.4. *Let the function $\gamma(y)$ be defined as*

$$\gamma(y) = y\left(\frac{1-e^{-y}}{e^{-y}}\right)^b$$

and $k = \gamma(y)$ for some positive y. Then the saturation of the original Kies distribution is

$$d(y) = e^{-y}. \tag{5.4}$$

Proof. Having in mind Proposition 2.5 we reach the equation

$$e^{-k\left(\frac{t}{1-t}\right)^b} = t$$

and changing the variables via formulas (4.3) and (4.4) we derive the statement $k = \gamma(y)$. Estimating k in equation (4.4) we obtain formula (5.4). □

Remark 5.5. The duality can be considered via Proposition 2.5 and the change of variables (4.11).

6 Numerical results

We present in Figure 1 the PDFs of the four trigonometrical corrected distributions as well as the PDF of the original Kies distribution. The chosen parameters' values are amongst $k \in \{1; 2\}$ and $b \in \{0.5; 1; 2\}$. We can see a confirmation of Proposition 3.1 for the left endpoints. All PDFs, except cos-Kies, tend to infinity when $b = 0.5$. The initial value for the cos-Kies distribution obtained via the second statement of Proposition 3.1 is $h_{cos}(0) = \frac{\pi^2 k^2}{8}$ – these values are marked by red circles in Figures 1a and 1b. On the contrary, when $b = 1$ cos-Kies PDF is zero for $t = 0$. All the rest are finite but non-zero: the original Kies value is k (blue circles), sin-Kies and cot-Kies PDFs tend to $h_{sin}(0) = h_{cot}(0) = \frac{\pi}{2}k$ (red circles), and tan-Kies initial PDF value is $h_{cot}(0) = \frac{\pi}{4}k$ (purple circles) – see Figures 1c and 1d. Also, all PDF left endpoints' values are zero when $b = 2$ – see Figures 1e and 1f.

The CDFs of the mentioned above distributions are presented in Figure 2. The Hausdorff saturations are marked as the red points. The geometrical meaning of Proposition 2.5 is the following: if the saturation is denoted by d, then the points $(0, 1-d)$, $(0, 1)$, $(d, 1)$, and $(d, 1-d)$ form a square. These squares are plotted by the green dashed lines.

Finally, we present in Table 1 some particular values of the saturations for the trigonometric-Kies distributions as well as the original one. We use the mentioned above parameters $k \in \{1; 2\}$ and $b \in \{0.5; 1; 2\}$. We report also in this table the results for some large and extremely large values of k – $k \in \{35; 55; 80; 100; 150; 200\}$. The second parameter is assumed to be $b = 1.5$. The saturations of the dual distributions can be derived by the use of Theorem 2.7 as one minus the corresponding trigonometric Kies saturation.

7 Figures and Tables

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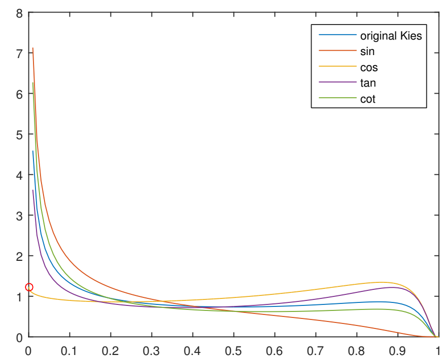
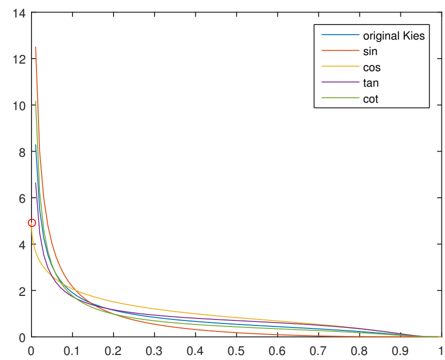
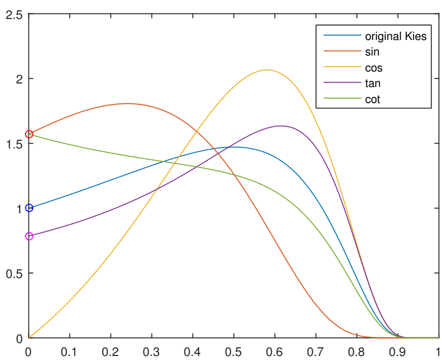
Figure 1. Probability density functions



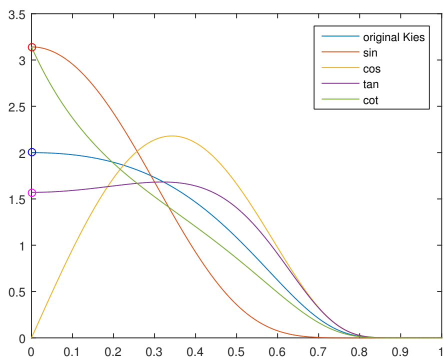
(a) $k = 1,\ b = 0.5$



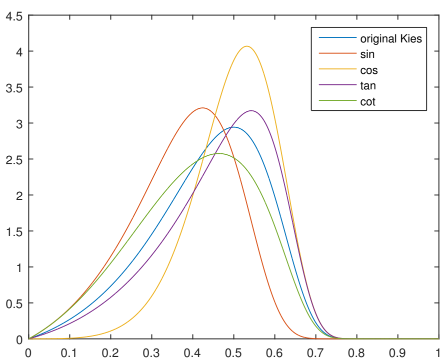
(b) $k = 2,\ b = 0.5$



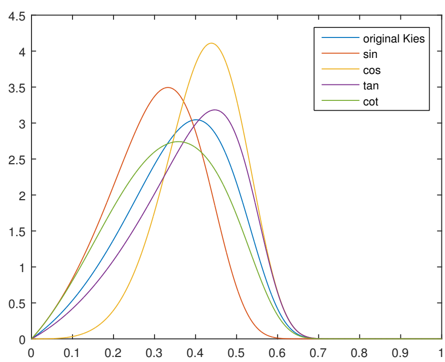
(c) $k = 1,\ b = 1$



(d) $k = 2,\ b = 1$

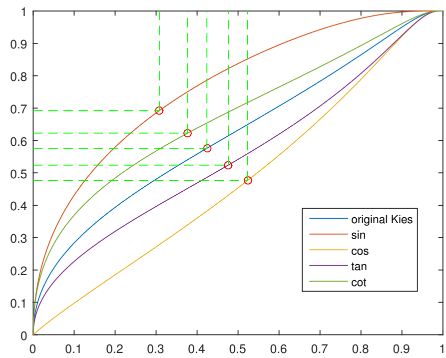


(e) $k = 1,\ b = 2$

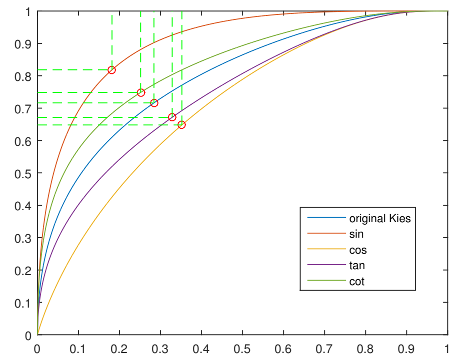


(f) $k = 2,\ b = 2$

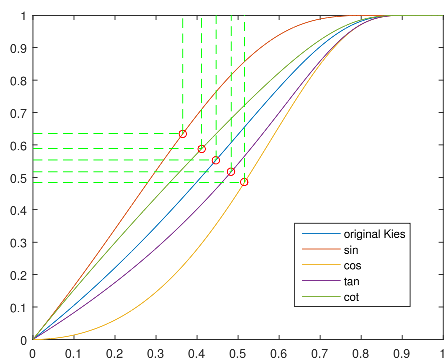
Figure 2. Cumulative distribution functions with Hausdorff saturations



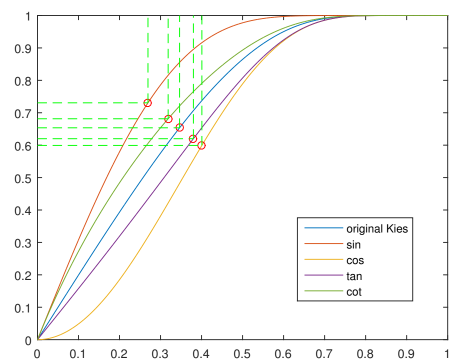
(a) $k = 1,\ b = 0.5$



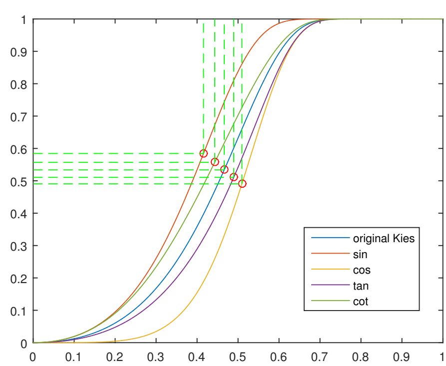
(b) $k = 2,\ b = 0.5$



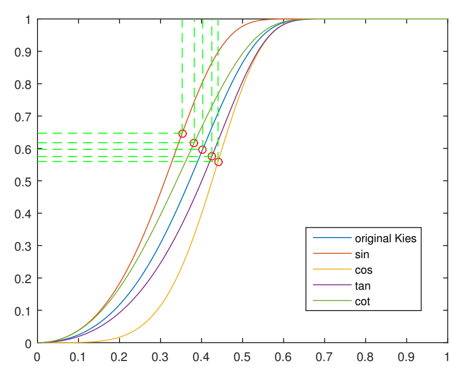
(c) $k = 1,\ b = 1$



(d) $k = 2,\ b = 1$



(e) $k = 1,\ b = 2$



(f) $k = 2,\ b = 2$

k	b	$original$	sin	cos	tan	cot
1	0.5	0.4240	0.3078	0.5234	0.4761	0.3772
2	0.5	0.2839	0.1816	0.3520	0.3284	0.2515
1	1	0.4464	0.3654	0.5156	0.4832	0.4115
2	1	0.3464	0.2693	0.4007	0.3801	0.3185
1	2	0.4662	0.4157	0.5094	0.4894	0.4431
2	2	0.4027	0.3528	0.4402	0.4249	0.3823
35	1.5	0.1305	0.0982	0.1432	0.1413	0.1236
55	1.5	0.1060	0.0786	0.1158	0.1146	0.1006
80	1.5	0.0886	0.0650	0.0964	0.0956	0.0843
100	1.5	0.0794	0.0579	0.0863	0.0856	0.0757
150	1.5	0.0648	0.0468	0.0701	0.0697	0.0619
200	1.5	0.0559	0.0401	0.0604	0.0601	0.0535

Table 1. Saturations for various values of k and b.

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