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BLOW-UP RESULTS FOR A REACTION-DIFFUSION EQUATION WITH A DIRICHLET BOUNDARY CONDITION

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Abstract This work is devoted to study the blow-up phenomenon in a reaction-diffusion equation, defined on a ball, with a Dirichlet boundary condition. It is shown that the classical solutions of this problem have to blow up in finite time at only the center point. Moreover, the blow-up growth rate estimate is investigated. Furthermore, to confirm and support the obtained analytic blow-up results, we use a finite difference scheme for estimating numerically the blow-up time and the blow-up growth rate constant, for a one-dimensional-space equation as a special case of the considered problem.

1 Introduction

We consider the initial-boundary problem:

$$\left\{\begin{array}{ll} u_t = \Delta u + \lambda(u^p + e^{qu}), & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R \end{array}\right\} \tag{1.1}$$

where $q, \lambda > 0$; $p > 1$, and B_R is a ball in R^n, and $u_0 \in C^2(R^n)$, is non-negative non-zero, radially non-increasing function, vanished on the boundary,
i.e. $u_0(x) = 0 \ \forall x \in \partial B_R$.
Many real-world phenomena can be modeled mathematically by partial differential equations. Therefore, since the last century, many authors have studied various mathematical problems involving partial differential equations, see for instance [1, 2, 3].
In time-dependent problems, it is known that, the blow-up phenomenon occurs in many real life problems such as heat propagation and combustion theory [4, 5]. Therefore, since last decades, this phenomenon has been investigated extensively by many authors, see [6, 7, 8, 9]. Generally, we say that the solution blows up, when it is not continuous globally in time. Which means, the solution is unbounded after a finite time [5]. Mathematically, for a time-dependent problem, it is said that the solution $u(x,t)$ blows up, if there exists $T > 0$, such that

$$||u(x,t)||_\infty \overset{t\to T^-}{\longrightarrow} \infty, \quad \text{where} \quad ||u(x,t)||_\infty = Sup_x |u(x,t)|$$

One of the common study problems, is the semilinear parabolic equation, defined in a bounded domain Ω with zero Dirichlet boundary conditions [10, 11, 12, 13, 14]:

$$\left\{\begin{array}{ll} u_t = \Delta u + f(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \end{array}\right\} \tag{1.2}$$

where f is differentiable, positive, non-decreasing function in $(0,\infty)$.
In [10], Kaplan has proved the sufficient conditions for blow-up, which state that for a large size initial function, the blow-up can occur in a finite time, if the function f satisfies the convexity property, in $(0,\infty)$, and the following condition holds:

$$\int_u^\infty \frac{dv}{f(v)} < \infty, \quad u \geq 1 \tag{1.3}$$

Later, Friedman and McLeod [11] have considered two special cases: where f a power or exponential function, and the domain is a ball in R^n, and u_0 is a large-size function and defined as in problem (1.1).
It was shown that the blow-up occurs only at the center point $(x = 0)$, in a finite time T.
i.e. $u(0,t) \to \infty, \quad as\, t \to T^-$.
For the polynomial type function: $f(u) = u^p$, $p > 1$, the upper point-wise estimate is derived in [11]:

$$u(x,t) \leq \frac{C}{|x|^\alpha}, \qquad x \epsilon B_R/\{0\}, \quad t \in (0,T), \alpha \geq \frac{2}{p-1}$$

Additionally, the upper blow-up rate estimate is established as well:

$$u(0,t) \leq \frac{C}{(T-t)^\beta}, \quad t \in (0,T), \quad \beta = \frac{1}{p-1}, \ C > 0$$

For the exponential type function: $f(u) = e^u$, it was shown that there exist $C > 0$, $\alpha \in (0,1)$ such that the upper point-wise estimate is established as follows:

$$u(x,t) \leq \log C + \frac{2}{\alpha}\log\left(\frac{1}{|x|}\right), \quad x \in B_R/\{0\}, \quad t \in (0,T) \tag{1.4}$$

Furthermore, for some $C_2 > 0$, the upper blow-up rate estimate can take the form:

$$u(x,t) \leq \log C_2 - \log(T-t), \quad x \in B_R, \; t \in (0,T), \tag{1.5}$$

In fact, problem (1.1) is considered a combination of the two special cases studied in [11]. This paper aims is to extend some known blow-up results of previous studied problems to problem (1.1), Moreover, we show that the blow up in this problem can only appear at the centre point. Additionally, for some $C > 0$, we prove that

$$\frac{(T-t)}{E_1(u)} \leq C, \quad \text{where} \quad E_1(u) = \int_u^\infty \frac{dv}{ve^v}$$

In addition, for a one-dimensional-space equation as a special case of the considered problem. We follow the numerical finite difference techniques, used in [15, 16, 17, 18, 19], to estimate numerically both the blow-up growth rate constant and the blow-up time.
This paper contains six sections: In the second section, it was proven that blow-up in problem (1.1) can occur in a finite time. Then, some solutions' properties of problem (1.1) are stated in section two. In the third section, the blow-up set of problem (1.1) is investigated. Section four is devoted to studying the blow-up growth rate of problem (1.1). In section five, a numerical experiment in one-dimensional space is given to support the obtained results. Finally, based on the obtained results, some conclusions are pointed out in the last section.

2 Preliminaries

The uniqueness and existence of local classical solution to problem (1.1) are guaranteed by standard parabolic existence theorems [20]. While, the next theorem states that, for a large-size initial function, the blow-up in problem (1.1) occurs in a finite time.

Theorem 2.1. *For a large size initial function, the blow-up of the solution occurs in a finite time. Moreover, the center point belongs to the blow-up set.*

Proof. Clearly, the function $f(u) = \lambda(u^p + e^{qu})$ is convex in $(0,\infty)$, due to the positivity of its second derivatives in this domain. In addition, we can show that

$$E_1(u) = \int_u^\infty \frac{dv}{\lambda(v^p + e^{qv})} < \infty, \quad u \geq 1$$

Therefore, by Kaplan condition (1.3), the solution of problem (1.1) blows up in a finite time.
According to [11], the blow-up in the flowing problem occurs in finite time at only the center point.

$$\left\{\begin{array}{lr} u_t = \Delta u + \lambda e^{qu}, & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R \end{array}\right\} \tag{2.1}$$

Since the solution of problem (1.1) is considered an upper-solution to the problem (2.1), it follows that the center point is a blow-up point to problem (1.1).

□

The following lemma shows that the classical solution of problem (1.1) is increasing in time, positive, and radially non-increasing.

Lemma 2.2. *[13] Any classical-solution: $u(x,t)$ to problem (1.1) satisfies:*

(i) $u > 0$, *in* $B_R \times (0,T)$

(ii) $u(x,t) = u(r,t)$, *where* $r = |x| = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$.

(iii) $u_r < 0$ in $(0,R] \times (0,T)$.

(iv) $u_t > 0$, *in* $B_R \times (0,T)$.

3 Blow-up Set

In this section, under certain assumption on initial function, we show that the center $(x = 0)$ is the only possible blow-up point to problem (1.1). For this, firstly, we need to recall this lemma:

Lemma 3.1. *[12]*
Let u be a blow-up solution to problem (1.2), with $f \in C^2(0,\infty)$, positive, and increasing. Assume that the initial function satisfies the condition:

$$u_{0r}(r) \leq -\theta r, \quad \text{for} \quad 0 < r \leq R, \quad \text{where} \quad \theta > 0 \tag{3.1}$$

If there exists a function $F \in C^2(0,\infty)$, such that $F, F', F'' > 0$ in R^+, and the following condition holds

$$f'F - fF' \geq 2\varepsilon FF' \text{in} R^+, \tag{3.2}$$

then for some $0 < \varepsilon$, $J = r^{n-1}u_r + \varepsilon r^n F(u) \leq 0$ in $(0,R) \times (0,T)$.

Theorem 3.2. *Assume that the condition (3.1) holds. Then blow-up in problem (1.1) occurs only at the center point.*

Proof. Set $f(u) = \lambda(u^p + e^{qu})$, and $F(u) = u^\delta + e^{\alpha u}$, $\delta, \alpha \in (0,1)$
It is clear that $F, F', F'' > 0$ in R^+. In order to apply Lemma 3.1, it is only left to prove that the inequality (3.2) holds. The left hand side of inequality (3.2) takes the form:

$$f'(u)F(u) - f(u)F'(u) = \lambda\left(pu^{p-1} + qe^{qu}\right)\left(u^\delta + e^{\alpha u}\right) - \lambda(u^p + e^{qu})(\delta u^{\delta-1} + \alpha e^{\alpha u})$$
$$= \lambda(p-\delta)u^{p+\delta-1} + \lambda(q-\alpha)e^{(q+\alpha)u} + \lambda(p-\alpha u)u^{p-1}e^{\alpha u} + \lambda(qu-\delta)u^{\delta-1}e^{qu}$$

On the other hand, one can show that

$$2\varepsilon F(u)F'(u) = 2\varepsilon(u^\delta + e^{\alpha u})(\delta u^{\delta-1} + \alpha e^{\alpha u})$$
$$= 2\varepsilon\delta u^{2\delta-1} + 2\varepsilon\alpha e^{2\alpha u} + 2\varepsilon\alpha u^\delta e^{\alpha u} + 2\varepsilon\delta u^{\delta-1}e^{\alpha u}$$

By choosing $\delta, \alpha, \varepsilon$ are small enough such that the following conditions are held:

(i) $p \geq \left(1 + \frac{2\varepsilon}{\lambda}\right)\delta, \quad q \geq \left(1 + \frac{2\varepsilon}{\lambda}\right)\alpha, p \geq 1+\delta, \frac{p}{\alpha} \geq \frac{2\varepsilon}{\lambda}$,

(ii) $\left(\frac{\delta}{q} + \frac{2\varepsilon\delta}{\lambda}\right) \leq u \leq \left(\frac{p}{\alpha} - \frac{2\varepsilon}{\lambda}\right)$, in $B_R \times (0,\ t)$, $t < T$,

we obtain that
$f'(u)F(u) - f(u)F'(u) \geq 2\varepsilon F(u)F'(u)$, in $(0, s)$, $s < \infty$
Thus the condition (4.1) holds, and by Lemma 3.1 , it follows that

$$J = r^{n-1}u_r + \varepsilon r^n(u^\delta + e^{\alpha u}) \leq 0, \quad (r,t) \in (0,R) \times (0,T)$$

It follows that
$$r^{n-1}u_r + \varepsilon r^n e^{\alpha u} \leq 0, \quad (r,t) \in (0,R) \times (0,T)$$

Which yields that $\frac{-du}{e^{\alpha u}} \geq \varepsilon r dr$
Integrating the both sides of last inequality, yields that

$$\frac{1}{\alpha e^{\alpha u}} \geq \frac{1}{2}\varepsilon r^2 \quad \text{or} \quad e^{\alpha u} \leq \frac{2}{\alpha\varepsilon} \cdot \frac{1}{r^2}$$

Thus $\quad u \leq \frac{1}{\alpha}\log(\frac{2}{\alpha\varepsilon}) + \frac{1}{\alpha}\log(\frac{1}{r^2})$

Hence $\quad u \leq \frac{1}{\alpha}\log(\frac{2}{\alpha\varepsilon}) + \frac{1}{\alpha}\log(1.1) - \frac{2}{\alpha}\log(r)$
It follows that the upper point-wise estimate is has the form:

$$u(x,t) \leq \log C + \frac{2}{\alpha}\log\left(\frac{1}{|x|}\right), \quad x \in B_R, \ t \in (0,T), C > 0,$$

Hence, at any time value, for nonzero spatial vectors, the solution is bounded. □

Remark 3.3. The assumption (3.1) is the sufficient but unnecessary condition to guarantee that the blow-up in problem (1.1) occurs only at the center point.

4 Growth Rate Estimate

This section is devoted to deriving the upper blow-up growth-rate estimate to problem (1.1), under the assumption (3.1). Firstly, we recall the following lemma.

Lemma 4.1. *[12] Let u be a blow-up solution to problem (1.2), with $f \in C^2(0,1)$, increasing, and $f,\ f',\ f'' > 0$ in R^+. In addition, we assume that the condition (3.1) holds.*
Set $F(x,t) = u_t - \alpha f(u)$, $x \in B_R$, $t \in (0,T)$, $\alpha > 0$
Then $F(x,t) \geq 0$, for $(x,t) \in \overline{B}_\varepsilon \times (\tau, T)$, where $tau \in (0,T)$, $\varepsilon \in (0,R)$.

Theorem 4.2. *Let u be a blow-up solution of problem (1.1), in addition, assume that condition (3.1) holds. Then there exists $0 < C$ such that:*

$$\frac{(T-t)}{E_1(u)} \leq C, \tag{4.1}$$

where $\quad E_1(u) = \int_u^\infty \frac{dv}{(v^p + e^{qv})}$, *for t close to T, $x = 0$.*

Proof. Set $f(u) = \lambda(u^p + e^{qu})$. By Lemma 4.1 it follows that
$u_t(0,t) \geq \alpha f(u(0,t)) = \alpha\lambda((u(0,t))^p + e^{qu(0,t)})$, for t close to , $\alpha > 0$
Thus

$$\frac{du}{(u(0,t))^p + e^{qu(0,t)}} \geq \lambda\,\alpha\, dt$$

Taking the Integration to the last inequality from t to T, and since $u(0,t) \xrightarrow{t\to T} \infty$, yields that

$$E_1(u) \geq \lambda\,\alpha\,(T-t), \quad \text{or} \quad \frac{(T-t)}{E_1(u)} \leq C,$$

where $x = 0$, t is close to T, $C > 0$

□

Remark 4.3. (i) The assumption (3.1) is the sufficient but unnecessary condition to guarantee that the upper blow-up growth rate of problem (1.1) takes the form (4.1).

(ii) Since $E_1(u(0,t)) \xrightarrow{t\to T} 0$, it follows that $\frac{1}{E_1(u(0,t))} \xrightarrow{t\to T} \infty$, while $(T-t) \xrightarrow{t\to T} 0$.

(iii) The constant C is considered the growth rate constant for problem (1.1). In the next section, this constant will be estimated numerically for a one-dimensional space problem.

5 Numerical Example

In order to support the theoretical results, the explicit Euler finite-difference scheme is used in this section to compute the numerical solutions for a one-dimensional space problem, as a special case of problem (1.1), where $p = 2; q = \lambda = 1$, namely:

$$\left\{ \begin{array}{c} u_t = u_{xx} + (u^2 + e^u), \quad (x,t) \in (-1,1) \times (0,T), \\ u(x,t) = 0, \qquad x = \mp 1 \quad , \quad t \in (0,T), \\ u(x,0) = u_0(x) = 100\left(1 - x^2\right), \qquad x \in [-1,1] \end{array} \right\} \tag{5.1}$$

Clearly, u_0 is non-negative non-zero, symmetric, non-increasing function, and it is vanished on the boundary, and it takes its maximum value at the centre point $x = 0$. In addition, the condition (3.1) holds. Therefore, by Theorem 2, the blow-up in problem (5.1) can only occur at the centre point.
For problem (5.1). The main focus of this section is on estimating the numerical blow-up time and the blow-up growth rate constant.

5.1 The Discrete Problem

We define the spatial grids points as follows:

$$x_j = x_{j-1} + h \ , \ \ j = 1, 2, \ldots, J-1, \ \ x_0 = -1, \ x_J = 1,$$

where $h = 2/J$, $J \in N$
While, the time grids points are defined as follows:

$$t_0 = 0, \ \ t_{n+1} = t_n + k = \text{nk}, \quad \text{for} \ \ \text{n} = 0, 1, 2, \ldots\ldots, \ k \in R^+$$

The approximate value of $u(x_j, t_n)$, is denoted by U_j^n, for $1 \le j \le J-1$, and $n > 0$, By approximating $u_{xx}(x_j, t_n)$ by the standard second order finite difference operator, and approximating $u_t(x_j, t_n)$ by the forward finite difference operator, the discrete problem of problem (5.1) becomes as follows:

$$\left\{ \begin{array}{c} U_j^{n+1} = (1-2r)\,U_j^n + r\left(U_{j+1}^n + U_{j-1}^n\right) + kf\left(U_j^n\right) \\ U_0^n = U_J^n = 0, \\ U_j^0 = u(x_j, 0) = u_0(x_j) \ , \end{array} \right\} \tag{5.2}$$

where $r = k/h^2$, $1 \le j \le J-1$, $n > 0$, $f\left(U_j^n\right) = (U_j^n)^2 + e^{U_j^n}$.
Moreover, we define $||U^n||_\infty = \max_{j=0,\ldots,J} |U_j^n| \ \ n \in N$.
It is known that the order of this proposed method is $O\left(k + h^2\right)$, [15]. Moreover, the space and time steps should be chosen such that this the stability condition $(2r \le 1)$ holds. In fact, the explicit Euler finite-difference scheme has been used by some authors to estimate the numerical blow-up times for some types of one-dimensional semi-linear parabolic equations [15, 16, 17, 18].
The next theorem [15], confirms that the solution of the discrete problem (5.2) is convergent to the exact solution of problem (5.1) when the space-step, h , is small enough.

Theorem 5.1. *Let $U^n = [U_0^n, U_1^n, \ldots\ldots, U_J^n]$, be a solution to problem (11), where the time step $k \in (0, \frac{h^2}{2}]$. Then*

(i) $U_j^n \ge 0$; *and* $U_{J-j}^n = U_j^n \quad U_j^{n+1} \ge U_j^n, \quad n \ge 0, \ 0 \le j \le J$

(ii) $||U^n - u^n|| \le Ch^2, \quad n \ge 0$, *where* $u^n = [u_0(x_j), \ u(x_1, t_n), \ldots\ldots, u(x_J, t_n)]$.

i.e. $\quad U^n \to u^n, \quad$ *as* $\quad h \to 0$,

It follows that the numerical solution obtained by (5.2) converges to the exact solution of problem (5.1).
In fact, the solution of problem (5.2) may not exist for any time-level, because they become unbounded. These solutions are called blow-up solutions.

Definition 5.2. [15] If there exists $m \in N$ such that $||U^n||_\infty$ is unbounded, $\forall \ n \ge m$, then $\{U^n\}$ is called blow-up solution to problem (5.1), and $T_J^m = mk$ is considered the blow up time.

Remark 5.3. The blow-up solution and blow-up time, for problem (5.2) are considered the numerical blow-up solution, and the numerical blow-up time, respectively, for problem (5.1). It is known that the numerical blow-up time is determined by some factors, such as the type of nonlinearity, the size of initial function, and the values of space (time) steps.

The following theorem confirms that the numerical blow-up time obtained by problem (5.2) is convergent to the theoretical blow-up time of problem (5.1).

Theorem 5.4. *[15]: Let $\{U^n\} = \left(U_0^n, \ U_1^n, \ldots..U_J^n\right)$, $n \in N$, be a solution to problem (11), where $k \in (0, \frac{h^2}{2})$. Then $\{U^n\}$, blows up in the finite time T_J^m, for some $m \in N$. Moreover, then $T_J^m \to T$, as $J \to \infty$, where T is the blow-up time of problem (5.1).*

5.2 Numerical Results

In this sub-section, we use the discrete problem (5.2) with using the Matlab software (R2020) to estimate numerically, the blow-up time for problem (5.1). The blow-up time is taken at once when the condition: $||U^n||_\infty \geq 10^6$ is satisfied.
Moreover, for achieving the convergence, the time-step is chosen as $k = h^2/2$.
Clearly, the obtained numerical solution is symmetric, and it takes its maximum at $x_{J/2} = 0$, Furthermore, we compute numerically the blow-up growth rate constant of problem (5.1), which is given by (4.1) as follows:

$$C_J = \max_{0 \leq n \leq m-1} C(n) \,, \quad \text{where} \quad C\,(n) = (T_J^m - t_n)\,/E_1(U_{J/2}^n).$$

For different space size- meshes ($J = 150, 200, 250$,300 and 350), we presents in Table 1, the estimated numerical blow-up times ($T_h = mk$), and the blow-up growth rate constant: C_J, and the iterative errors bounds, obtained by using the error-formula: $E_J = |T_{J+50}^m - T_J^m|$, where $m = m(J)$ denotes the first number of iteration, when the numerical solution blows up.
The Figures 1, 2, 3, 4 and 5 present the evolution overtime in the numerical blow-up solutions of problem (5.1), for J =150, 200, 250 ,300 and 350 respectively, and $0 \leq n \leq m$.
Based on the numerical results presented in Table 1 and in (Figure 1, 2, 3, 4 and 5, the following observations are obtained:

(i) The centre point is the only possible numerical blow-up point, and that supports the theoretical blow-up results, shown in section 3.

(ii) The iterative errors are decreasing, as we refine the space (time) steps, which confirms that the numerical blow-up times sequence is convergent.

(iii) The numerical blow-up growth rate can be estimated by the approximate constant C_J.

Table 1. Numerical blow-up times, estimated growth rates constants, and time errors bounds

J	k	m	T_h	C_J	E_J
150	8.8889E-05	17	0.001511113	3.2239E+06	1.1113E-04
200	5.0000E-05	28	0.001400000	1.3394E+99	8.800E-05
250	3.2000 E-05	41	0.001312000	1.5517E+14	4.534E-05
300	2.2222E-05	57	0.001266654	1.9269E+07	9.475E-06
350	1.6327E-05	77	0.001257179	2.0642E+06	………..

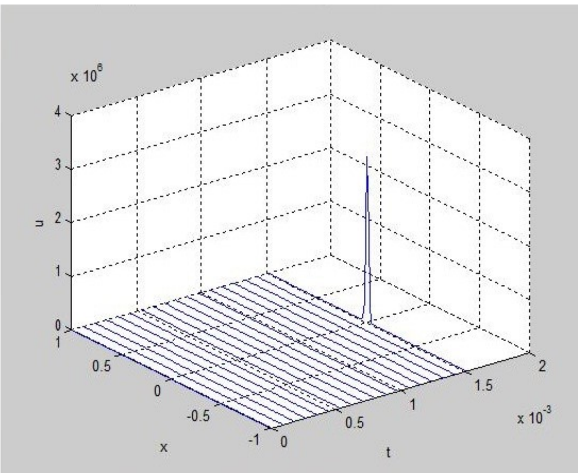

Figure 1. The evolutions in the numerical blow-up solutions overtime with $J = 150$

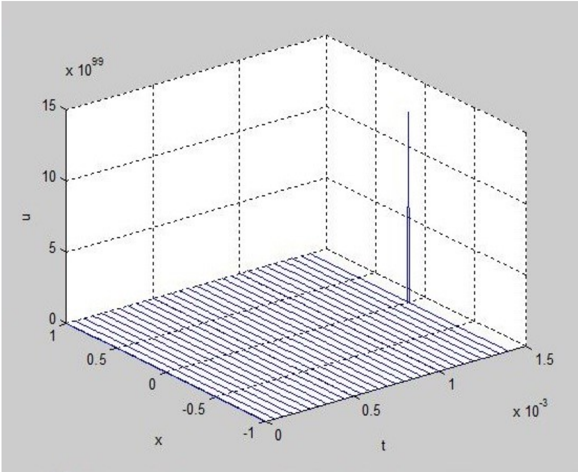

Figure 2. The evolutions in the numerical blow-up solutions overtime with $J = 200$

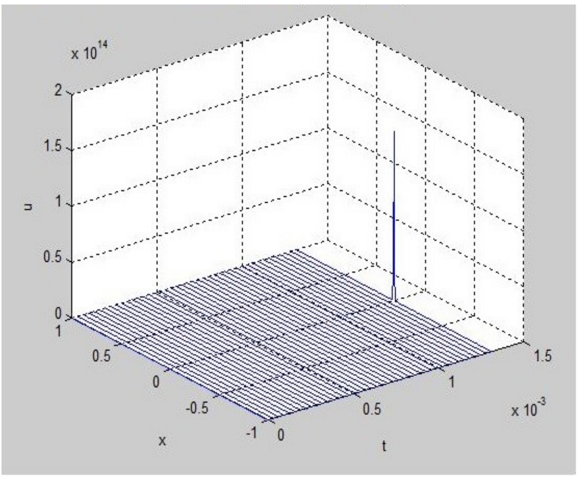

Figure 3. The evolutions in the numerical blow-up solutions overtime with $J = 250$

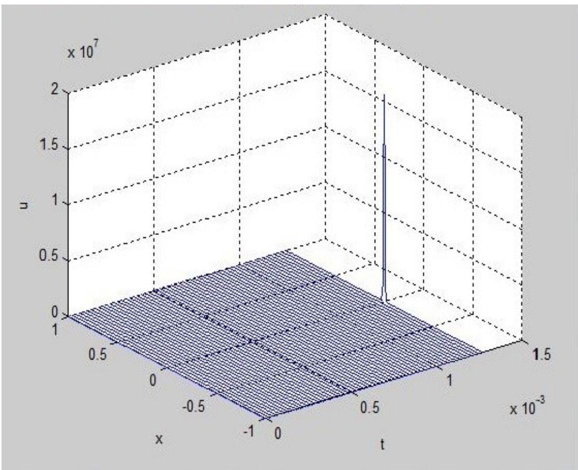

Figure 4. The evolutions in the numerical blow-up solutions overtime with $J = 300$

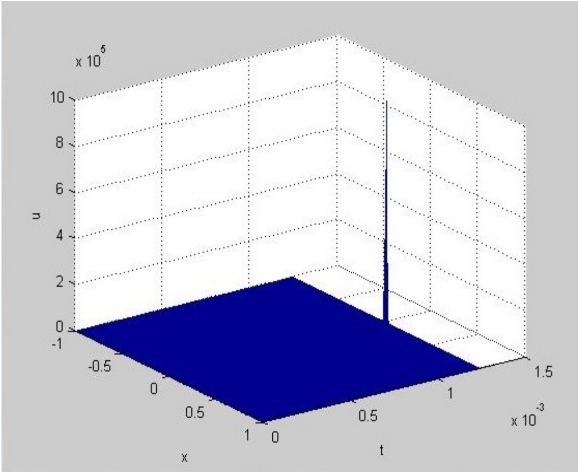

Figure 5. The evolutions in the numerical blow-up solutions overtime with $J = 350$

6 Conclusions

This paper deals with the blow-up solutions of a semi-linear parabolic equation associated with Dirichlet boundary condition. Namely, the blow-up set and growth rates are considered. Moreover, the numerical blow-up time and estimated growth rate constant are computed, to a one-dimensional space semi-linear heat equation, using a finite difference scheme. The obtained results show the solution of blow-up in problem (1.1) may occur in a finite time, and it can only occur at the origin point. Moreover, on a numerical point of view, it is observed that the growth rate constant can determine the size of blow-up in the considered problem.

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