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Normal category arising from the semigroup OX_n

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Abstract The cross-connection theory proposed by K. S. S. Nambooripad provides the construction of a regular semigroup from its principal left (right) ideals using a certain category called a normal category. In the present work, we study the structure of the semigroup OX_n, of singular order-preserving transformations on a finite chain $X_n = \{1 < 2 < \cdots < n\}$, using the normal category. Here, we characterize all of Green's equivalences on the semigroup OX_n and hence prove that OX_n is a fundamental regular semigroup. Further, we construct a normal category called the powerset category $P_o(X_n)$ and prove that $P_o(X_n)$ is isomorphic to the principal left ideal category of OX_n, which is denoted as $\mathcal{L}(OX_n)$. Also, we construct the cone semigroup $T\mathcal{L}(OX_n)$ and prove that $T\mathcal{L}(OX_n)$ is isomorphic to OX_n.

1 Introduction

In describing the structure of regular semigroups, Grillet proposed the idea of cross-connections of regular partially ordered sets [5]. However, Grillet's theory was only intended to describe the structure of fundamental regular semigroups, a subclass of regular semigroups. In the early 1990s, K. S. S. Nambooripad generalized Grillet's cross-connection theory to the general class of regular semigroups, and in this context, he proposed the notion of "normal categories" [9]. These normal categories are further characterized as the principal left ideal category $\mathcal{L}(S)$ of some regular semigroup S. A normal category $\mathcal{C}$ is a small category with subobjects in which an idempotent normal cone is associated with each object of $\mathcal{C}$, and each morphism has a normal factorization. All the normal cones in a normal category form a regular semigroup $T\mathcal{C}$ known as the semigroup of normal cones. In this article, we consider the semigroup OX_n of singular order-preserving transformations on a finite chain X_n and characterize the category $\mathcal{L}(OX_n)$ with the power set category $P_o(X_n)$. The power set category $P_o(X_n)$ is a normal category constructed from the chain X_n.

This article is organized as follows. In section 2, we discuss the important concepts and results regarding the general theory of cross-connections proposed by K. S. S. Nambooripad. In section 3, Green's equivalences in the semigroup OX_n have been provided. In section 4, we prove that the power set category $P_o(X_n)$ is normal and isomorphic to $\mathcal{L}(OX_n)$. Further, we obtained the semigroup of normal cones in $P_o(X_n)$. We illustrate our results on OX_3, the semigroup of singular order-preserving transformations that preserve order on a chain X_3 with length 3.

2 Preliminaries

In the sequel, we assume familiarity with the basic concepts in category theory [11] and semigroup theory [5, 6, 7, 10]. Also, the definitions and results on cross-connections are as in [4, 9]. Throughout this paper, we write transformations to the right of their argument and take the composition from left to right. For category $\mathcal{C}$, $v\mathcal{C}$ denote the set of objects of $\mathcal{C}$ and $\mathcal{C}(a, b)$ the morphisms from a to b. We assume that the categories under consideration are small unless otherwise stated.

A category $\mathcal{P}$ is said to be a preorder category if every hom-set of $\mathcal{P}$ has at most one morphism. This property of a preorder category induces certain quasi-order relation " $\subseteq$ " on $v\mathcal{P}$ and is given by $p \subseteq p'$ if $\mathcal{P}(p, p') \neq \emptyset$. Moreover, $\mathcal{P}$ is said to be a strict preorder if " $\subseteq$ " is a partial order.

Definition 2.1. A small category $\mathcal{C}$ is said to be a category with subobjects if there is a strict preorder subcategory $\mathcal{P}$ of $\mathcal{C}$ with $v\mathcal{P} = v\mathcal{C}$ having the following properties:

(1) every morphisms of $\mathcal{P}$ is a monomorphism in $\mathcal{C}$.

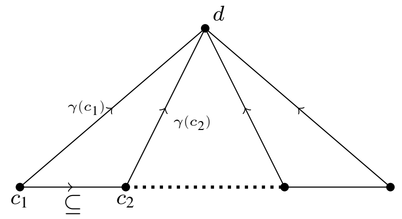
(2) if $h = h'k$ for $h, k \in \mathcal{P}$, then $h' \in \mathcal{P}$.

The pair $(\mathcal{C}, \mathcal{P})$ is called the category with subobjects. If $c' \subseteq c$, the unique morphism from c' to c is called the inclusion morphism and is denoted by $j^c_{c'}$. An inclusion $j^c_{c'}$ splits if there exists $e : c \to c' \in \mathcal{C}$ such that $j^c_{c'} q = 1_{c'}$ and the morphism q is called a retraction. A factorization of a morphism $f \in \mathcal{C}$ of the form $f = ewj$ where e is a retraction, w is an isomorphism and j is an inclusion is called the normal factorization of f. The morphism ew is known as the epimorphic component of f and is denoted by f°.

Definition 2.2. Let $\mathcal{C}$ be a category with subobjects and $d \in v\mathcal{C}$. A map $\gamma : v\mathcal{C} \to \mathcal{C}$ is called a cone with vertex d if

(1) $\gamma(c) \in \mathcal{C}(c, d)$ for all $c \in v\mathcal{C}$.

(2) If $c_1 \subseteq c_2$, then $j_{c_1}^{c_2}\gamma(c_2) = \gamma(c_1)$



For a cone γ, let c_γ denote the vertex of γ. For $c \in v\mathcal{C}$, the morphism $\gamma(c) : c \to c_\gamma$ is called the component of γ at c. A cone γ is said to be normal if there exists $c \in v\mathcal{C}$ such that $\gamma(c) : c \to c_\gamma$ is an isomorphism. We denote by $\mathcal{TC}$, the set of all normal cones in $\mathcal{C}$ and by M_γ, the set

$$M_\gamma = \{c \in v\mathcal{C} : \gamma(c) \text{ is an isomorphism } \}.$$

Definition 2.3. A category $\mathcal{C}$ with subobjects is called a normal category if the following holds

(1) Any morphism in $\mathcal{C}$ has a normal factorization.

(2) Every inclusion in $\mathcal{C}$ splits.

(3) For each $c \in v\mathcal{C}$ there is a normal cone γ with vertex c and $\gamma(c) = 1_{c_\gamma}$.

Observe that given a normal cone γ and an epimorphism $f : c_\gamma \to d$, the map $\gamma * f : a \to \gamma(a)f$ from $v\mathcal{C}$ to $\mathcal{C}$ is a normal cone with vertex d. Consider two normal cones γ and σ, then

$$\gamma \cdot \sigma = \gamma * (\sigma(c_\gamma))^\circ$$

where $(\sigma(c_\gamma))^\circ$ is the epimorphic part of $\sigma(c_\gamma)$, defines a binary composition on $\mathcal{TC}$.

Theorem 2.4. (Theorem III.2 [9]) *Let $\mathcal{C}$ be a normal category. Then $\mathcal{TC}$, the set of all normal cones in $\mathcal{C}$ is a regular semigroup with the binary operation*

$$\gamma \cdot \sigma = \gamma * (\sigma(c_\gamma))^\circ \tag{2.1}$$

and $\gamma \in TC$ is idempotent if and only if $\gamma(c_\gamma) = 1_{c_\gamma}$.

Normal categories of a regular semigroup: There are two normal categories associated with a regular semigroup S, namely the principal left ideal category $\mathcal{L}(S)$ and the principal right ideal category $\mathcal{R}(S)$. The objects of $\mathcal{L}(S)$ are principal left ideals Se generated by idempotents $e \in E(S)$. The morphisms are partial right translations $\rho(e, u, f) : Se \to Sf : u \in eSf$ such that for every $x \in Se,\ \rho(e, u, f) : x \mapsto xu$. Dually, the objects of the category $\mathcal{R}(S)$ of principal right ideals are eS, generated by $e \in E(S)$ and the morphisms are partial left translations $\lambda(e, v, f) : eS \to fS :\ v \in fSe$, which maps $x \mapsto vx$ for any $x \in eS$.

Proposition 2.5. *Let S be a regular semigroup. Then $\mathcal{L}(S)$ is a normal category. $\rho(e, u, f) = \rho(e', v, f')$ if and only if $e\mathcal{L}e', f\mathcal{L}f', u \in eSf, v \in e'Sf'$ and $v = e'u$. Let $\rho = \rho(e, u, f)$ be a morphism in $\mathcal{L}(S)$. For any $g \in \mathcal{R}_u \cap \omega(e)$ and $h \in E(\mathcal{L}_u), \rho = \rho(e, g, g)\rho(g, u, h)\rho(h, h, f)$ is a normal factorization of ρ.*

Proposition 2.6. *Let S be a regular semigroup, $a \in S$ and $f \in E(\mathcal{L}_a)$. Then for each $e \in E(S)$, let $\rho^a(Se) = \rho(e, ea, f)$. Then ρ^a is a normal cone in $\mathcal{L}(S)$ with vertex Sf called the principal cone generated by a.*

$$M\rho^a = \{Se : e \in E(\mathcal{R}_a)\}.$$

ρ^a is an idempotent in $\mathcal{TL}(S)$ iff $a \in E(S)$. The mapping $a \mapsto \rho^a$ is a homomorphism from S to $\mathcal{TL}(S)$.

3 Semigroup of order-preserving transformations on a finite chain

Let $X_n = \{1 < 2 < \cdots < n : n \in \mathbb{N}\}$ be a finite chain of length n. A transformation $f : X_n \to X_n$ is called order-preserving if $(i)f \leq (j)f$ whenever $i \leq j$. A transformation is said to be singular if it is not invertibele(not one-one and onto). The semigroup of all singular order-preserving mappings from X_n to itself under function composition is denoted by OX_n. To consider nontrivial cases only, we assume $n \geq 3$. The Green's relations in the semigroup OX_n are characterized entirely by their images and kernels. It is known that OX_n is a regular subsemigroup of $\mathcal{T}X_n$, the full transformation semigroup of X_n. The following proposition characterizes all the Green's equivalences in OX_n.

Lemma 3.1. *The semigroup OX_n, of singular order-preserving transformations on a finite chain $X_n = \{1 < 2 < \cdots < n : n \in \mathbb{N}\}$ is a regular semigroup. Let $f, g \in OX_n$, then the following holds.*

(1) *$f \leq_{\mathcal{R}} g$ if and only if $ker\ g \subseteq ker\ f$.*

(2) *$f \leq_{\mathcal{L}} g$ if and only if $Im\ f \subseteq Im\ g$.*

Proof. Let f be an order-preserving function on a finite chain X_n and let $Im\ f = \{x_1 < x_2 < \cdots < x_k : x_i \in X_n, i = 1, 2, \cdots k\}$. Then there exists $n_1 < n_2 < \cdots < n_k = n \in \mathbb{N}$, such that

$$(x)f = \begin{cases} x_1, & \text{if } x = 1, 2, \cdots, n_1, \\ x_{i+1}, & \text{if } n_i < x \leq n_{i+1},\ i = 1, 2, \cdots k-1. \end{cases} \tag{3.1}$$

Now define $g : X_n \to X_n$ by

$$(x)g = \begin{cases} n_i, & \text{if } x = x_i,\ i = 1, 2, \cdots, k, \\ n_1, & \text{if } x < x_1, \\ n_i, & \text{if } x_i < x < x_{i+1},\ i \in \{1, 2, \cdots k-1\}, \\ n_k, & \text{if } x > x_k. \end{cases} \tag{3.2}$$

Then clearly, g is an order-preserving singular transformation on X_n and thus $g \in OX_n$. Now consider,

$$(x)fgf = (x_1)gf = (n_1)f = x_1 = (x)f, \text{ if } 1 \leq x \leq n_1$$

and

$$(x)fgf = (x_i)gf = (n_{i+1})f = x_{i+1} = (x)f, \text{ if } n_i < x \leq n_{i+1}, \quad \text{where } i = 1, 2, \cdots k-1.$$

Hence $fgf = f$ and g is a generalized inverse of f. Hence OX_n is a regular semigroup.

To prove the first assertion, suppose $f \leq_{\mathcal{R}} g$, then there exists some $h \in OX_n$ such that $f = gh$. Let $(x, y) \in ker\ g$ then $(x)g = (y)g$. Then $(x)f = (y)f$ and $(x, y) \in ker\ f$. Conversely, suppose that $ker\ g \subseteq ker\ f$ and let $Im\ g = \{x_1 < x_2 < \cdots < x_k : x_i \in X_n, i = 1, 2, \cdots, k\}$. Since g is an order-preserving function $(x_i)g^{-1}$ is an interval for each $i = 1, 2, \cdots, k$. Therefore let $A_i = (x_i)g^{-1}$ for $i = 1, 2, \cdots, k$. Then $A_1 \cup A_2 \cup \cdots \cup A_k = X_n$, and $A_i \cap A_j = \phi$ for $i \neq j$. Choose exactly one representative a_i from each interval. Since $ker\ g \subseteq ker\ f$, f is a constant on each A_i. Now define $h : X_n \to X_n$ by

$$(x)h = \begin{cases} (a_i)f, & \text{if } x = x_i,\ i = 1, 2, \cdots, k, \\ (a_1)f, & \text{if } x < x_1, \\ (a_i)f, & \text{if } x_i < x < x_{i+1},\ i \in \{1, 2, \cdots k-1\}, \\ (a_k)f, & \text{if } x > x_k. \end{cases} \tag{3.3}$$

Since both f and g are order-preserving h is also order- preserving and $h \in OX_n$. Let $x \in X_n$ then x is an element of exactly one A_i where $i = 1, 2, \cdots, k$. Let $x \in Aj$ then $(x)g = x_j$ and

$$(x)gh = (x_j)h = (a_j)f = (x)f$$

thus $f = gh$. To prove the second assertion, assume $f \leq_{\mathcal{L}} g$, then it is obvious that $Im\ f \subseteq Im\ g$. Conversely, assume that $Im\ f \subseteq Im\ g$ and let $Im\ f = \{y_1 < y_2 < \cdots < y_m : y_i \in X_n, i = 1, 2, \cdots, m\}$. Now let $B_i = (y_i)g^{-1}$ for $i = 1, 2, \cdots, m$ then each B_i is an interval. Fix exactly one element from each B_i, say b_i and define $h : X_n \to X_n$ by $(x)h = b_i$ with $(f(x))g^{-1} \in B_i$. Now it can be seen that $h \in OX_n$ and $f = hg$. Hence $f \leq_{\mathcal{L}} g$. □

Proposition 3.2. *Let f and g be elements of the semigroup OX_n of singular order-preserving transformations on a finite chain X_n. Then,*

(1) *$f\ \mathcal{R}\ g$ if and only if $ker\ g = ker\ f$,*

(2) *$f\ \mathcal{L}\ g$ if and only if $Im\ f = Im\ g$,*

(3) *$f\ \mathcal{H}\ g$ if and only if $f = g$,*

(4) *$f\ \mathcal{D}\ g$ if and only if $|Im\ f| = |Im\ g|$.*

Proof. The proof of the first and second assertions follows immediately from Lemma 3.1. Now suppose that $f\ \mathcal{H}\ g$ then $f\ \mathcal{L}\ g$ and $f\ \mathcal{R}\ g$. Using (1) and (2) we have $ker\ f = ker\ g$ and $Im\ f = Im\ g$. Since f and g are order-preserving, f and g must be identical. Now suppose $f\ \mathcal{D}\ g$ then by definition, there exists $h \in OX_n$ such that $f\ \mathcal{L}\ h\ \mathcal{R}\ g$. Then it follows from (1) and (2) of above that $Im\ f = Im\ h$ and $ker\ g = ker\ h$. Since $ker\ g = ker\ h$ we have $|Im\ g| = |Im\ h|$ thus $|Im\ g| = |Im\ f|$. Conversely, assume that $|Im\ g| = |Im\ f| = m \leq n$. Let $Im\ f = \{x_1 < x_2 < \cdots < x_m\}$ and $Im\ g = \{y_1 < y_2 < \cdots < y_m\}$. For $n_1 < n_2 < \cdots < n_k = n,\ m_1 < m_2 < \cdots < m_k = n \in \mathbb{N}$, let

$$(x)f = \begin{cases} x_1, & \text{if } x = 1, 2, \cdots, n_1, \\ x_{i+1}, & \text{if } n_i < x \leq n_{i+1},\ i = 1, 2, \cdots, k-1, \end{cases} \tag{3.4}$$

and

$$(x)g = \begin{cases} y_1, & \text{if } x = 1, 2, \cdots, m_1, \\ y_{i+1}, & \text{if } m_i < y \leq m_{i+1},\ i = 1, 2, \cdots, k. \end{cases} \tag{3.5}$$

Now define,

$$(x)h = \begin{cases} x_1, & \text{if } x = 1, 2, \cdots, m_1, \\ x_{i+1}, & \text{if } m_{i-1} < y \leq m_i,\ i = 2, 3, \cdots k. \end{cases} \tag{3.6}$$

and it is easy to observe that $Im\ f = Im\ h$ and $ker\ g = ker\ h$ and thus $f\mathcal{D}g$. □

Remark 3.3. Since the Green's $\mathcal{H}$ relation in $OX_n$is identity, the semigroup OX_n is a fundamental regular semigroup which is a subsemigroup of full transformation semigroup of X_n.

Example 3.4. Let $X_3 = 1 < 2 < 3$ be the finite chain of length three. Then the semigroup OX_3 is given by

$$OX_3 = \left\{ \begin{pmatrix} 1&2&3 \\ 1&2&2 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 1&3&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 2&3&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 1&1&2 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 1&1&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 2&2&3 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 1&1&1 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 2&2&2 \end{pmatrix}, \begin{pmatrix} 1&2&3 \\ 3&3&3 \end{pmatrix} \right\}.$$

We denote the elements in OX_3 as follows.

$f = \begin{pmatrix} 1&2&3 \\ 1&2&2 \end{pmatrix}, g = \begin{pmatrix} 1&2&3 \\ 1&3&3 \end{pmatrix}, h = \begin{pmatrix} 1&2&3 \\ 2&3&3 \end{pmatrix}, u = \begin{pmatrix} 1&2&3 \\ 1&1&2 \end{pmatrix}, v = \begin{pmatrix} 1&2&3 \\ 1&1&3 \end{pmatrix}, w = \begin{pmatrix} 1&2&3 \\ 2&2&3 \end{pmatrix}, k_1 = \begin{pmatrix} 1&2&3 \\ 1&1&1 \end{pmatrix}, k_2 = \begin{pmatrix} 1&2&3 \\ 2&2&2 \end{pmatrix},$ $k_3 = \begin{pmatrix} 1&2&3 \\ 3&3&3 \end{pmatrix}$.

Then we have $E(OX_3) = \{k_1, k_2, k_3, f, g, v, w\}$. Now we identify the Green's relations of OX_3.

$$\begin{array}{ll} Im\ k_1 = \{1\} & Im\ f = Im\ u = \{1, 2\} \\ Im\ k_2 = \{2\} & Im\ g = Im\ v = \{1, 3\} \\ Im\ k_3 = \{3\} & Im\ h = Im\ w = \{2, 3\} \end{array}$$

$\mathcal{L}(OX_3) = \{(k_1, k_1), (k_2, k_2), (k_3, k_3), (f, u), (g, v), (h, w)\}$

$$ker\ k_1 = ker\ k_2 = ker\ k_3 = X_3 \times X_3$$

$$ker\ f = ker\ g = ker\ h = \{(1, 1)(2, 2), (3, 3), (2, 3)\}$$

$$ker\ u = ker\ v = ker\ w = \{(1, 1)(2, 2), (3, 3), (1, 2)\}$$

In OX_3 we get $k_1\ \mathcal{R}\ k_2\ \mathcal{R}\ k_3$, $\ f\ \mathcal{R}\ g\ \mathcal{R}\ h$ and $u\ \mathcal{R}\ v\ \mathcal{R}\ w$ and the egg box diagram becomes

f	g	h	$\mathcal{D}_2$
u	v	w	

k_1	k_2	k_3	$\mathcal{D}_1$.

4 The category of Principal left ideals of OX_n

In this section, we characterize the normal category $\mathcal{L}(OX_n)$ associated with the principal left ideals of OX_n. Here, we use S and OX_n mutually to denote the semigroup of order-preserving singular transformations on X_n. For any proper nontrivial subchain$A$ of X_n, let e_A denote the idempotent transformation with image A. Note that e_A is not uniquely determined by A.

Lemma 4.1. *Let $A, B \subsetneq X_n$ and $\rho(e_A, u, e_B)$ be a morphism from Se_A to Se_B. Then for any $x \in A$, $xu \in B$. Also $\rho(e_A, u, e_B) = \rho(e'_A, v, e'_B)$ if and only if $xu = xv$ for all $x \in A$, where $e_A,\ e'_A$ are idempotents with image A and $e_B,\ e'_B$ are idempotents with image B.*

Proof. By the definition of a morphism in $\mathcal{L}(S)$, $u \in e_ASe_B$ and $Xu \subseteq Xe_B = B$. In particular $xu \in B$ for all $x \in A$. To prove the second assertion, let $\rho(e_A, u, e_B) = \rho(e'_A, v, e'_B)$ then by Proposition 2.5 $u = e_Av$. Also since e_A is an idempotent map with image A it can be seen that $e_A|_A = 1_A$. Hence $xu = xv$ for all $x \in A$. Conversely, if $xu = xv$ for all $x \in A$, then since $u \in e_ASe_B$, $e_Au = u$ and by our assumption $e_Au = e_Av$. Hence $u = e_Av$ and using Proposition 2.5 we have $\rho(e_A, u, e_B) = \rho(e'_A, v, e'_B)$. ◻

Proposition 4.2. *All normal cones in the category $\mathcal{L}(OX_n)$ are principal cones.*

Proof. Let γ be a normal cone in $\mathcal{L}(OX_n)$, with $c_\gamma = Se_A$ for some $e_A \in E(OX_n)$. For any $x \in X_n$, e_x denotes the constant map whose image is x and $Se_x = \{e_x\}$. Consider $\gamma(Se_x)$ for $x \in X_n$. Let $\gamma(Se_x) = \rho(e_x, u_x, e_A)$ then by Lemma 4.1 $xu_x \in A$. Since $\gamma(Se_x)$ is uniquely determined by x, u_x is uniquely determined by x. Define α on X_n as follows.

$$x\alpha = xu_x \quad \text{for all } x \in X_n \text{ and } u_x \text{ as above .}$$

Since u_x is uniquely determined by x, α is well defined. Since $xu_x \in A$ for all $x \in X_n$, α is a function from X_n with image contained in A. Now we prove that α is an order-preserving transformation. If possible, assume that α is not an order-preserving function. Then there exists $x, y \in X_n$ such that $x\alpha < y\alpha$ for $x > y$. Now consider the set $Y = \{x, y\}$ such that $Sx, Sy \subseteq Se_Y$ and $\gamma(Se_Y) = \rho(e_Y, u, e_A)$. Since $Sx, Sy \subseteq Se_Y$ we have

$$\gamma(Se_x) = j_{Se_x}^{Se_Y}\gamma(Se_Y) \text{ and } \gamma(Sy) = j_{Sy}^{Se_Y}\gamma(Se_Y).$$

That is

$$\rho(e_x, u_x, e_A) = \rho(e_x, e_x, e_Y)\rho(e_Y, u, e_A) = \rho(e_x, e_xu, e_A).$$

Similarly we get $\rho(e_y, u_y, e_A) = \rho(e_y, e_yu, e_A)$. From these two equations, we get $x\alpha = xu_x = xe_xu = xu$ and $y\alpha = yu$. Hence $xu < yu$ for $x > y$, which contradicts that u is order-preserving. Therefore, $\alpha \in S$ is an order-preserving transformation with image α contained in A. Since γ is a normal cone, there is a component $\gamma(Se_C)$ is an isomorphism and let $\gamma(Se_C) = \rho(e_C, \beta, e_A)$. Then by Lemma 4.1 $x\beta \in A$ for all $x \in C$. Since $\gamma(Se_C)$ is an isomorphism $\beta\ \mathcal{L}\ e_A$, and $Im\ \beta = A$. Now, we show that $Im\ \alpha = A$. Let $y \in A$, then there exists $x \in C$ such that $x\beta = y$.

$$\rho(e_x, u_x, e_A) = \gamma(Sx) = j_{Sx}^{Se_C}\gamma(Se_C) = \rho(e_x, e_x\beta, e_A).$$

Thus $u_x = e_x\beta$(using Proposition 2.5), so that $x\alpha = xu_x = xe_x\beta = x\beta = y$. Hence α is onto. Now we prove that $\gamma = \rho^\alpha$. Since $Im\ \alpha = Se_A$ the vertex of ρ^α is $Se_A = c_\gamma$. For $B \subseteq X$, we prove that if $\gamma(Se_B) = \rho(e_B, v, e_A)$, then $\rho(e_B, v, e_A) = \rho(e_B, e_B\alpha, e_A)$. For that, it is sufficient to prove that $xv = xe_B\alpha$ for all $x \in B$. If $x \in B$, then $Sx \subseteq Se_B$ and by the definition of cones

$$\gamma(Sx) = j_{Sx}^{Se_B}\gamma(Se_B) = \rho(e_x, e_x, e_B)\rho(e_B, v, e_A) = \rho(e_x, e_xv, e_A).$$

But $\gamma(Sx) = \rho(e_x, u_x, e_A)$, equating these we get $xu_x = xe_xv = xv$. That is for all $x \in B$ we have $x\alpha = xv$. Therefore $\rho(e_B, v, e_A) = \rho(e_B, e_B\alpha, e_A)$. Hence $\gamma = \rho^\alpha$ and all normal cones are of the form ρ^α for some $\alpha \in S$. ◻

Theorem 4.3. *The semigroup of normal cones in $\mathcal{L}(OX_n)$ is isomorphic to $O(X_n)$.*

Proof. It is known that, the map $\phi : T\mathcal{L}(OX_n) \to \mathcal{L}(OX_n)$ defined by $(\alpha)\phi = \rho^\alpha$ is a semigroup homomorphism by Proposition 2.6. Using Proposition 4.2, the map ϕ is onto. Now we need to show that ϕ is injective. For, let $\alpha, \beta \in S$ such that $\rho^\alpha = \rho^\beta$. For any $x \in X_n$, $\rho^\alpha(Sx) = \rho(e_x, e_x\alpha, e_A)$ where $e_A \;\mathcal{L}\; \alpha$ and $\rho^\beta(Sx) = \rho(e_x, e_x\beta, e_B),\; e_B \;\mathcal{L}\; \beta$. Since $\rho^\alpha = \rho^\beta$, we have

$$\rho(e_x, e_x\alpha, e_B) = \rho(e_x, e_x\alpha, e_B).$$

By Lemma 4.1, $e_x\alpha = e_x\beta$. It follows that $x\alpha = x\beta$ for all $x \in X_n$ and $\alpha = \beta$. □

4.1 Power set category

Let $X_n = \{1 < 2 < \cdots < n\}$ be a non empty finite chain and to avoid trivialities, assume that $n \geq 3$. Given any finite chain, one can construct a category $P_o(X_n)$ from X_n whose objects are all proper subchains of X_n and morphisms are the order-preserving transformations between the subchains. $P_0(X_n)$ is called the power set category and it is a category with subobjects in which inclusions are set inclusions. That is we have the inclusion function $j = j_A^B : A \to B$ if $A \subseteq B$. In the following proposition we prove that $P_o(X_n)$ is a normal category.

Proposition 4.4. *The power set category $P_o(X_n)$ is a normal category.*

Proof. It is easy to see that $P_o(X_n)$ is a category with subobjects and the subobject relation is induced by the usual subchain relation. Given an inclusion $j_{A'}^A$ where $A' \subseteq A$, define a retraction $e : A \to A'$ as follows:
Let $A' = \{x_1 < x_2 < \cdots < x_k\}$ and $x_i \in X_n,\; i = 1, 2, \cdots, k$. Define

$$(x)e = \begin{cases} x, & \text{if } x \in A', \\ x_i, & \text{if } x_i < x < x_{i+1},\; i \in \{1, 2, \cdots k-1\}, \\ x_1, & \text{if } x < x_1, \\ x_k, & \text{if } x > x_k. \end{cases} \tag{4.1}$$

Clearly, $e \in S$ and $je = 1_{A'}$. Given any morphism(order-preserving transformation) $f : A \to B$; let $B' = Im\ f$ and A' is the cross-section of the partition of A determined by $ker\ f$. Then f has a normal factorization and $f = euj$, where $u = f|_{A'}$ is a bijection and $j = j_{B'}^B$. Given any $A \subseteq X_n$, let γ be a cone in $P_o(X_n)$ with vertex A is defined as follows. Let $u : X_n \to A$ be an order-preserving transformation such that $u(a) = a$ for all $a \in A$. For any $B \subseteq X_n$, define $\gamma(B) = u|_B : B \to A$. Then γ is a normal cone with $\gamma(A) = 1_A$. Thus $P_o(X_n)$ is a normal category. □

In the following theorem it is shown that the categories $P_o(X_n)$ and $\mathcal{L}(OX_n)$ are isomorphic. For that, we show that there exists an inclusion preserving functor from $\mathcal{L}(OX_n)$ to $P_o(X_n)$ which is an order isomorphism, v-injective, v-surjective and fully-faithful.

Theorem 4.5. *The categories $P_o(X_n)$ and $\mathcal{L}(OX_n)$ are isomorphic.*

Proof. Define a functor $F : \mathcal{L}(OX_n) \to P_o(X_n)$ as follows: For $Se_A \in v\mathcal{L}(OX_n)$ and a morphism $\rho(e_A, u, e_B) \in \mathcal{L}(OX_n)$ we have

$$vF(Se_A) = A \quad and \quad F(\rho(e_A, u, e_B)) = u|_A.$$

Clearly, F is well defined by Proposition 3.2 and Lemma 4.1. Now let $\rho(e_A, u, e_B), \rho(e_B, v, e_C)$ be two composable morphisms in $\mathcal{L}(OX_n)$. Then

$$\rho(e_A, u, e_B)\rho(e_B, v, e_C) = \rho(e_A, uv, e_C).$$

Now $F(\rho(e_A, uv, e_C)) = uv|_A = u|_A v|_B = F(\rho(e_A, u, e_B))F(\rho(e_B, v, e_C))$. Hence F is a functor. Using Proposition 3.2 it is easy to prove that F is inclusion preserving and vF is an order isomorphism.

Now we prove that vF is a bijection. For, Let $A \subseteq X_n$ such that $A = \{x_1 < x_2 < \cdots < x_k\}$. Then define

$$(x)e = \begin{cases} x, & \text{if } x \in A, \\ x_1, & \text{if } x < x_1, \\ x_i, & \text{if } x_{i-1} < x < x_i,\; i \in \{2, 3, \cdots k\}, \\ x_k, & \text{if } x > x_k. \end{cases} \tag{4.2}$$

Clearly, e is an idempotent order-preserving transformation with $Im\ e = A$. Now $F(Se) = Im\ e = A$. Hence vF is v-surjective. By Proposition 3.2 it follows that vF is injective. To complete the proof only need to prove F is fully-faithful. Now let f be an order-preserving transformation from A to B. Then $e_A f$ is an order-preserving transformation with the image contained in B and $e_A f|_A = f$. So $e_A f \in e_A S e_B$ and $\rho(e_A, e_A f, e_B) : Se_A \to Se_B$ such that $F(\rho(e_A, e_A f, e_B)) = f$. Hence F is full. The proof of F is faithfull follows from Lemma 4.1. Hence the Theorem. □

Since the category $P_o(X_n)$ is isomorphic to $\mathcal{L}(OX_n)$, the corresponding semigroups of normal cones $T\mathcal{L}(OX_n)$ and $TP_o(X_n)$ are isomorphic. But using Theorem 4.3 we get $TP_o(X_n)$ is isomorphic to OX_n. Summarising, we have the following theorem.

Theorem 4.6. *$TP_o(X_n)$ is isomorphic to the semigroup S of singular order-preserving transformation on a finite chain X_n.*

Remark 4.7. All normal cones in $P_o(X_n)$ can be described as follows. Let γ be a normal cone in $P_o(X_n)$ with vertex $A \subseteq X_n$. Then let $\alpha : X_n \to X_n$ be defined as follows.

$$(x)\alpha = (x)\gamma(\{x\}), \text{ for all } x \in X_n.$$

Then using a similar argument to the one in the proof of Proposition 4.2, we may observe that $\alpha \in S$ and $\gamma = \rho^{\alpha}$. Notice that the semigroup OX_n is represented by T $P_o(X_n)$.

Example 4.8. The semigroup OX_3 consists of singular transformations on a finite chain $X_3 = \{1 < 2 < 3\}$ of length three. In this example, we construct the categories $\mathcal{L}(OX_3)$ and $P_o(X_3)$. From Example 3.4, we have the semigroup $OX_3 = \{k_1, k_2, k_3, f, g, h, u, v, w\}$ and the egg box diagram of $\mathbb{O}X_3$ is given below.

f	g	h	$\mathcal{D}_2$
u	v	w	

k_1	k_2	k_3	$\mathcal{D}_1$.

$$E(OX_3) = \{k_1, k_2, k_3, f, g, v, w\}.$$

$\mathcal{L}(OX_3)$ is the category whose objects are the principal left ideals of OX_3. Since OX_3 has 6 distinct $\mathcal{L}$ classes, $\mathcal{L}(OX_3)$ has 6 objects and is given by $v\mathcal{L}(OX_3) = \{Sf, Sv, Sw, Sk_1, Sk_2, Sk_3\}$. To obtain the hom-sets in $\mathcal{L}(OX_3)$ we compute the following sets.

$$fSf = \{k_1, k_2, f\},\ fSv = \{k_1, k_3, u\},\ fSw = \{k_2, k_3, u\},$$
$$fSk_1 = \{k_1\}, \quad fSk_2 = \{k_2\}, \quad fSk_3 = \{k_3\},$$

$$vSf = \{k_1, k_2, u\}, \quad vSv = \{k_1, k_3, v\}, \quad vSw = \{k_2, k_3, w\},$$
$$vSk_1 = \{k_1\}, \quad vSk_2 = \{k_2\}, \quad vSk_3 = \{k_3\},$$

$$wSf = \{k_1, k_2, u\}, \quad wSv = \{k_1, k_3, v\}, \quad wSw = \{k_2, k_3, w\},$$
$$wSk_1 = \{k_1\}, \quad wSk_2 = \{k_2\}, \quad wSk_3 = \{k_3\},$$

$$k_1Sf = \{k_1, k_2\}, \quad k_1Sv = \{k_1, k_3\}, k_1Sw = \{k_2, k_3\},$$
$$k_1Sk_1 = \{k_1\}, \quad k_1Sk_2 = \{k_2\}, \quad k_1Sk_3 = \{k_3\},$$

$$k_2Sf = \{k_1, k_2\}, \quad k_2Sv = \{k_1, k_3\}, \quad k_2Sw = \{k_1, k_2, k_3\},$$
$$k_2Sk_1 = \{k_1\}, \quad k_2Sk_2 = \{k_2\}, \quad k_2Sk_3 = \{k_3\},$$

$$k_3Sf = \{k_1, k_2\}, \quad k_3Sv = \{k_1, k_3\}, \quad k_3Sw = \{k_2, k_3\},$$
$$k_3Sk_1 = \{k_1\}, \quad k_3Sk_2 = \{k_2\}, \quad k_3Sk_3 = \{k_3\}.$$

The hom-sets in the category $\mathcal{L}(OX_3)$ can be obtained as follows. By the definition of a morphism in $\mathcal{L}(OX_3)$ we get

$$\mathcal{L}(OX_3)(Sf, Sf) = \{\rho(f, u, f) : u \in fSf\}$$

and we have the set $fSf = \{k_1, k_2, f\}$ thus
$\mathcal{L}(OX_3)(Sf, Sf) = \{\rho(f, k_1, f), \rho(f, k_2, f), \rho(f, f, f)\}$.
In the similar manner we get all the morphisms in $\mathcal{L}(OX_3.)$

$\mathcal{L}(OX_3)(Sf, Sv) = \{\rho(f, k_1, v), \rho(k, k_3, v), \rho(f, u, v)\}$
$\mathcal{L}(OX_3)(Sf, Sw) = \{\rho(f, k_2, w), \rho(f, k_3, w), \rho(f, u, w)\}$
$\mathcal{L}(OX_3)(Sf, Sk_1) = \{\rho(f, k_1, k_1)\}$
$\mathcal{L}(OX_3)(Sf, Sk_2) = \{\rho(f, k_2, k_2)\}$
$\mathcal{L}(OX_3)(Sf, Sk_3) = \{\rho(f, k_3, k_3)\}$

$\mathcal{L}(OX_3)(Sv, Sf) = \{\rho(v, k_1, f), \rho(v, k_2, f), \rho(v, u, f)\}$
$\mathcal{L}(OX_3)(Sv, Sv) = \{\rho(v, k_1, v), \rho(v, k_3, v), \rho(v, v, v)\}$
$\mathcal{L}(OX_3)(Sv, Sw) = \{\rho(v, k_2, w), \rho(v, k_3, w), \rho(v, w, w)\}$
$\mathcal{L}(OX_3)(Sv, Sk_1) = \{\rho(v, k_1, k_1)\}$
$\mathcal{L}(OX_3)(Sv, Sk_2) = \{\rho(v, k_2, k_2)\}$
$\mathcal{L}(OX_3)(Sv, Sk_3) = \{\rho(v, k_3, k_3)\}$

$\mathcal{L}(OX_3)(Sw, Sf) = \{\rho(w, k_1, f), \rho(w, k_2, f), \rho(w, u, f)\}$
$\mathcal{L}(OX_3)(Sw, Sv) = \{\rho(w, k_1, v), \rho(w, k_3, v), \rho(w, v, v)\}$
$\mathcal{L}(OX_3)(Sw, Sw) = \{\rho(w, k_2, w), \rho(w, k_3, w), \rho(w, w, w)\}$
$\mathcal{L}(OX_3)(Sw, Sk_1) = \{\rho(w, k_1, k_1)\}$

$\mathcal{L}(OX_3)(Sw, Sk_2) = \{\rho(w, k_2, k_2)\}$
$\mathcal{L}(OX_3)(Sw, Sk_3) = \{\rho(w, k_3, k_3)\}$

$\mathcal{L}(OX_3)(Sk_1, Sf) = \{\rho(k_1, k_1, f), \rho(k, k_2, f)\}$
$\mathcal{L}(OX_3)(Sk_1, Sv) = \{\rho(k_1, k_1, v), \rho(k_1, k_3, v)\}$
$\mathcal{L}(OX_3)(Sk_1, Sw) = \{\rho(k_1, k_2, w), \rho(k_1, k_3, w)\}$
$\mathcal{L}(OX_3)(Sk_1, Sk_1) = \{\rho(k_1, k_1, k_1)\}$
$\mathcal{L}(OX_3)(Sk_1, Sk_2) = \{\rho(k_1, k_2, k_2)\}$
$\mathcal{L}(OX_3)(Sk_1, Sk_3) = \{\rho(k_1, k_3, k_3)\}$

$\mathcal{L}(OX_3)(Sk_2, Sf) = \{\rho(k_2, k_1, f), \rho(k_2, k_2, f)\}$
$\mathcal{L}(OX_3)(Sk_2, Sv) = \{\rho(k_2, k, v), \rho(k, k_3, v)\}$
$\mathcal{L}(OX_3)(Sk_2, Sw) = \{\rho(k_2, k_2, w), \rho(k_2, k_3, w)\}$
$\mathcal{L}(OX_3)(Sk_2, Sk_1) = \{\rho(k_2, k_1, k_1)\}$
$\mathcal{L}(OX_3)(Sk_2, Sk_2) = \{\rho(k_2, k_2, k_2)\}$
$\mathcal{L}(OX_3)(Sk_2, Sk_3) = \{\rho(k_2, k_3, k_3)\}$

$\mathcal{L}(OX_3)(Sk_3, Sf) = \{\rho(k_3, k_1, f), \rho(k_3, k_2, f)\}$
$\mathcal{L}(OX_3)(Sk_3, Sv) = \{\rho(k_3, k_1, v), \rho(k_3, k_3, v)\}$
$\mathcal{L}(OX_3)(Sk_3, Sw) = \{\rho(k_3, k_2, w), \rho(k_3, k_3, w)\}$
$\mathcal{L}(OX_3)(Sk_3, Sk_1) = \{\rho(k_3, k_1, k_1)\}$
$\mathcal{L}(OX_3)(Sk_3, Sk_2) = \{\rho(k_3, k_2, k_2)\}$
$\mathcal{L}(OX_3)(Sk_3, Sk_3) = \{\rho(k_3, k_3, k_3)\}$

The powerset category $P_o(X_3)$: The objects in $P_o(X_3)$ are proper subchains of X_3. $vP_o(X_3) = \left\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\right\}$ and morphisms are the order-preserving mappings between the subchains of X_3. The morphisms in $P_o(X_3)$ are described below.

$$P_o(X_3)(\{1\},\{1\}) = \{(\begin{smallmatrix}1\\1\end{smallmatrix})\}$$
$$P_o(X_3)(\{1\},\{2\}) = \{(\begin{smallmatrix}1\\2\end{smallmatrix})\}$$
$$P_o(X_3)(\{1\},\{3\}) = \{(\begin{smallmatrix}1\\3\end{smallmatrix})\}$$
$$P_o(X_3)(\{1\},\{1,2\}) = \{(\begin{smallmatrix}1\\1\end{smallmatrix}), (\begin{smallmatrix}1\\2\end{smallmatrix})\}$$
$$P_o(X_3)(\{1\},\{1,3\}) = \{(\begin{smallmatrix}1\\1\end{smallmatrix}), (\begin{smallmatrix}1\\3\end{smallmatrix})\}$$
$$P_o(X_3)(\{1\},\{2,3\}) = \{(\begin{smallmatrix}1\\3\end{smallmatrix}), (\begin{smallmatrix}1\\2\end{smallmatrix})\}$$

$$P_o(X_3)(\{2\},\{1\}) = \{(\begin{smallmatrix}2\\1\end{smallmatrix})\}$$
$$P_o(X_3)(\{2\},\{2\}) = \{(\begin{smallmatrix}2\\2\end{smallmatrix})\}$$
$$P_o(X_3)(\{2\},\{3\}) = \{(\begin{smallmatrix}2\\3\end{smallmatrix})\}$$
$$P_o(X_3)(\{2\},\{1,2\}) = \{(\begin{smallmatrix}2\\1\end{smallmatrix}), (\begin{smallmatrix}2\\2\end{smallmatrix})\}$$
$$P_o(X_3)(\{2\},\{1,3\}) = \{(\begin{smallmatrix}2\\1\end{smallmatrix}), (\begin{smallmatrix}2\\3\end{smallmatrix})\}$$
$$P_o(X_3)(\{2\},\{2,3\}) = \{(\begin{smallmatrix}2\\2\end{smallmatrix}), (\begin{smallmatrix}2\\3\end{smallmatrix})\}$$

$$P_o(X_3)(\{3\},\{1\}) = \{(\begin{smallmatrix}3\\1\end{smallmatrix})\}$$
$$P_o(X_3)(\{3\},\{2\}) = \{(\begin{smallmatrix}3\\2\end{smallmatrix})\}$$
$$P_o(X_3)(\{3\},\{3\}) = \{(\begin{smallmatrix}3\\3\end{smallmatrix})\}$$
$$P_o(X_3)(\{3\},\{1,2\}) = \{(\begin{smallmatrix}3\\1\end{smallmatrix}), (\begin{smallmatrix}3\\2\end{smallmatrix})\}$$
$$P_o(X_3)(\{3\},\{1,3\}) = \{(\begin{smallmatrix}3\\1\end{smallmatrix}), (\begin{smallmatrix}3\\3\end{smallmatrix})\}$$
$$P_o(X_3)(\{3\},\{2,3\}) = \{(\begin{smallmatrix}3\\3\end{smallmatrix}), (\begin{smallmatrix}3\\2\end{smallmatrix})\}$$

$$P_o(X_3)(\{1,2\},\{1\}) = \{(\begin{smallmatrix}1&2\\1&1\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,2\},\{2\}) = \{(\begin{smallmatrix}1&2\\2&2\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,2\},\{3\}) = \{(\begin{smallmatrix}1&2\\3&3\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,2\},\{1,2\}) = \{(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}), (\begin{smallmatrix}1&2\\2&2\end{smallmatrix}), (\begin{smallmatrix}1&2\\1&2\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,2\},\{1,3\}) = \{(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}), (\begin{smallmatrix}1&2\\3&3\end{smallmatrix}), (\begin{smallmatrix}1&2\\1&3\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,2\},\{2,3\}) = \{(\begin{smallmatrix}1&2\\2&2\end{smallmatrix}), (\begin{smallmatrix}1&2\\3&3\end{smallmatrix}), (\begin{smallmatrix}1&2\\2&3\end{smallmatrix})\}$$

$$P_o(X_3)(\{1,3\},\{1\}) = \{(\begin{smallmatrix}1&3\\1&1\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,3\},\{2\}) = \{(\begin{smallmatrix}1&3\\2&2\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,3\},\{3\}) = \{(\begin{smallmatrix}1&3\\3&3\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,3\},\{1,2\}) = \{(\begin{smallmatrix}1&3\\1&1\end{smallmatrix}), (\begin{smallmatrix}1&3\\2&2\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,3\},\{1,3\}) = \{(\begin{smallmatrix}1&3\\1&1\end{smallmatrix}), (\begin{smallmatrix}1&3\\3&3\end{smallmatrix})\}$$
$$P_o(X_3)(\{1,3\},\{2,3\}) = \{(\begin{smallmatrix}1&3\\3&3\end{smallmatrix}), (\begin{smallmatrix}1&3\\2&2\end{smallmatrix})\}$$

$$P_o(X_3)(\{2,3\},\{1\}) = \{(\begin{smallmatrix}2&3\\1&1\end{smallmatrix})\}$$

$$
\begin{aligned}
P_o(X_3)(\{2,3\},\{2\}) &= \{\left(\begin{smallmatrix}2&3\\2&2\end{smallmatrix}\right)\} \\
P_o(X_3)(\{2,3\},\{3\}) &= \{\left(\begin{smallmatrix}2&3\\3&3\end{smallmatrix}\right)\} \\
P_o(X_3)(\{2,3\},\{1,2\}) &= \{\left(\begin{smallmatrix}2&3\\1&1\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\2&2\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\1&2\end{smallmatrix}\right)\} \\
P_o(X_3)(\{2,3\},\{1,3\}) &= \{\left(\begin{smallmatrix}2&3\\1&1\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\3&3\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\1&3\end{smallmatrix}\right)\} \\
P_o(X_3)(\{2,3\},\{2,3\}) &= \{\left(\begin{smallmatrix}2&3\\2&2\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\3&3\end{smallmatrix}\right),\left(\begin{smallmatrix}2&3\\2&3\end{smallmatrix}\right)\}.
\end{aligned}
$$

It can be seen that the categories $P_o(X_3)$ and $\mathcal{L}(OX_3)$ are isomorphic, and the following is the equivalent vertex mapping,.

$$k_1 \mapsto \{1\}, k_2 \mapsto \{2\}, k_3 \mapsto \{3\}, f \mapsto \{1,2\}, v \mapsto \{1,3\}, w \mapsto \{2,3\}.$$

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