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The Golden Spiral of Order k

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Abstract The golden spiral is a logarithmic spiral that grows outward by a factor of the golden ratio for every 90 degrees of rotation. We introduce the notion of the θ-type and b-type golden spirals of order k for any real number $k \geq 0$ as a generalization of the golden spiral. The θ-type golden spiral of order k is a logarithmic spiral that grows outward by a factor of ϕ_k, the golden ratio of order k, for every θ_k radians of rotation. The b-type golden spiral of order k is a logarithmic spiral that grows outward by a factor of ϕ_k for each quarter turn $\theta = \frac{\pi}{2}$ with b_k being the rate of increase of the spiral. The golden ratio of order k is the root of a quadratic polynomial and it is a general form of the golden ratio ϕ for any real number $k \geq 0$. It is also shown that $\theta_k = \frac{\pi}{2}$ [resp. $b_k = b = \frac{2}{\pi} \ln \phi$] if and only if $k = 0$.

1 Introduction

The authors in [1] introduced the notion of the *golden ratio of order* k [1, Definition 3.1], denoted ϕ_k, as a generalized form of the *golden ratio* ϕ for any real number $k \geq 0$. Thus, it is natural to introduce the notion of the *golden spiral of order* k as a generalized form of *golden spiral* since the definition of the golden spiral is directly related to the notion of the golden ratio. The main goal of this paper is to generalize the notion of the golden spiral to the golden spiral of order k for any real number $k \geq 0$ in two different ways; namely the θ-type and b-type (Definitions 2.1 and 2.2, respectively). The golden spiral is a *logarithmic spiral* that *grows outward* by a factor of the golden ratio for every 90 degrees of rotation. The θ-type [resp. b-type] golden spiral of order k is a special type of logarithmic spiral that *grows outward by a factor of* ϕ_k for each $\theta = \theta_k$ radians of rotation [resp. for each *quarter turn* $\theta = \frac{\pi}{2}$ when b_k is the *rate of increase of the spiral*] (see the next section for the definitions of θ_k and b_k).

• The authors assume that the reader is familiar with the results in number theory that we use in this article. Actually, the internet search will provide all necessary sources that are required and mentioned in this paper without any direct reference.

In the rest of this section, we recall the definitions of the golden ratio and the golden ratio of order k, respectively, and then recall some results related to the golden ratio and the golden ratio of order k (Remark 1.1) for the sake of completeness. In the next section, we will recall the notion of the logarithmic spiral and then define the two types of golden spirals of order k (as a special case of the logarithmic spiral) and write some properties of θ_k and b_k.

We now start with the definition of the golden ratio, denoted by ϕ.

• Two quantities, real numbers $a > b > 0$, are in the golden ratio if their ratio is the same as the *ratio of their sum to the larger of the two quantities*. That is,

$$\frac{a+b}{a} = \frac{a}{b} = \phi,$$

where *the Greek letter "phi"* (ϕ) denotes the golden ratio and from the above identity, we get *the quadratic equation*

$$x^2 - x - 1 = 0$$

by assuming $x = \frac{a}{b}$.

Thus,

$$\phi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

are *two distinct roots* of the above quadratic equation and they are *algebraic numbers* (i.e. a number is algebraic if it is the root of a *polynomial equation with integer coefficients*).

Clearly, the *constant* ϕ satisfies the quadratic equation $\phi^2 = \phi + 1$, and is an *irrational number* with a value of

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618033988749\cdots$$

and

$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.618033\cdots.$$

Because ϕ is *a ratio between positive quantities*, ϕ is necessarily the *positive root*. The *negative root* is in fact the *negative inverse* $-\frac{1}{\phi}$, which *shares many properties* with the golden ratio and we have

$$\psi = \frac{1-\sqrt{5}}{2} = 1 - \phi = -\frac{1}{\phi} \approx -0.6180339887\cdots.$$

We now recall the definition of the golden ratio of order k ([1, Definition 3.1]).

• Let $a > b > 0$ be two real numbers and $k \geq 0$ a fixed real number. We say that a/b is a golden ratio of order k, denoted ϕ_k, if it satisfies the following identity

$$\frac{a}{b} = \frac{(2k+1)(k+1)a - (k^2+k-1)b}{(k+1)^2 a}.$$

From the above identity, we get the quadratic equation

$$(k+1)^2 x^2 - (2k+1)(k+1)x + (k^2+k-1) = 0$$

by assuming $x = \frac{a}{b}$.

Thus,

$$\phi_k = \frac{(2k+1)+\sqrt{5}}{2(k+1)} \text{ and } \psi_k = \frac{(2k+1)-\sqrt{5}}{2(k+1)}$$

are *two distinct roots* of the above quadratic equation and they are *algebraic numbers* when $k \geq 0$ is an integer (i.e. a number is algebraic if it is the root of a polynomial equation with integer coefficients). Obviously, $\phi_0 = \phi$ and $\psi_0 = \psi$ (see the definition of ϕ and ψ above). Also, $\phi_k = \frac{a}{b}$ since a/b is larger than 1 and $\psi_k \neq \frac{a}{b}$ since $\psi_k < 1$ for all $k \geq 0$ from the fact that $\psi \approx -0.6180339887\cdots < 0$ (see [1, Theorem 3.6(c)] or Remark 1.1(b) below).

Finally, we end this section by recalling some results from [1] for the sake of reference and completeness.

Remark 1.1. The following are true:

(a) For any real $k \geq 0$, $\phi_k = \frac{1}{k+1}(k+\phi)$ ([1, Remark 3.2]).

(b) For any real $k \geq 0$, $\psi_k = \frac{1}{k+1}(k+\psi)$ ([1, Remark 3.2]).

(c) Let $k \geq 0$ be a real number. Then $\phi_k = \phi$ if and only if $k = 0$ ([1, Theorem 3.4]).

(d) If $k \geq 0$ is a real number, then $\phi_k \neq 1$ ([1, Theorem 3.4]).

(e) If $k \neq 0$ is a positive real number, then $1 < \phi_k < \phi$ ([1, Theorem 3.4]).

(f) $\lim_{k\to\infty}\phi_k = 1$ ([1, Theorem 3.4]).

(g) Let $k \geq 0$ be a real number. Then $\phi_k : \mathbb{R}^+ = [0,\infty) \to (1,\phi]$ is a bijection ([1, Proposition 3.5]).

2 Two Special Types of the Logarithmic Spirals

In this section, we first recall the *polar equation* of the logarithmic spiral in general and then introduce the notion of two types of the golden spirals of order k for any real number $k \geq 0$. One for the case when the *angle of rotation* $\theta = \theta_k$ and the *rate of increase of the spiral* $b = \frac{2}{\pi}\ln\phi$; and the other one when $\theta = \frac{\pi}{2}$ with b_k being the rate of increase of the spiral (Definitions 2.1 and 2.2, respectively).

A logarithmic spiral is a curve whose equation in Polar Coordinates (r,θ) is given by

$$r = ae^{b\theta},$$

where r is the distance from the Origin, θ is the angle from the x-axis, e is the base of natural logarithm, $a > 0$ (the initial radius of the spiral), and $b \neq 0$ (the rate of increase of the spiral) are arbitrary constants. But, in our work (related to the golden spirals), we will study the cases when $a = 1$ and $b > 0$. The logarithmic spiral is also known as the Growth Spiral or Equiangular Spiral. Clearly, it can also be expressed in Cartesian coordinates as follows:

$$x = r\cos\theta = a\cos\theta e^{b\theta}$$

$$y = r\sin\theta = a\sin\theta e^{b\theta}.$$

In addition, any *Radius from the origin* meets the spiral at distances which are in *Geometric Progression*. The polar equation for a golden spiral is the same as for other logarithmic spirals, but with a special value of the growth factor b. That is, $r = \phi$ when $\theta = \frac{\pi}{2}$ in the equation $r = e^{b\theta}$, which implies that

$$b = \frac{2}{\pi}\ln\phi.$$

• Each of the following two definitions provides a generalization of the golden spiral. See Remark 1.1 for the value of ϕ_k with respect to k and ϕ.

Definition 2.1. Let $k \geq 0$ be a real number, ϕ_k the golden ratio of order k, ϕ the golden ratio, and $b = \frac{2}{\pi}\ln\phi$. The θ-type golden spiral of order k is a logarithmic spiral $r = e^{b\theta}$ that grows outward by a factor of ϕ_k for every θ_k radians of rotation, that is,

$$\phi_k = e^{b\theta_k},$$

where

$$\theta_k = \frac{1}{b}[(\ln(k+\phi) - \ln(k+1)].$$

Definition 2.2. Let $k \geq 0$ be a real number, ϕ_k the golden ratio of order k, and ϕ the golden ratio. The b-type golden spiral of order k is a logarithmic spiral $r = e^{b_k\theta}$ that grows outward by a factor of ϕ_k for every quarter turn $\theta = \frac{\pi}{2}$ radians of rotation, that is,

$$\phi_k = e^{b_k\frac{\pi}{2}},$$

where

$$b_k = \frac{2}{\pi}[(\ln(k+\phi) - \ln(k+1)].$$

We now write the relationship between θ_k and b_k.

Remark 2.3. Clearly, from the two above definitions, we have

$$e^{b\theta_k} = \phi_k = e^{b_k \frac{\pi}{2}},$$

which implies that $b\theta_k = b_k \frac{\pi}{2}$.

We end the paper with some special values of θ_k and b_k.

Proposition 2.4. *The following are true:*

(a) $\theta_k = \frac{\pi}{2}$ *if and only if* $k = 0$.

(b) $b_k = b$ *if and only if* $k = 0$.

(c) $\lim_{k\to\infty} \theta_k = 0$.

(d) $\lim_{k\to\infty} b_k = 0$.

Proof. We just give a proof for Parts (a) and (c) since the proof of (b) and (d) is similar to the proof of (a) and (c), respectively. Moreover, the sufficiency of (a) and (b) is clear. (a):

$$\frac{\pi}{2} = \theta_k = \frac{1}{b}[\ln(k+\phi) - \ln(k+1)]$$

implies

$$\ln \phi = b\frac{\pi}{2} = \ln \frac{k+\phi}{k+1}$$

implies

$$k\phi + \phi = k + \phi$$

implies

$k(\phi - 1) = 0$, which implies $k = 0$ since $\phi \neq 1$.

(c): Since, by Remark 1.1(f),

$$\lim_{k\to\infty} \phi_k = 1$$

and $b\theta_k = \ln \phi_k$, then the result follows from the fact that $\ln 1 = 0$. □

References

[1] E. Mehdi-Nezhad, A. M. Rahimi, *A note on Fibonacci numbers and the golden ratio of order k*, Palestine Journal of Mathematics, to appear.

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