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Fixed Point Theorems in M-metric Spaces via Rational Expression and Application

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Abstract In this article, rational expressions are used to prove some fixed-point theorems of Meir-Keeler type contraction in the context of M-metric spaces. The article's findings enhance and expand on several recent results in fixed-point theory. Concluding remarks and the primary theorem ensure the extension. At the end, the primary theorem's applicability in establishing theorems for certain integral equations and integral-type contractive conditions is established.

1 Introduction

One of the most intriguing study areas that exists at the nexus of three different branches of mathematics: topology, applied mathematics, and nonlinear functional analysis, is metric fixed point theory. It was widely considered that Banach [1] established the first metric fixed-point theorem in 1922. However, it was recognized that the concept of the fixed-point theorem was employed before Banach's publication. Furthermore, in certain research publications by Liouville, Picard, Poincare, and others, fixed point results were utilized to demonstrate the existence and uniqueness of solutions to initial value problems. Regardless, the main objective of Banach's theorem is to demonstrate the existence and uniqueness of the fixed point. Plenty of novel discoveries in the context of metric fixed point theory have been established during the three decades preceding this one.

A fascinating result that extends the well-known Banach contraction principle was made in 1969 by Meir and Keeler [2]. Based on results established by Meir-Keeler; numerous authors have expanded on this contraction, see [3]-[8]. In order to achieve many impressive results, a huge amount of mathematical work was carried out to demonstrate the applicability of fixed point theory [9]-[14].

Dass and Gupta [15] were the first to use contractive conditions with rational expressions to prove fixed point theorems. Jaggi [16] prove fixed point theorems with the help of rational expression. Further, Gupta [17] and Samet [18] generalized the results due to [15]. Main fixed point result of Najeh et al. [19] generalized the finding of Gupta and Saxena [17]. Koti et al. [20] findings significantly enhance those of Najeh et al. [19] and Samet [18]. Recently, a number of authors have generalized Das-Gupta and Jaggi's findings by taking into different types of mathematical metric space concepts. See, e.g. [21]-[27] and references therein

In 2014, Asadi et al. [28] broadened the definition of partial metric space and offered some instances to demonstrate that their concepts are a valid generalization of partial metric space. This new concept was termed as M-metric space and fixed point theorems for the Banach contraction principle and Meir-Keeler contraction have been established. Later, in 2016 Monfared [29]-[30] applied covered Ciric-contraction to established Matkowski's fixed point theorem and

Boyd-Wong fixed point theorem for M-metrics. In 2018, Ozgur [31] introduced the rectangular M-metric space and various fixed point theorem proven therein, which extending both rectangular and M-metric space. Recently Asim et al. [32] proved some fixed point theorem for Meir-Keeler type contraction in rectangular M-metric spaces. In the literature of fixed point theory numerous novel developments in M-metric space have been observed, see [33]-[37].

2 Preliminaries

Definition 2.1. [38] Consider a non-empty set E and a mapping $\sigma : E \times E \to \mathbb{R}^+$ satisfies the following axioms:

(i) $\sigma(r, r) = \sigma(s, s) = \sigma(r, s)$ iff $r = s$;

(ii) $\sigma(r, r) \leq \sigma(r, s), \forall r, s \in E$;

(iii) $\sigma(r, s) = \sigma(s, r), \forall r, s \in E$;

(iv) $\sigma(r, s) \leq \rho(r, \kappa) + \sigma(\kappa, s) - \sigma(\kappa, \kappa), \forall r, s, \kappa \in E$.

Then, (E, σ) is referred as partial metric space on E.

Note: The following notation will be useful for the discussion that follows:

(i) $\Delta_{rs} = \min\{\Delta(r, r), \Delta(s, s)\}$,

(ii) $\Delta'_{rs} = \max\{\Delta(r, r), \Delta(s, s)\}$.

Definition 2.2. [28] Consider a non-empty set E and a mapping $\Delta : E \times E \to \mathbb{R}^+$ satisfies the following axioms:

(i) $\Delta(r, r) = \Delta(s, s) = \Delta(r, s)$ iff $r = s$;

(ii) $\Delta_{rs} \leq \Delta(r, s)$, $\forall r, s \in E$;

(iii) $\Delta(r, s) = \Delta(s, r)$, $\forall r, s \in E$;

(iv) $(\Delta(r, s) - \Delta_{rs}) \leq (\Delta(r, \kappa) - \Delta_{r\kappa}) + (\Delta(\kappa, s) - \Delta_{\kappa s})$, $\forall r, s, \kappa \in E$.

Thus, pair (E, Δ) is called an M-metric space.

Remark 2.3. [28] For every $r, s, \kappa \in E$,

(i) $0 \leq \Delta'_{rs} + \Delta_{rs} = \Delta(r, r) + \Delta(r, s)$;

(ii) $0 \leq \Delta'_{rs} - \Delta_{rs} = |\Delta(r, r) - \Delta(s, s)|$;

(iii) $\Delta'_{rs} - \Delta_{rs} \leq (\Delta'_{r\kappa} - \Delta_{r\kappa}) + (\Delta'_{\kappa s} - \Delta_{\kappa s})$.

The relationship between the M-metric and the conventional metric is shown below.

Example 2.4. [28] Let Δ be an M-metric. Put

(i) $\Delta^w(r, s) = \Delta(r, s) - 2\Delta_{rs} + \Delta'_{rs}$,

(ii) $\Delta^s(r, s) = \Delta(r, s) - \Delta_{rs}$ when $r \neq s$ and $\Delta^s(r, s) = 0$ if $r = s$.

Thus, the pair Δ^w and Δ^s follows conventional metric axioms.

Remark 2.5. [28] For every $r, s \in E$,

(i) $\Delta(r, s) - \Delta'_{r\varsigma} \leq \Delta^w(r, s) \leq \Delta(r, s) + \Delta'_{rs}$,

(ii) $\Delta(r, s) - \Delta'_{rs} \leq \Delta^s(r, s) \leq \Delta(r, s)$.

Example 2.6. [28] Let (E, d) be a metric space. Then $\Delta(r, s) = ad(r, s) + b$ where $a, b > 0$, is an M-metric, because we can put $\pi(t) = at + b$.

Lemma 2.7. [28] Every partial metric is an M-metric.

Proof. Consider a partial metric $\Delta : E \to E$. We are only obligated to consider the subsequent conditions:

(i) $\Delta(r,r) = \Delta(s,s) = \Delta(\kappa,\kappa)$,

(ii) $\Delta(r,r) < \Delta(s,s) < \Delta(\kappa,\kappa)$,

(iii) $\Delta(r,r) = \Delta(s,s) < \Delta(\kappa,\kappa)$,

(iv) $\Delta(r,r) = \Delta(s,s) > \Delta(\kappa,\kappa)$,

(v) $\Delta(r,r) < \Delta(s,s) = \Delta(\kappa,\kappa)$,

(vi) $\Delta(r,r) > \Delta(s,s) = \Delta(\kappa,\kappa)$.

For example, to prove (iv), we have
$\Delta(r,s) \leq \Gamma(r,\kappa) + \Delta(\kappa,s) - \Delta(\kappa,\kappa)$,
$\Delta(r,s) \leq \Gamma(r,\kappa) + \Delta(\kappa,s) - \Delta(s,s)$,
$\Delta(r,s) - \Delta(r,r) \leq \Delta(r,\kappa) - \Delta(r,r) + \Delta(\kappa,s) - \Delta(s,s)$,
$\Delta(r,s) - \Delta_{rs} \leq \Delta(r,\kappa) - \Delta_{r\kappa} + \Delta(\kappa,s) - \Delta_{\kappa s}$. □

Example 2.8. Let $E = \{1,2\}$, then

$$\Delta(1,1) = 1, \Delta(2,2) = 8, \Delta(1,2) = \Delta(2,1) = 7.$$

So, this implies that Δ is not partial metric but M-metric.

Example 2.9. Suppose that $E = \mathbb{R}_+$ and $\Delta : E \times E \to \mathbb{R}_+$ defined as

$$\Delta(r,s) = \frac{r+s}{2}$$

$\forall\, r,s \in E$. This implies that Δ is not partial metric but M-metric.

Example 2.10. Let $E = [0,\infty)$ and function $\Delta : E \times E \to \mathbb{R}_+$ given by

$$\Delta(r,s) = \frac{r^2+s^2}{2}$$

$\forall\, r,s \in E$. This implies that Δ is not partial metric but M-metric.

Above examples show that converse of Lemma 2.7 is not true.

3 Topology concerning M-metric space

The set

$$\{B_\Delta(r,\epsilon) : r \in E, \epsilon > 0\},$$

where

$$B_\Delta(r,\epsilon) = \{s \in E : \Delta(r,s) < \Delta_{pq} + \epsilon\},$$

$\forall r \in E$, $\epsilon > 0$, is a base for τ_Δ.

Definition 3.1. [28] Consider a sequence $\{r_n\}$ in a M-metric space (E,Δ), then

(a)

$$\{r_n\} \to r \Leftrightarrow \lim_{n\to\infty} (\Delta(r_n,r) - \Delta_{r_n r}) = 0; \tag{3.1}$$

(b) sequence $\{r_n\}$ is said to be M-Cauchy sequence if

$$\lim_{n,\Delta\to\infty} (\Delta(r_n,r_\Delta) - \Delta_{r_n r_\Delta}) \text{ and } \lim_{n,\Delta\to\infty} (\Delta'_{r_n r_\Delta} - \Delta_{r_n r_\Delta}) \tag{3.2}$$

both exists finitely;

(c) if for every M-Cauchy sequence $\{r_n\} \to r$ such that

$$\lim_{n\to\infty} \Delta(r_n, r) - \Delta_{r_n r} = 0 \text{ and } \lim_{n\to\infty} \Delta'_{r_n r} - \Delta_{r_n r} = 0,$$

then, pair (E, Δ) is called complete M-metric space.

Lemma 3.2. [28] Consider two sequences $\{r_n\}$ and $\{s_n\}$ in a M-metric space (E, Δ). If $\lim_{n\to\infty} r_n = r$ and $\lim_{n\to\infty} s_n = s$, then

$$\lim_{n\to\infty} (\Delta(r_n, s_n) - \Delta_{r_n s_n}) = \Delta(r, s) - \Delta_{rs}.$$

Following lemma is observed from Lemma 3.2.

Lemma 3.3. [28] Consider a sequence $\{r_n\}$ in a M-metric space (E, Δ). If $\lim_{n\to\infty} r_n = r$, then

$$\lim_{n\to\infty} (\Delta(r_n, s) - \Delta_{r_n s}) = \Delta(r, s) - \Delta_{rs}.$$

Lemma 3.4. [28] Consider a sequence $\{r_n\}$ in a M-metric space (E, Δ). If $r_n \to r$ and $r_n \to s$, then $\Delta(r, s) = \Delta_{pq}$. Again if, $\Delta(r, r) = \Delta(s, s)$ then $r = s$.

Lemma 3.5. [28] Let $\{r_n\}$ be a sequence in an M-metric space (E, Δ) if there exists $r \in [0, 1)$ and $\forall n \in \mathbb{N}$ such that

$$\Delta(r_{n+1}, r_n) \leq r\Delta(r_n, r_{n-1}), . \tag{3.3}$$

Then,

(a) $\Delta(r_n, r_{n-1}) = 0$ as $n \to \infty$,

(b) $\Delta(r_n, r_n) = 0$ as $n \to \infty$,

(c) $\Delta_{r_n r_n} = 0$ as $n \to \infty$,

(d) $\{r_n\}$ is an M-Cauchy sequence.

Let Φ be the collection of all continuous monotone increasing functions $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\pi(t) < kt, \forall t > 0$, $k \in (0, 1)$ and $\pi(0) = 0$.

4 Main Results

We will start this part of the article by establishing the fixed point theorem below.

Theorem 4.1. Let (E, Δ) be a complete M-metric space. If a continuous self-map T on the set E, fulfills with following axiom for all $\epsilon > 0$, there exists $\delta > 0$ as well as

$$\epsilon \leq \pi\Big(\max\Big\{\frac{(1 + \Delta(r, Tr))\Delta(s, Ts)}{1 + \Delta(r, s)}, \frac{\Delta(r, Tr)\Delta(s, Ts)}{\Delta(r, s)}, \Delta(r, s)\Big\}\Big) < \epsilon + \delta(\epsilon) \Rightarrow \Delta(Tr, Ts) < \epsilon, \tag{4.1}$$

for all $r, s \in E$, $r \neq s$ *or* $s \neq Ts$, where $\pi \in \Phi$ then, $\exists$ a unique point $s^* \in E$ such that $s^* = Ts^*$. Also, for all $r \in E$, the sequence $\{T^n(r)\} \to s^* \in E$.

Proof. From (4.1), we observe that

$$\Delta(Tr, Ts) < \mathbb{J}(r, s), \tag{4.2}$$

where

$$\mathbb{J}(r, s) = \pi\Big(\max\Big\{\frac{(1 + \Delta(r, Tr))\Delta(s, Ts)}{1 + \Delta(r, s)}, \frac{\Delta(r, Tr)\Delta(s, Ts)}{\Delta(r, s)}, \Delta(r, s)\Big\}\Big).$$

Let $r_0 \in E$ and define $r_n = Tr_{n-1}$, we have

$$\begin{aligned}\mathbb{J}(r_{n-1}, r_n) &= \pi\Big(\max\Big\{ \frac{(1+\Delta(r_{n-1}, Tr_{n-1}))\Delta(r_n, Tr_n)}{1+\Delta(r_{n-1}, r_n)}, \\ &\qquad \frac{\Delta(r_{n-1}, Tr_{n-1})\Delta(r_n, Tr_n)}{\Delta(r_{n-1}, r_n)}, \Delta(r_{n-1}, r_n)\Big\}\Big) \\ &= \pi\Big(\max\Big\{ \frac{(1+\Delta(r_{n-1}, r_n))\Delta(r_n, r_{n+1})}{1+\Delta(r_{n-1}, r_n)}, \\ &\qquad \frac{\Delta(r_{n-1}, r_n)\Delta(r_n, r_{n+1})}{\Delta(r_{n-1}, r_n)}, \Delta(r_{n-1}, r_n)\Big\}\Big) \\ &= \pi\Big(\max\Big\{ \Delta(r_n, r_{n+1}), \Delta(r_n, r_{n+1}), \Delta(r_{n-1}, r_n)\Big\}\Big) \\ &= \pi\Big(\max\Big\{ \Delta(r_n, r_{n+1}), \Delta(r_{n-1}, r_n)\Big\}\Big).\end{aligned}$$

If

$$\max\Big\{\Delta(r_n, r_{n+1}), \Delta(r_{n-1}, r_n)\Big\} = \Delta(r_n, r_{n+1}),$$

then,

$$\mathbb{J}(r_{n-1}, r_n) = \pi(\Delta(r_n, r_{n+1})).$$

Hence, (4.2) implies that

$$\begin{aligned}\Delta(r_n, r_{n+1}) &= \Delta(Tr_{n-1}, Tr_n) \\ &< \mathbb{J}(r_{n-1}, r_n) \\ &= \pi(\Delta(r_n, r_{n+1})) \\ &< k\Delta(r_n, r_{n+1}),\end{aligned}$$

this leads to a contradiction. Thus,

$$\mathbb{J}(r_{n-1}, r_n) = \pi(\Delta(r_{n-1}, r_n)),$$

implies that

$$\Delta(r_n, r_{n+1}) < km(r_{n-1}, r_n). \tag{4.3}$$

Now, by Lemma 3.5, we can say that $\{r_n\}$ is a Cauchy sequence. Since E is complete, we have

$$Tr_{n-1} = r_n \to s^* \in E.$$

Since T is continuous mapping, then $Tr_n \to Ts^*$ in E.
From Lemma 3.2, we have

$$\Delta(s^*, Ts^*) = \Delta_{s^*Ts^*},$$

and

$$\begin{aligned}0 &= \lim_{n\to\infty} (\Delta(r_n, Tr_n) - \Delta_{r_nTr_n}) \\ &= \Delta(s^*, s^*) - \Delta_{s^*Ts^*} \\ &= \Delta(Ts^*, s^*) - \Delta_{s^*Ts^*}.\end{aligned}$$

By Lemma 3.4 and

$$\Delta(s^*, Ts^*) = \Delta_{s^*Ts^*} = \Delta(Ts^*, Ts^*) = \Delta(s^*, s^*),$$

so that $s^* = Ts^*$. Now, if $s^* = Ts^*$, then

$$\begin{aligned}\Delta(s^*, s^*) &= \Delta(Ts^*, Ts^*)\\ &< \mathbb{J}(s^*, s^*)\\ &= \pi\Big(\max\Big\{\frac{(1+\Delta(s^*, Ts^*))(\Delta(s^*, Ts^*))}{(1+\Delta(s^*, s^*))} + \frac{(\Delta(s^*, Ts^*))(\Delta(s^*, Ts^*))}{(\Delta(s^*, s^*))}, \Delta(s^*, s^*)\Big\}\Big)\\ &= \pi\Big(\max\Big\{\frac{(1+\Delta(s^*, s^*))(\Delta(s^*, s^*))}{(1+\Delta(s^*, s^*))} + \frac{(\Delta(s^*, s^*))(\Delta(s^*, s^*))}{(\Delta(s^*, s^*))}, \Delta(s^*, s^*)\Big\}\Big)\\ &= k\Delta(s^*, s^*),\end{aligned}$$

this implies that

$$\Delta(s^*, s^*) = 0. \tag{4.4}$$

For uniqueness, let $s(\neq s^*) \in E$ such that $Ts = s$, then

$$\begin{aligned}\Delta(s^*, s) &= \Delta(Ts^*, Ts)\\ &< \mathbb{J}(s^*, s)\\ &= \pi\Big(\max\Big\{\frac{(1+\Delta(s^*, Ts^*))(\Delta(s, Ts))}{(1+\Delta(s^*, s))} + \frac{(\Delta(s^*, Ts^*))(\Delta(s, Ts))}{(\Delta(s^*, s))}, \Delta(s^*, s)\Big\}\Big)\\ &= \pi\Big(\max\Big\{\frac{(1+\Delta(s^*, s^*))(\Delta(s, s))}{(1+\Delta(s^*, s))} + \frac{(\Delta(s^*, s^*))(\Delta(s, s))}{(\Delta(s^*, s))}, \Delta(s^*, s)\Big\}\Big)\\ &< k\Delta(s^*, s)\end{aligned}$$

which is contradiction, hence $s^* = s$. □

Example 4.2. Consider $E = [0, 1]$ and a mapping $T : E \to E$ such that $Tr = \frac{r}{2}$, where $r \in E$, and mapping $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $\pi(t) = \frac{t}{\sqrt{2}}$ for all $t \in \mathbb{R}^+$. Now, we verify the condition (4.2). Assume that if $r < s$, then

$$\frac{r+s}{4} = \frac{1}{2}(Tr + Ts) = \Delta(Tr, Ts) < \pi(\mathbb{J}(r, s)) = \frac{1}{\sqrt{2}}(\frac{r+s}{2})$$

It results in the conclusion that condition (4.2) is true and $r = 0$ is a unique fixed point of T with $k \in (\frac{1}{\sqrt{2}}, 1)$.

Remark 4.3. The above main theorem is M-metric generalization of primary results of Koti $et\ al.$ [20], Samet $et\ al.$ [18] and Najeh $et\ al.$ [19].

Remark 4.4. Samet $et\ al.$ [18] generalized main theorem of Das and Gupta [15], Najeh $et\ al.$ [19] and also generalized Gupta and Saxena [17]. The main theorem stated above generalize results in [17] and [15] in the settings of M-metric space.

Remark 4.5. The above main theorem is a generalized version of Banach contraction principle and Asadi et al. [28] in the setting of M-metric spaces.

Corollary 4.6. Let $T : E \to E$ be a continuous mapping and suppose, the following axiom satisfies that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \pi\Big(\frac{(1+\Delta(r, Ts))\Delta(s, Ts)}{1+\Delta(r, s)} + \frac{\Delta(r, Tr)\Delta(s, Ts)}{\Delta(r, s)} + \Delta(r, s)\Big) < \epsilon + \delta(\epsilon) \Rightarrow \Delta(Tr, Ts) < \epsilon,$$

for all $r, s \in E,\ \ r \neq s\ \ or\ \ s \neq Ts$, where $\pi \in \Phi$, then there exists unique $s^* \in E$ such that $s^* = Ts^*$. Also, for all $r \in E$, the sequence $\{T^n(r)\} \to r^* \in E$.

Corollary 4.7. Let $T: E \to E$ be a continuous mapping and suppose that for all $r, s \in E,\ r \neq s\ or\ s \neq Ts$, T satisfies that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \alpha\Big(\frac{(1+\Delta(r,Tr))\Delta(s,Ts)}{1+\Delta(r,s)} + \frac{\Delta(r,Tr)\Delta(s,Ts)}{\Delta(r,s)} + \Delta(r,s)\Big) < \epsilon + \delta(\epsilon) \Rightarrow \Delta(Tr,Ts) < \epsilon,$$

where $\alpha \in (0, \frac{1}{3})$, then there exists unique $s^* \in E$ such that $s^* = Ts^*$. Also, for all $r \in E$, the sequence $\{T^n(r)\} \to s^* \in E$.

Corollary 4.8. Let $T: E \to E$ be a continuous mapping and suppose, the following axiom satisfies that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \pi\Big(\max\Big\{\frac{(1+\Delta(r,Ts))\Delta(s,Ts)}{1+\Delta(r,s)}, \Delta(r,s)\Big\}\Big) < \epsilon + \delta(\epsilon) \Rightarrow \Delta(Tr,Ts) < \epsilon,$$

for all $r, s \in E,\ r \neq s\ or\ s \neq Ts$, where $\pi \in \Phi$, then there exists unique $s^* \in E$ such that $s^* = Ts^*$. Also, for all $r \in E$, the sequence $\{T^n(r)\} \to s^* \in E$.

Corollary 4.9. Let $T: E \to E$ be a continuous mapping and suppose, the following axiom satisfies that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \pi\Big(\frac{(1+\Delta(r,Ts))\Delta(s,Ts)}{1+\Delta(r,s)} + \Delta(r,s)\Big) < \epsilon + \delta(\epsilon) \Rightarrow \Delta(Tr,Ts) < \epsilon,$$

for all $r, s \in E,\ r \neq s\ or\ s \neq Ts$, where $\pi \in \Phi$, then there exists unique $s^* \in E$ such that $s^* = Ts^*$. Also, for all $r \in E$, the sequence $\{T^n(r)\} \to s^* \in E$.

Corollary 4.10. Let $T: E \to E$ be a continuous mapping and suppose that for all $r, s \in E,\ r \neq s\ or\ s \neq Ts$, T satisfies for all $\epsilon > 0$, there exists $\delta > 0$ as well as

$$\epsilon \leq \alpha\Big(\frac{(1+\Delta(r,Ts))\Delta(s,Ts)}{1+\Delta(r,s)} + \Delta(r,s)\Big) < \epsilon + \delta(\epsilon) \Rightarrow \Delta(Tr,Ts) < \epsilon,$$

where $\alpha \in (0, \frac{1}{2})$, then there exists unique $s^* \in E$ such that $s^* = Ts^*$. Also, for all $r \in E$, the sequence $\{T^n(r)\} \to s^* \in E$.

Theorem 4.11. Let (E, Δ) be a complete M-metric space and $\Omega: \mathbb{R}_+ \to \mathbb{R}_+$ be a function meets the following axioms:

(i) for $t > 0 \Rightarrow \Omega(t) > 0$ and $\Omega(0) = 0$

(ii) for all $\alpha \leq \beta, \Omega(\alpha) \leq \Omega(\beta)$ and right continuous,

(iii) for $\epsilon > 0$, there exists $\delta > 0$ as well as

$$\epsilon \leq \Omega(\mathbb{J}(r,s)) \leq \epsilon + \delta \Rightarrow \Omega(\Delta(Tr,Ts)) < \epsilon \tag{4.5}$$

for all $r, s \in E,\ r \neq s$, where

$$\mathbb{J}(r,s) = \max\Big\{\frac{(1+\Delta(r,Tr))\Delta(s,Ts)}{1+\Delta(r,s)}, \frac{\Delta(r,Tr)\Delta(s,Ts)}{\Delta(r,s)}, \Delta(r,s)\Big\}$$

and $\pi \in \Phi$. Then equation (4.1) is meet.

Proof. Fix $\epsilon > 0$, so $\Omega(\epsilon) > 0$. Hence by (4.5), there exists $\delta > 0$ such that for all $r, s,\ r \neq s$,

$$\Omega(\epsilon) \leq \Omega(\mathbb{J}(r,s)) < \Omega(\epsilon) + \delta \Rightarrow \Omega(\Delta(Tr,Ts)) < \Omega(\epsilon).$$

According to right continuity of Ω, there exists $\delta_1 > 0$ such that

$$\Omega(\epsilon + \delta_1) < \Omega(\epsilon) + \delta,$$

fix $r, s \in E$ with $r \neq s$ such that

$$\epsilon \leq \mathbb{J}(r,s) < \epsilon + \delta.$$

Since, Ω is a non-decreasing mapping, we have

$$\Omega(\epsilon) \leq \Omega(\mathbb{J}(r,s)) < \Omega(\epsilon + \delta_1) < \Omega(\epsilon) + \delta,$$

and so

$$\Omega(\Delta(Tr, Ts)) < \Omega(\epsilon)$$

which implies that

$$\Delta(Tr, Ts) < \epsilon.$$

□

Example 4.12. Consider a set $E = [0, 1]$ and mapping $T : E \to E$ such that $Tr = \frac{r}{2}$, where $r \in E$, and a mapping $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $\pi(t) = \frac{t}{\sqrt{2}}$ and $\Omega(t) = \frac{t}{2}$ for all $t \in \mathbb{R}^+$. Assume that if $r < s$, then

$$\frac{r+s}{8} = \frac{1}{4}(Tr + Ts) = \Omega(\Delta(Tr, Ts)) < \Omega(\mathbb{J}(r,s)) = \frac{1}{2\sqrt{2}}(\frac{r+s}{2})$$

It results in the conclusion that condition (4.2) is true and $r = 0$ is a unique fixed point of T with $k \in (\frac{1}{\sqrt{2}}, 1)$.

Corollary 4.13. Consider a M-metric space (E, Δ) and $\aleph : \mathbb{R}_+ \to \mathbb{R}_+$ be a locally integrable function such that

$$(i) \quad t > 0 \Rightarrow \int_0^t \aleph(s)ds > 0$$

$$(ii) \quad \text{for every } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$

$$\epsilon \leq \int_0^{\mathbb{J}(r,s)} \aleph(s)ds < \epsilon + \delta \Rightarrow \int_0^{\Delta(Tr,Ts)} \aleph(s)ds < \epsilon$$

for all $r, s \in E$ with $r \neq s$. Then equation (4.1) is satisfied.

Corollary 4.14. Let $\mathbb{D}$ be the collection of functions $O : [0, \infty) \to [0, \infty)$ satisfying

(i) O is non-decreasing and continuous,

(ii) for $t > 0, O(t) > 0$ and $O(0) = 0$.

Consider a M-metric space (E, Δ) and for each $\epsilon > 0$, there exists $\delta(\epsilon)$ such that

$$\epsilon \leq O(\mathbb{J}(r,s)) < \epsilon + \delta(\delta) \Rightarrow O(\Delta(Tr, Ts)) < \epsilon$$

for all $r, s \in E$ with $r \neq s$, where $O \in \mathbb{D}$. Then equation (4.1) is satisfied.

Corollary 4.15. Consider a M-metric space (E, Δ) and $\Xi : \mathbb{R}^+ \to \mathbb{R}^+$be a locally integrable function such that

$$\int_0^t \Xi(s)ds > 0, \text{ for all } t > 0.$$

Suppose that T meets all of the following criteria for all $r, s \in E$ and $r \neq s$

$$\int_0^{\Delta(Tr,Ts)} \Xi(t)dt \leq \mu \int_0^{\mathbb{J}(r,s)} r(t)dt,$$

where $\mu \in (0, 1)$, implies that there exists a fixed point $s^* \in E$ of T, which is unique. Furthermore, for any $r \in E$, the sequence $\{T^n\}$ converges to s^*.

Proof. Let $\epsilon > 0$, it is straightforward to identify that (4.1) meets the requirements for $\delta(\epsilon) = \epsilon(\frac{1}{\mu} - 1)$. Then (4.1) is true and this concludes the proof. □

Remark 4.16. Theorem 4.1 of Asadi [9] is a corollary of Theorem 4.11.

Remark 4.17. Corollary 4.15 is generalization of Corollary 4.1 of Asadi [9] and Corollary 3.3 of Najeh *et al.* [19] in the setting of M-metric space.

Remark 4.18. Corollary 4.14 is generalization of Corollary 3.4 of Najeh *et al.* [19].

5 Application

Consider

$$\mho(t) = \int_a^b K(t, s, \mho(s))ds + g(t), \text{ for all } t \in [a, b] \tag{5.1}$$

where $t \in [a, b]$, $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $b > a \geq 0$.

This section of article, establishes an existence theorem for a solution of the integral equation of form given above, where solution belongs to the set $E = (C[a, b], \mathbb{R})$ which contains continuous functions defined on $I = [a, b]$ using the obtained main result in Theorem 4.1.

Consider

$$(T\mho)(t) = \int_a^b \Theta(t, s, \mho(s))ds + g(t), u \in X, t \in [a, b].$$

The occurrence of solution of (5.1) is identical to the occurrence of a fixed point of T. Since E equipped with M-metric given by

$$\Delta(\mho, \Omega) = \sup_{t\in I} |\mho(t) - \Omega(t)|, \text{ for all } \mho, \Omega \in E$$

forms a complete metric space as every complete metric is complete M-metric space.

Assume that the aforementioned condition are true:

(i) $\Theta : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g : [a, b] \to \mathbb{R}$;

(ii) $\displaystyle\int_a^b \Theta(t, s, .) : \mathbb{R} \to \mathbb{R}$ is increasing for all $t, s \in I$;

(iii) for all $\mho, \Omega \in E,\ s, t \in I$ and $\alpha \in (0, 1),$ we have

$$\Big|\Theta(t, s, \mho(s)) - \Theta(t, s, \Omega(s))\Big|^2 \leq \frac{\alpha^2}{4(b-a)}\Big(\Big|\mho(s) - \Omega(s)\Big|\Big)^2.$$

Theorem 5.1. Assume that conditions (i)-(iii) are satisfied. Then integral equation (5.1) has a unique solution.

Proof. For $\mho, \Omega \in E$, we have

$$\begin{aligned}
\Big|(T\mho)(t) - (T\Omega)(t)\Big|^2 &\leq \Big|\int_a^b \Big(\Theta(t, s, \mho(s)) - \Theta(t, s, \Omega(s))\Big)ds\Big|^2 \\
&\leq \int_a^b \int_a^b \Big|\Big(\Theta(t, s, \mho(s)) - \Theta(t, s, \Omega(s))\Big)\Big|^2 ds^2 \\
&\leq (b-a)\int_a^b \Big|\Big(\Theta(t, s, \mho(s)) - \Theta(t, s, \Omega(s))\Big)\Big|^2 ds \\
&\leq (b-a)\frac{\alpha^2}{4(b-a)}\int_a^b \Big|\mho(s) - \Omega(s)\Big|^2 ds \\
&\leq \Big[\frac{\alpha}{2}\Delta(\mho, \Omega)\Big]^2 \\
\leq \Big[\frac{\alpha}{2} \max\Big\{&\frac{(1 + \Delta(\mho, T\mho))\Delta(\Omega, T\Omega)}{1 + \Delta(\mho, \Omega)}, \frac{\Delta(\mho, T\mho)\Delta(\Omega, T\Omega)}{\Delta(\mho, \Omega)}, \Delta(\mho, \Omega)\Big\}\Big]^2,
\end{aligned}$$

and so

$$\Delta(T\mho, T\Omega) \leq \frac{\alpha}{2} \max\Big\{\frac{(1 + \Delta(\mho, T\mho))\Delta(\Omega, T\Omega)}{1 + \Delta(\mho, \Omega)}, \frac{\Delta(\mho, T\mho)\Delta(\Omega, T\Omega)}{\Delta(\mho, \Omega)}, \Delta(\mho, \Omega)\Big\}.$$

Hence by equation (4.1) in Theorem 4.1, the integral equation (5.1) has a unique solution in E. □

6 Conclusion

Taking into account for fixed point theorems in M-metric spaces involving contractive conditions have received considerable attention through the last few years. In this connection, the main aim of this paper is to present the concept of Meir-Keeler type contractions via rational expression in M-metric spaces. Moreover, we proved fixed point theorems, which ensure the existence and uniqueness of the fixed point for this new type of contractive mapping. For instance, using the ideas orthogonality, it is possible to extend our result in orthogonal metric spaces.

References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fundam. Math., 3(1), 133-181, (1922).

[2] E. Keeler and A. Meir, *A theorem on contraction mappings*, J. Math. Anal. Appl, 28, 326-329, (1969).

[3] H. Aydi and E. Karapinar, *A Meir-Keeler common type fixed point theorem on partial metric spaces*, J. Fixed Point Theory Appl., 1, 1-10, (2012).

[4] R.P. Agarwal, D. O'Regan, and N. Shahzad, *Fixed point theory for generalized contractive maps of Meir-Keeler type*, Math. Nachr., 276(1), 3-22, (2004).

[5] S. Karpagam and S. Agrawal, *Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps*, Nonlinear Anal Theory Methods Appl, 74(4), 1040-1046, (2011).

[6] M. Akkouchi, *A Meir-Keeler type common fixed point theorem for four mappings*, Opusc. Math., 31(1), 5-14, (2011).

[7] A. Alsaadi, B. Singh, V. Singh and I. Uddin, *Meir-Keeler type contraction in orthogonal M-metric spaces*, Symmetry, 14(9), 1856, (2022).

[8] N. Mlaiki, N. Souayah, K. Abodayeh and T. Abdeljawad, *Meir-Keeler contraction mappings in M_b-metric Spaces*, J. Comput. Anal. Appl., 27(1), (2021).

[9] Asadi M, *fixed Point Theorem for Meir-Keeler type Mappings in M-metric Spaces with Application*, Fixed Point Theory and Application, 210, 1-10, (2015).

[10] B. Singh, V. Singh, I. Uddin and O. Acar, *Fixed point theorems on an orthogonal metric space using Matkowski type contraction*, Carpathian Math. Publ., 14(1), 127-134, (2022).

[11] D. Gopal, M. Abbas, D.K. Patel and C. Vetro, *Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation*, Acta Math. Sci., 36(3), 957-970, (2016).

[12] H. Lakzian, D. Gopal and W. Sintunavarat, *New fixed point results for mappings of contractive type with an application to nonlinear fractional differential equations*, JJ. Comput. Anal. Appl., 18, 251-266, (2016).

[13] L. Budhia, H. Aydi, A. Ansari and D. Gopal, *Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations*, Nonlinear Anal.: Model. Control, 25(4), 580-597, (2020).

[14] V. Singh and B. Singh, *A Study of Ciric Type Generalized Contraction Via $\mathcal{B}$-Contraction with Application*, In International Conference on Frontiers in Industrial and Applied, Springer Nature Singapore, Singapore, 605-614, (2021).

[15] B.K. Dass and S. Gupta, *An extension of Banach contraction principle through rational expression*, Indian J. pure appl. Math, 6(12), 1455-1458, (1975).

[16] D.S. Jaggi, *Some unique fixed point theorems*, Indian J. Pure Appl. Math, 8(2), 223-230, (1977).

[17] A.N. Gupta and A. Saxena,*A unique fixed point theorem in metric spaces*, Math. Stud, 52, 156-158, (1984).

[18] B. Samet, C. Vetro and H. Yazidi, *A fixed point theorem for a Meir-Keeler type contraction through rational expression*, J. Nonlinear Sci. Appl, 6(3), 162-169, (2013).

[19] N. Redjel, A. Dehici and I.M. Erhan, *A fixed point theorem for Meir-Keeler type contraction via Gupta-Saxena expression*, Fixed Point Theory Appl., 1-9, (2015).

[20] N. Koti, V. Vara and A.K. Singh, *Meir-Keeler type contraction via rational expression*, Acta Math. Univ. Comenianae., 1, 19-28, (2020).

[21] M. Arshad, E. Karapınar and J. Ahmad, *Some unique fixed point theorems for rational contractions in partially ordered metric spaces*, J. Inequal Appl., 1, 1-16, (2013).

[22] S. Chandok, B.S. Choudhury and N. Metiya, *Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions*, J. Egypt. Math. Soc., 23(1), 95-101, (2015).

[23] J. Harjani, B. Lopez and K. Sadarangani, *A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space*, Abstr. Appl. Anal., (2010).

[24] J. Ahmad,A. Al-Rawashdeh and A. Azam, *New fixed point theorems for generalized F-contractions in complete metric spaces*, J. Fixed Point Theory Appl., 1, 1-18, (2015).

[25] M Sarwar, M. Bahadur Zada and S. Radenovic, *Rational type inequality with applications to Voltera–Hammerstein nonlinear integral equations*, Int. J. Nonlinear Sci. Numer. Simul., 21(5), 465-473, (2020).

[26] M.B. Zada and M. Sarwar, *Common fixed point theorems for rational F_R-contractive pairs of mappings with applications*, J. Inequal Appl., 1-14, (2019).

[27] M. Sarwar, W. Ahmad and T. Abdeljawad, *Fixed-point theorems for rational interpolative-type operators with applications*, J. Funct. Spaces, 1-6, (2020).

[28] M. Asadi, E. Karapınar and P. Salimi, *New extension of p-metric spaces with some fixed-point results on M-metric spaces*, J. Inequal Appl., 1-9, (2014).

[29] H. Monfared, M. Azhini and M. Asadi, *Fixed point results on M-metric spaces*, J. Math. Anal, 7(5), 85-101, (2016).

[30] H. Monfared, M. Azhini and M. Asadi, *A generalized contraction principle with control function on M-metric spaces*, Nonlinear Funct. Anal. Appl., 22(2), 395-402, (2017).

[31] N.Y. Ozgur, N. Mlaiki, N. Tas and N. Souayah, *A new generalization of metric spaces: rectangular M-metric spaces*, Math. Sci., 12, 223-233, (2018).

[32] M. Asim, S. Mujahid and I. Uddin, *Meir-Keeler Contraction In Rectangular M_b-Metric Space*, Topol. Algebra Appl., 9(1), 96-104, (2021).

[33] M. Asim, I. Uddin and M. Imdad, *Fixed point results in M_v-metric spaces with an application*, J. Inequal Appl., 1-19, (2019).

[34] M. Asim, A.R. Khan and M. Imdad, *Rectangular M_b-metric spaces and fixed point results*, J. Math. Anal, 10(1), 10-18, (2019).

[35] M. Asim, K.S. Nisar, A. Morsy and M. Imdad, *Extended rectangular $M_{r\xi}$-metric spaces and fixed point results*, Mathematics, 7(12), 1136, (2019).

[36] H. Monfared, M. Azhini and M. Asadi, *C-class and $F(\psi,\varphi)$-contractions on M-metric spaces*, Int. J. Nonlinear Anal. Appl., 8(1), 209-224, (2017).

[37] K. Abodayeh, N. Mlaiki, T. Abdeljawad and W. Shatanawi, *Relations Between partial Metric Spaces and M-Metric Spaces, Caristi Kirk's Theorem in M- Metric Type Spaces*, 7(3), 1-12, (2016).

[38] S.G. Matthews, *Partial metric topology*, Ann. N. Y. Acad. Sci., 728(1), 183-197, (1994).

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