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A NEW SORT OF CONDENSING MULTIVALUED MAPPINGS AND RELATED FIXED POINT RESULTS

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Abstract In this development, we present some fixed point theorems for condensing multivalued mappings in the setting of Banach spaces via measure of noncompactness, without adding regularity. Our results upgrade and extend many theorems in the literature. Moreover, an application to differential inclusions is given here to illustrate the usability of the obtained results.

1 Introduction and Preliminary results

Throughout this paper, let $(X, ||.||)$ be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. Denote

$$\mathcal{P}_p(X) := \{C \in \mathcal{P}(X) : C \text{ is nonempty and has property } p\}. \tag{1.1}$$

In particular, $\mathcal{P}_{cl,bd}(X)$, $\mathcal{P}_{cl,cv}(X)$ and $\mathcal{P}_{cp,cv}(X)$ denote the classes of closed-bounded, closed-convex and compact-convex subsets of X, respectively.
The function $d_H : \mathcal{P}_{cl,bd}(X) \times \mathcal{P}_{cl,bd}(X) \to \mathbb{R}^+$ defined by

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}, \tag{1.2}$$

satisfies all the conditions of a metric on $\mathcal{P}_{cl,bd}(X)$ and is called the Hausdorff-Pompeiu metric on X, where $d(a, B) = \inf\{||a-b|| : b \in B\}$. A point-to-set mapping $T : X \to \mathcal{P}_p(X)$ is simply referred to as a multivalued mapping $T : X \to X$. A point $x \in X$ is called a fixed point of T if $x \in Tx$. If $T : C_1 \to C_2$ is a multivalued mapping, then the graph $Gr(T)$ of the mapping T is defined by

$$Gr(T) = \{(x, y) \in C_1 \times C_2 : y \in Tx\}. \tag{1.3}$$

A multivalued mapping $T : C_1 \to C_2$ is said to be closed if its graph is closed in the product topology on $C_1 \times C_2$. For more information on this subject, we recommend interested readers to consult [6, 12, 13].
In what follows, we restrict ourselves only to the fixed point theory related to closed multivalued mappings. A particular case of Himmelberg fixed point theorem [7] is the following:

Theorem 1.1. *(O'Regan,[11]) Let C be a closed convex and bounded subset of a Banach algebra X and let $T : C \to \mathcal{P}_{cl,cv}(C)$ be a compact and closed multivalued mapping. Then T has a fixed point.*

The compactness of T in Theorem 1.1 is further weakened by condensing mappings with the help of measure of noncompactness in the Banach space X. The first try is given by Darbo [4] in 1955, by modeling the classical Banach contraction principle [2] with the well-known Kuratowskii measure of noncompactness $\alpha : \mathcal{P}_{bd}(X) \to \mathbb{R}^+$ of a bounded set in the Banach space X [9], which is the functions defined by

$$\alpha(\Omega) = \inf\Big\{\, \varepsilon > 0 \,:\, \Omega \subset \bigcup_{k=1}^{n} B_k,\ B_k \subset X,\ \text{Diam}\,(B_k) \le \varepsilon : k = 1, 2, \cdots, n \in \mathbb{N}\Big\}, \tag{1.4}$$

where Diam (B) denotes the diameter of a bounded set B.
In the same direction, let $\mathcal{M}_X$ and $\mathcal{N}_X$ stand for the collection of all nonempty and bounded subsets of X, and the collection of all relatively compact subsets of X, respectively. We write $\overline{B}$ and Cov (B) to denote the closure and closed convex hull of $B \subset X$, respectively. Banaŝ et al [3] gave a new axiomatic for the measure of noncompactness as follows:

Definition 1.2. ([3]) A map $\mu : \mathcal{M}_X \to [0, +\infty[$ is called measure of non-compactness defined on X if it satisfies the following properties:
1) The family $\ker \mu = \{B \in \mathcal{M}_X : \mu(B) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_X$,
2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$,
3) $\mu(B) = \mu(\overline{B}) = \mu(\text{Cov}\,(B))$,
4) $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B)$ for all $\lambda \in [0,1]$ and $A, B \in \mathcal{M}_X$,
5) if $\{B_n\}$ is a decreasing sequence of nonempty, closed and bounded subsets of X with $\lim \mu(B_n) = 0$, then $B_\infty = \cap_n B_n \neq \emptyset$.

Definition 1.3. ([3]) Let μ be a measure of noncompactness in a Banach space X. The measure μ is homogeneous if $\mu(\lambda A) = |\lambda|\mu(A)$ for $\lambda \in \mathbb{R}$. If the measure μ satisfied the condition $\mu(A + B) \leq \mu(A) + \mu(B)$ it is called subadditive.
The measure μ being both homogeneous and subadditive is said to be sublinear.

Definition 1.4. ([3]) We say that a measure of non-compactness μ has the maximum property if $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.

Definition 1.5. ([3]) A sublinear measure of non-compactness μ that has the maximum property and is such that $\ker \mu = \mathcal{N}_X$ is called a regular measure.

Note that the functions α defined by (1.4) enjoys the conditions (1) through (5) in Definition 1.2, then α is measure of noncompactness in the sense of Banaŝ and Goebel [3]. In addition, α is regular measure of noncompactness on X.
Now, we state a key result which is a version of Darbo's fixed point result [4] in the setting of multivalued mappings:

Theorem 1.6. *(Dhage [5]) Let C be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : C \to \mathcal{P}_{cl,cv}(C)$ be a closed mapping. Assume that there exists a constant $k \in (0,1)$ such that*

$$\mu(T(\Omega)) \leq k\mu(\Omega), \tag{1.5}$$

for any subset Ω of C. Then T has a fixed point in C.
Where μ is measure of noncompactness in the sense of Definition 1.2.

The following fixed point theorem for condensing multivalued mappings is a natural extension of the previous Theorem 1.6. See Hu and Papageorgiou [8]:

Theorem 1.7. *Suppose that C is a nonempty, bounded, closed, and convex subset of a Banach space X and $T : C \to \mathcal{P}_{cl,cv}(C)$ a closed mapping. If for any nonempty subset Ω of C with $\mu(\Omega) > 0$ we have*

$$\mu(T(\Omega)) < \mu(\Omega), \tag{1.6}$$

where μ is regular measure of noncompactness in X, then T has a fixed point in C.

However, Theorem 1.7 does not ensure the existence of fixed points unless the measure is assumed regular (in the sense of Definition 1.5). Furthermore, it is rather difficult to find the mappings satisfying the conditions on given Banach spaces.
In this paper, using a measure of noncompactness, we prove a new fixed point theorem for a new class of condensing multivalued mappings in Banach spaces satisfying:

$$\inf_{\substack{\Omega \in \mathcal{P}(C) \\ \mu(\Omega) > 0}} \big\{\mu(\Omega) - \mu(T(\Omega))\big\} > 0. \tag{1.7}$$

Moreover, we show a result for a new class of condensing multivalued mappings, we call it μE-condensing multivalued mappings, with the aid of an auxiliary function ϕ satisfying $\phi(1) = 0$

and $\inf_{t>1}\phi(t) > 0$. Compared to Theorem 1.7, we mention that our results are proved without using regularity of the measure.
Finally, to show the importance of lack of regularity, we apply our results to obtain an existence theorem for Cauchy-Lipschitz-type differential inclusions in Banach spaces.

2 Main results

Before stating the main fixed point result of this section, we need the following Lemma which appears in [17]:

Lemma 2.1. *If μ is a measure of noncompactness, then also $\nu = e^{\mu} - 1$ is a measure of noncompactness.*

Theorem 2.2. *Let C be a nonempty bounded, closed and convex subset of a Banach space X and $T : C \to \mathcal{P}_{cl,cv}(C)$ be a closed multivalued mapping such that*

$$\inf_{\substack{\Omega\in\mathcal{P}(C)\\ \mu(\Omega)>0}} \big\{\mu(\Omega) - \mu(T(\Omega))\big\} > 0,$$

where μ is an arbitrary measure of noncompactness.
Then T has a fixed point in C.

Proof. Letting

$$A = \inf_{\substack{\Omega\in\mathcal{P}(C)\\ \mu(\Omega)>0}} \big\{\mu(\Omega) - \mu(T(\Omega))\big\}. \tag{2.1}$$

Then

$$\mu(T(\Omega))) \leq \mu(\Omega) - A, \tag{2.2}$$

for all $\Omega \in \mathcal{P}(C)$, with $\mu(\Omega) > 0$.
Thus

$$e^{\mu(T(\Omega))} \leq k e^{\mu(\Omega)}, \tag{2.3}$$

where $k = e^{-A} \in (0, 1)$.
Therefore, we have

$$\nu(T(\Omega)) \leq k\nu(\Omega), \tag{2.4}$$

for all $\Omega \subset C$, where $\nu = e^{\mu} - 1$.
By using Lemma 2.1, ν is a measure of noncompactness in the sense of Definition 1.2. Then according to Theorem 1.6, we deduce that T has a fixed point in C. □

Example 2.3. Consider the Hilbert space $X = l_2$ over $\mathbb{R}$ with basis $\{e_n : n \in \mathbb{N}\}$ and let $C = \{x \in X : ||x|| \leq 1\}$. Then C is nonempty bounded, convex and closed in X.
Define the mapping

$$\begin{array}{rccl} T: & C & \to & \mathcal{P}_{cl,cv}(C) \\ & x & \mapsto & \left\{ \begin{array}{ll} \{\sum_{i=1}^{\infty} \frac{\alpha_i}{2} e_i\} & \text{for} \quad x \in \{x \in X : ||x|| < 1\}, \\ \{\sum_{i=1}^{\infty} \frac{\alpha_i}{3} e_i\} & \text{for} \quad x \in \{x \in X : ||x|| = 1\}. \end{array} \right. \end{array} \tag{2.5}$$

Define the measure of noncompactness μ by

$$\mu(\Omega) = \left\{ \begin{array}{ll} 0 & \text{if } \Omega \text{ is a precompact,} \\ 1 & \text{else.} \end{array} \right. \tag{2.6}$$

It is easy to see that μ has the maximum property, invariant under passage to the convex hull and it is not homogeneous. Then μ is not regular.
Now, let $\Omega \subset C$ with $\mu(\Omega) > 0$, we have

$$\mu(\Omega) - \mu(T\Omega) = 1 > 0. \tag{2.7}$$

In other words, we have

$$\inf_{\substack{\Omega\in\mathcal{P}(C)\\ \mu(\Omega)>0}} \{\mu(\Omega)-\mu(T(\Omega))\} > 0. \tag{2.8}$$

Therefore, all conditions of Theorem 2.2 are satisfied and $(0,0,0,...)$ is the only fixed point of T in C.

Note that Theorem 2.2 extends the proven Theorem 3.2 in [17] for singlevalued mappings. Also, Theorem 2.2 yields a version of Theorem 3 in [14] for the case of the diameter measure of noncompactness in Banach spaces: $\mu(\Omega) := \text{Diam}\,(\Omega) = \sup_{x,y\in\Omega} ||x-y||$. Namely, we assert the following:

Corollary 2.4. *([17]) Let C be a nonempty bounded, closed and convex subset of a Banach space X and $T : C \to C$ be a continuous mapping such that*

$$\inf_{\substack{\Omega\in\mathcal{P}(C)\\ \mu(\Omega)>0}} \{\mu(\Omega)-\mu(T(\Omega))\} > 0, \tag{2.9}$$

where μ is an arbitrary measure of noncompactness.
Then T has a fixed point in C.

Theorem 2.5. *Let C be a nonempty bounded, closed and convex subset of a Banach space X and $T : C \to C$ be continuous a mapping such that* $\inf_{x\neq y\in\Omega\subset C}\{||x-y|| - ||Tx-Ty||\} > 0$. *Then T has a fixed point.*

Proof. Let $\Omega \subset C$ and $x \neq y \in \Omega$, putting

$$\alpha = \inf_{x\neq y\in\Omega\subset C}\{||x-y|| - ||Tx-Ty||\}. \tag{2.10}$$

Then

$$||Tx-Ty|| \leq ||x-y|| - \alpha, \tag{2.11}$$

for all $x \neq y \in \Omega$.
We apply supremum on the left-hand side and the right-hand side of (2.11), we obtain

$$\text{Diam}\,(\Omega) \leq \text{Diam}\,(\Omega) - \alpha, \tag{2.12}$$

for all $\Omega \in \mathcal{P}(C)$, with $\text{Diam}\,(\Omega) > 0$.
Thus

$$\inf_{\substack{\Omega\in\mathcal{P}(C)\\ \text{Diam}\,(\Omega)>0}} \{\text{Diam}\,(\Omega) - \text{Diam}\,(T(\Omega))\} > 0. \tag{2.13}$$

This finishes the proof. □

Remark 2.6. *The above theorem illustrates the situation where a measure of noncompactness explicitly derives from a norm.*

Definition 2.7. Let C be a nonempty bounded, closed and convex subset of a Banach space X and $T : C \to \mathcal{P}_{cl,cv}(C)$ be a multivalued mapping. T will be said a μE-weakly condensing if it is closed and

$$\mu(T(\Omega)) \leq \mu(\Omega) - \phi(1+\mu(\Omega)),$$

for all $\Omega \in \mathcal{P}(C)$, with $\mu(\Omega) > 0$ and $\phi : [1,+\infty[\to [0,+\infty[$ is a function satisfying $\phi(1) = 0$ and $\inf_{t>1}\phi(t) > 0$.

Theorem 2.8. *Let C be a nonempty bounded, closed and convex subset of a Banach space X and $T : C \to \mathcal{P}_{cl,cv}(C)$ be a μE-weakly condensing multivalued mapping. Then T has a fixed point in C.*

Proof. Let $\Omega \subset C$, from Definition 2.7, we have

$$0 < \inf_{t>1} \phi(t) \leq \phi(1+\mu(\Omega)) \leq \mu(\Omega) - \mu(T(\Omega)). \tag{2.14}$$

Hence, we get

$$\inf_{\substack{\Omega \in \mathcal{P}(C) \\ \mu(\Omega)>0}} \{\mu(\Omega) - \mu(T(\Omega))\} > 0. \tag{2.15}$$

According to Theorem 2.2, the mapping T has a fixed point. □

Corollary 2.9. *([17]) Let C be a nonempty bounded, closed and convex subset of a Banach space X and $T : C \to C$ be a μE-weakly condensing mapping. Then T has a fixed point in C.*

3 Application

In this section, inspired by the works of the authors [1, 15, 16, 18], we investigate the existence for differential inclusions under new and weak conditions. For this aim, let $X = \mathcal{C}([0,1], \mathbb{R})$ be the space of all continuous functions from $[0,1]$ into $\mathbb{R}$, equipped with the norm $||x|| = \max_{t\in[0,1]} |x(t)|$ such that for all $x \in X$, $||x|| \leq \xi$ for some ξ. Consider the differential inclusion:

$$\begin{cases} x'(t) \in K(t, x(t)), t \in [0,1] \\ x(0) = 0, \end{cases} \tag{3.1}$$

where $x \in X$ and $K : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,cv}$ is a lower semicontinuous multivalued mapping.
Note that (3.1) is equivalent to the Volterra-type integral inclusion:

$$x(t) \in \int_0^t K(s, x(s))ds, \tag{3.2}$$

for all $t \in [0,1]$.
Define the multivalued operator T from X into $\mathcal{P}(X)$ by:

$$Tx(t) = \{y \in X : y(t) \in \int_0^t K(s, x(s))ds, t \in [0,1]\}, \tag{3.3}$$

for all $x \in X$.
Let $x \in X$, so according to Michael's selection Theorem [10], there exits a continuous operator $k_x : [0,1] \to \mathbb{R}$ such that $k_x(s) \in K(s, x(s))$ for any $s \in [0,1]$, which implies that $\int_0^t k_x(s)ds \in Tx(t)$, then $Tx \neq \emptyset$. On the other hand, it is clear to see that Tx is a closed set.
On the other side, let μ the measure of noncompactness of the norm defined as follows (see [3])

$$\mu(\Omega) = \sup_{x\in\Omega} ||x||, \tag{3.4}$$

for all $\Omega \in \mathcal{M}_X$.
Define the function θ by

$$\begin{aligned} \theta : [0,1] &\to \mathbb{R} \\ t &\mapsto 0 \end{aligned}$$

Note that μ is a sublinear measure of noncompactness with maximum property and $\ker \mu = \{\theta\} \neq \mathcal{N}_X$, so μ is not regular.
Under the above assumptions, we can state the following theorem.

Theorem 3.1. *If there exists $A > 0$ such that*

$$|k_x(t)| \leq |x(t)| - A, \tag{3.5}$$

for all $t \in [0,1]$ and $x \in X$. Then the differential inclusion (3.1) has a solution.

Proof. Let $\Omega \subset X \setminus \{0\}$ and $x \in \Omega$ such that $a \in Tx$, hence there exists $k_x(s) \in K(s, x(s))$ for $s \in [0, 1]$ with $a(t) = \int_0^t k_x(s)ds$, we have

$$\begin{aligned} |a(t)| &\leq \int_0^t |k_x(s)|ds \\ &\leq \int_0^t (|x(s)| - A)ds \\ &\leq \int_0^1 (||x|| - A)ds \\ &\leq ||x|| - A. \end{aligned}$$

Therefore

$$||a|| \leq ||x|| - A. \tag{3.6}$$

Then

$$\mu(T\Omega) \leq \mu(\Omega) - A. \tag{3.7}$$

Thus

$$\inf_{\substack{\Omega \in \mathcal{P}(X) \\ \mu(\Omega) > 0}} \{\mu(\Omega) - \mu(T(\Omega))\} > 0. \tag{3.8}$$

By applying Theorem 2.2, we deduce that T has a fixed point. □

4 Data Availability

No data were used to support this study.

5 Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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