Palestine Journal of Mathematics

Vol 13(2)(2024) , 30–40

Non-self Ćirić $\alpha^+(\theta, \phi)$-proximal contractions with best proximity point

Khairul Habib Alam and Yumnam Rohen

Communicated by Harikrishnan Panackal

MSC 2020 Classifications: Primary 47H10, 54H25; Secondary 41A65, 46B20, 47H05, 47H09.

Keywords and phrases: α^+-proximal admissible mapping, Weak p-property, (θ, ϕ)-contraction, Ćirić $\alpha^+(\theta, \phi)$-proximal contraction.

Acknowledgment: *The authors express their gratitude to the editor and reviewers for their insightful criticism and helpful recommendations, which helped to elevate the superiority of our work. For the essential assistance, Khairul Habib Alam was extremely appreciative of the University Grant Commission, New Delhi.*

Corresponding Author: Khairul Habib Alam (almkrlhb@gmail.com/alamkh@gmx.co.uk)

Abstract. This study aims to provide a new class of Ćirić $\alpha^+(\theta, \phi)$-proximal contraction, a non-self generalized proximal contraction mapping on a non-empty closed subset of any metric space. Also, we prove that such contractions satisfying some conditions must have a unique best proximity point if we take the base space as complete. For some particular values of the constants, that we have used to generalize the proximal contraction, we conclude different types of proximal contractions. We substantiate the deduced findings with examples. An application to show the solution of an integral equation in the context of proximity point results over the space of all real-valued continuous functions is discussed. By presenting these findings, we aim to encourage a new wave of scholars to keep exploring this exciting topic that has so many applications ahead of it.

1 Introduction and Preliminaries

Mathematical study on fixed point theory for self-operators covers a large area. It can be used for a wide range of topics both inside and outside of mathematics, including different kinds of real-world word problems. To be sure, the traditional Banach contraction principle [3] which is at the heart of metric fixed point theory continues to inspire scientists to demonstrate groundbreaking results for self-operators that expand upon it in a variety of ways (see [1, 4, 6, 10, 16]). Instead of self-operators if we are able to deal with the mappings that are non-self is one method to make this theory better.

Let us consider a non-self-mapping $H : P \longrightarrow Q$ for a metric space (V, d) with non-void subsets P and Q. $H\vartheta = \vartheta$ is unlikely to have a solution because H is not a self-mapping. Therefore, it is crucial to look for an $\vartheta \in P$ that is in some way most closest to $H\vartheta \in Q$. Numerous academics have investigated this issue and tried to determine whether there is a point $\vartheta^* \in P$ with the least amount of error, where $d(\vartheta^*, H\vartheta^*)$ is the smallest globally. If ϑ^* corresponds to $d(\vartheta^*, H\vartheta^*) = d(P, Q)$, where $d(P, Q) = \inf\{d(\vartheta_1, \vartheta_2) : \vartheta_1 \in P, \vartheta_2 \in Q\}$, then $\vartheta^* \in P$ is a best proximity point [2] of the non-self-mapping $H : P \longrightarrow Q$. Furthermore, the mapping H is a self-mapping if $P = Q$, which leads to the topic fixed point theory that can be solved using any of the extensions or itself, Banach's [3] fixed point contraction theorem, and this will lead to the best proximity point. As a result, one of the intriguing issues in fixed point theory is the best proximity point theory. For more results in this direction, authors can see [5, 7, 8, 14].

In order to properly read this article, the following notation [2] should be fixed:

$$P_0 = \{\vartheta_1 \in P : d(\vartheta_1, \vartheta_2) = d(P, Q) \text{ for a } \vartheta_2 \in Q\},$$

$$Q_0 = \{\vartheta_2 \in Q : d(\vartheta_1, \vartheta_2) = d(P, Q) \text{ for a } \vartheta_1 \in P\}.$$

We now review a number of concepts, examples, and findings that are necessary to comprehend this study.

Definition 1.1. [9] Let $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ be any mapping and $H : P \longrightarrow Q$ is so that

$$\left.\begin{array}{rcl} \alpha(u_1, u_2) & \geq & 0 \\ d(u_1, H\vartheta_1) & = & d(P, Q) \\ d(u_2, H\vartheta_2) & = & d(P, Q) \end{array}\right\} \Rightarrow \alpha(\vartheta_1, \vartheta_2) \geq 0,$$

for any $u_1, u_2, \vartheta_1, \vartheta_2 \in P$. Then H is called α^+-proximal admissible.

Definition 1.2. [11] Let (P, Q) represents a pair of non-empty subsets of the metric space (V, d). For any $u_1, u_2 \in P$ and $\vartheta_1, \vartheta_2 \in Q$, we have

$$\left.\begin{array}{rcl} d(u_1, \vartheta_1) & = & d(P, Q) \\ d(u_2, \vartheta_2) & = & d(P, Q) \end{array}\right\} \Rightarrow d(u_1, u_2) \leq d(\vartheta_1, \vartheta_2)$$

if and only if the pair (P, Q) said to satisfy weak p-property.

Definition 1.3. [12] Let Θ be the collection of mappings $\theta : \mathbb{R}_+ \longrightarrow (1, +\infty)$ that satisfies

(θ1) $r < s \Rightarrow \theta(r) < \theta(s),\ for\ all\ r, s \in \mathbb{R}_+$, i.e., θ is strictly increasing,

(θ2) $\lim_{n\to\infty} r_n = 0$ iff $\lim_{n\to\infty} \theta(r_n) = 1$, where $\{r_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathbb{R}_+$,

(θ3) θ is continuous.

Definition 1.4. [13] Let Φ consists the collection of mappings $\phi : [1, +\infty) \longrightarrow [1, +\infty)$ that satisfies

(ϕ1) $r \leq s \Rightarrow \phi(r) \leq \phi(s),\ for\ all\ r, s \in [1, +\infty)$, i.e., ϕ is increasing,

(ϕ2) $\lim_{n\to\infty} \phi^n(r) = 1$, for any $r \in (1, +\infty)$,

(ϕ3) ϕ is continuous.

Lemma 1.5. *[15] For all $\phi \in \Phi$, we have $\phi(1) = 1$ and $\phi(r) < r, \forall r > 1$.*

Definition 1.6. [13] A mapping $H : V \longrightarrow V$, where (V, d) is any metric space, will be called as (θ, ϕ)-contraction if for any $\vartheta_1, \vartheta_2 \in V$ there exist functions $\phi \in \Phi, \theta \in \Theta$ that satisfies

$$d(H\vartheta_1, H\vartheta_2) > 0 \Rightarrow \theta[d(H\vartheta_1, H\vartheta_2)] \leq \phi[\theta(d(\vartheta_1, \vartheta_2))].$$

2 Main Results

Here we define Ćirić $\alpha^+(\theta, \phi)$-proximal contraction, a new class of non-self contractions. Also, we state and prove results, for mappings that satisfy such contraction conditions must have a best proximity point. Consequently, we define Chatterjea $\alpha^+(\theta, \phi)$-proximal contraction and present some corollaries that are analogous to Reich, Kannan, Banach, and many more.

Definition 2.1. A mapping $H : P \longrightarrow Q$ is known as a Ćirić $\alpha^+(\theta, \phi)$-proximal contraction when it applies to non-empty subsets P, Q of any metric space (V, d) if for some $\theta \in \Theta, \phi \in \Phi$ we have

$$d(\vartheta_1, \vartheta_2) > 0 \Rightarrow \theta(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) \leq \phi(\theta(M(\vartheta_1, \vartheta_2))), \tag{2.1}$$

for all $\vartheta_1, \vartheta_2 \in P$. Where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$\begin{aligned} M(\vartheta_1, \vartheta_2) \ = \ & \zeta_1 d(\vartheta_1, \vartheta_2) + \zeta_2 \Big\{ d(\vartheta_1, H\vartheta_1) - d(P, Q) \Big\} + \zeta_3 \Big\{ d(\vartheta_2, H\vartheta_2) \\ & - d(P, Q) \Big\} + \zeta_4 \Big\{ d(\vartheta_1, H\vartheta_2) - d(P, Q) \Big\} + \zeta_5 \Big\{ d(\vartheta_2, H\vartheta_1) \\ & - d(P, Q) \Big\}, \end{aligned}$$

with $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 \geq 0, \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5 < 1, \zeta_4 = \zeta_5, \zeta_3 + \zeta_4 \neq 1$.

The following is an example of a Ćirić $\alpha^+(\theta,\phi)$-proximal contraction.

Example 2.2. Consider two non-empty subsets

$$P=\{(\vartheta_1,\vartheta_2)\in\mathbb{R}^2:\vartheta_1=1,0\le\vartheta_2\le 1\},$$

$$Q=\{(\vartheta_1,\vartheta_2)\in\mathbb{R}^2:\vartheta_1=\frac{1}{2},0\le\vartheta_2\le 1\}$$

of the metric space $(\mathbb{R}^2,d_u)$, where d_u is usual metric of $\mathbb{R}^2$. Suppose $H:P\longrightarrow Q$ is given by

$$H(\vartheta_1,\vartheta_2)=\Big(\frac{\vartheta_1}{2},\frac{\vartheta_2}{4}\Big),\forall(\vartheta_1,\vartheta_2)\in P$$

and $\alpha^+:P\times P\longrightarrow(-\infty,+\infty)$ is given by $\alpha(r_1,r_2)=0$. Then for $\theta(r)=1+\frac{r}{2}\in\Theta$ and $\phi(r)=\frac{1+r}{2}\in\mathbf{\Phi}$, H is a Ćirić $\alpha^+(\theta,\phi)$-proximal contraction for $\zeta_1=\frac{3}{5},\zeta_2=\frac{1}{7},\zeta_3=\frac{1}{9},\zeta_4=\frac{1}{17},\zeta_5=\frac{1}{17}$.

Now we present our main result.

Theorem 2.3. *Consider two non-empty subsets P,Q satisfying weak p-property of any complete metric space (V,d) with $P_0\ne\emptyset$. Assume that a continuous mapping $H:P\longrightarrow Q$ is a Ćirić $\alpha^+(\theta,\phi)$-proximal contraction so that $H(P_0)\subseteq Q_0$, $d(\vartheta_1,H\vartheta_0)=d(P,Q)$ and $\alpha(\vartheta_0,\vartheta_1)\ge 0$ for some $\vartheta_0,\vartheta_1\in P_0$. Then unique best proximity point ϑ^* of H exists in P so that $d(\vartheta^*,H\vartheta^*)=d(P,Q)$.*

Proof. According to the statement of the theorem, there are some points $\vartheta_0,\vartheta_1\in P_0$ so that $d(\vartheta_1,H\vartheta_0)=d(P,Qa)$ and $\alpha(\vartheta_0,\vartheta_1)\ge 0$. Since $H(P_0)\in Q_0$, for $\vartheta_1\in P_0$ there will be $\vartheta_2\in P_0$ so that $d(\vartheta_2,H\vartheta_1)=d(P,Q)$. Being H an α^+-proximal admissible, we have $\alpha(\vartheta_1,\vartheta_2)\ge 0$. Again since $H(P_0)\in Q_0$, for $\vartheta_2\in P_0$ there will be $\vartheta_3\in P_0$ so that $d(\vartheta_3,H\vartheta_2)=d(P,Q)$. Being H an α^+-proximal admissible, we have $\alpha(\vartheta_2,\vartheta_3)\ge 0$. Proceeding similarly, we will have a sequence $\{\vartheta_n\}$ of points in P_0 with

$$d(\vartheta_{n+1},H\vartheta_n)=d(P,Q)\text{ and }\alpha(\vartheta_n,\vartheta_{n+1})\ge 0,\forall n\in\mathbb{N}. \tag{2.2}$$

Since the pair (P,Q) is fulfilling the weak p-property, we have

$$d(\vartheta_n,\vartheta_{n+1})\le d(H\vartheta_{n-1},H\vartheta_n),\forall n\in\mathbb{N}.$$

If for some $n\in\mathbb{N},\vartheta_n=\vartheta_{n+1}$, then

$$d(\vartheta_{n+1},H\vartheta_n)=d(P,Q)\Rightarrow d(\vartheta_n,H\vartheta_n)=d(P,Q).$$

So that ϑ_n will be a best proximity point of H in $P_0\subseteq P$.

Hence we can assume that $\vartheta_n\ne\vartheta_{n+1},\forall n\in\mathbb{N}$ i.e., the sequence consists of distinct points, i.e., $d(\vartheta_n,\vartheta_{n+1})\ge 0$. Since H is Ćirić $\alpha^+(\theta,\phi)$-proximal contraction

$$\begin{aligned}\theta(d(\vartheta_n,\vartheta_{n+1}))&\le\theta(d(H\vartheta_{n-1},H\vartheta_n)),\ \text{since }\theta\text{ is strictly increasing}\\&\le\theta(d(H\vartheta_{n-1},H\vartheta_n))+\alpha(\vartheta_{n-1},\vartheta_n),\ \text{since }\alpha(\vartheta_{n-1},\vartheta_n)\ge 0\\&\le\phi(\theta(M(\vartheta_{n-1},\vartheta_n))).\end{aligned}$$

Where

$$
\begin{aligned}
M(\vartheta_{n-1},\vartheta_n) &= \zeta_1 d(\vartheta_{n-1},\vartheta_n)+\zeta_2\Big\{d(\vartheta_{n-1},H\vartheta_{n-1})-d(P,Q)\Big\}\\
&\quad +\zeta_3\Big\{d(\vartheta_n,H\vartheta_n)-d(P,Q)\Big\}+\zeta_4\Big\{d(\vartheta_{n-1},H\vartheta_n)\\
&\quad -d(P,Q)\Big\}+\zeta_5\Big\{d(\vartheta_n,H\vartheta_{n-1})-d(P,Q)\Big\}\\
&\le \zeta_1 d(\vartheta_{n-1},\vartheta_n)+\zeta_2\Big\{\Big(d(\vartheta_{n-1},\vartheta_n)+d(\vartheta_n,H\vartheta_{n-1})\Big)\\
&\quad -d(P,Q)\Big\}+\zeta_3\Big\{\Big(d(\vartheta_n,\vartheta_{n+1})+d(\vartheta_{n+1},H\vartheta_n)\Big)\\
&\quad -d(P,Q)\Big\}+\zeta_4\Big\{\Big(d(\vartheta_{n-1},\vartheta_n)+d(\vartheta_n,\vartheta_{n+1})\\
&\quad +d(\vartheta_{n+1},H\vartheta_n)\Big)-d(P,Q)\Big\}+\zeta_5\Big\{d(\vartheta_n,H\vartheta_{n-1})\\
&\quad -d(P,Q)\Big\}\\
&= \zeta_1 d(\vartheta_{n-1},\vartheta_n)+\zeta_2\Big\{d(\vartheta_{n-1},\vartheta_n)+\Big(d(\vartheta_n,H\vartheta_{n-1})\\
&\quad -d(P,Q)\Big)\Big\}+\zeta_3\Big\{d(\vartheta_n,\vartheta_{n+1})+\Big(d(\vartheta_{n+1},H\vartheta_n)\\
&\quad -d(P,Q)\Big)\Big\}+\zeta_4\Big\{\Big(d(\vartheta_{n-1},\vartheta_n)+d(\vartheta_n,\vartheta_{n+1})\Big)\\
&\quad +\Big(d(\vartheta_{n+1},H\vartheta_n)-d(P,Q)\Big)\Big\}+\zeta_5\Big\{d(\vartheta_n,H\vartheta_{n-1})\\
&\quad -d(P,Q)\Big\}\\
&= \Big(\zeta_1+\zeta_2+\zeta_4\Big)d(\vartheta_{n-1},\vartheta_n)+\Big(\zeta_3+\zeta_4\Big)d(\vartheta_n,\vartheta_{n+1}),\ \text{by 2.2.}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\theta(d(\vartheta_n,\vartheta_{n+1})) &\le \phi(\theta(M(\vartheta_{n-1},\vartheta_n)))\\
&< \theta(M(\vartheta_{n-1},\vartheta_n)),\ \text{since } \phi\in\Phi \text{ and by Lemma 1.5}\\
\Rightarrow \theta(d(\vartheta_n,\vartheta_{n+1})) &< \theta\Big(\Big(\zeta_1+\zeta_2+\zeta_4\Big)d(\vartheta_{n-1},\vartheta_n)+\Big(\zeta_3+\zeta_4\Big)d(\vartheta_n,\vartheta_{n+1})\Big).
\end{aligned}
$$

Since θ is strictly increasing

$$d(\vartheta_n,\vartheta_{n+1})<\Big(\zeta_1+\zeta_2+\zeta_4\Big)d(\vartheta_{n-1},\vartheta_n)+\Big(\zeta_3+\zeta_4\Big)d(\vartheta_n,\vartheta_{n+1})$$

$$\Rightarrow \Big(1-\zeta_3-\zeta_4\Big)d(\vartheta_n,\vartheta_{n+1})<\Big(\zeta_1+\zeta_2+\zeta_4\Big)d(\vartheta_{n-1},\vartheta_n)$$

$$\Rightarrow d(\vartheta_n,\vartheta_{n+1})<\frac{\Big(\zeta_1+\zeta_2+\zeta_4\Big)}{\Big(1-\zeta_3-\zeta_4\Big)}d(\vartheta_{n-1},\vartheta_n).$$

Now since $\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5\ge 0, \zeta_1+\zeta_2+\zeta_3+2\zeta_4<1, \zeta_3+\zeta_4\ne 1$, we have

$$\frac{\Big(\zeta_1+\zeta_2+\zeta_4\Big)}{\Big(1-\zeta_3-\zeta_4\Big)}<1$$

$$\Rightarrow d(\vartheta_n,\vartheta_{n+1})<d(\vartheta_{n-1},\vartheta_n)$$

$$\Rightarrow \theta(M(\vartheta_{n-1},\vartheta_n)))<\theta(d(\vartheta_{n-1},\vartheta_n)).$$

Hence

$$\Rightarrow 1 \le \theta(d(\vartheta_n, \vartheta_{n+1})) \le \phi(\theta(M(\vartheta_{n-1}, \vartheta_n))) \le \phi(\theta(d(\vartheta_{n-1}, \vartheta_n)))$$

$$\begin{aligned}\Rightarrow 1 \le \theta(d(\vartheta_n, \vartheta_{n+1})) &\le \phi(\theta(d(\vartheta_{n-1}, \vartheta_n)))\\ &\le \phi(\phi(\theta(d(\vartheta_{n-2}, \vartheta_{n-1})))) = \phi^2(\theta(d(\vartheta_{n-2}, \vartheta_{n-1})))\\ &\vdots\\ &\le \phi^n(\theta(d(\vartheta_0, \vartheta_1))).\end{aligned}$$

Limiting as $n \to +\infty$ and as $\theta(d(\vartheta_0, \vartheta_1)) \ge 1$, we have $\lim_{n\to+\infty} \phi^n(\theta(d(\vartheta_0, \vartheta_1))) = 1$ and so

$$\lim_{n\to+\infty} \theta(d(\vartheta_n, \vartheta_{n+1})) = 1$$

$$\Rightarrow \lim_{n\to+\infty} d(\vartheta_n, \vartheta_{n+1}) = 0, \text{ since } \theta \in \Theta. \tag{2.3}$$

By the contrary method we will prove $\lim_{n\to+\infty} d(\vartheta_n, \vartheta_m) = 0$, that is $\{\vartheta_n\}$ is a Cauchy sequence, as follows:

Suppose the sequence $\{\vartheta_n\}$ is not Cauchy, then there must be $\varepsilon > 0, k \in \mathbb{N}$ and sub-sequences $\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ with $p_n > q_n > k$ so that

$$d(\vartheta_{p_n}, \vartheta_{q_n}) \ge \varepsilon \text{ and } d(\vartheta_{p_n-1}, \vartheta_{q_n}) < \varepsilon \text{ for all } n > k.$$

Now

$$\begin{aligned}\varepsilon \le d(\vartheta_{p_n}, \vartheta_{q_n}) &\le d(\vartheta_{p_n}, \vartheta_{p_n-1}) + d(\vartheta_{p_n-1}, \vartheta_{q_n})\\ &< d(\vartheta_{p_n-1}, \vartheta_{p_n}) + \varepsilon.\end{aligned}$$

$$\Rightarrow \lim_{n\to+\infty} d(\vartheta_{p_n}, \vartheta_{q_n}) = \varepsilon, \text{ by 2.3}. \tag{2.4}$$

Again

$$\begin{aligned}\varepsilon \le d(\vartheta_{p_n}, \vartheta_{q_n}) &\le d(\vartheta_{p_n}, \vartheta_{q_n+1}) + d(\vartheta_{q_n}, \vartheta_{q_n+1})\\ \Rightarrow \varepsilon &\le d(\vartheta_{p_n}, \vartheta_{q_n+1}) \le d(\vartheta_{p_n}, \vartheta_{q_n}) + d(\vartheta_{q_n}, \vartheta_{q_n+1}),\\ &\quad \text{by limiting } n \to +\infty \text{ and triangle inequality.}\end{aligned}$$

$$\Rightarrow \lim_{n\to+\infty} d(\vartheta_{p_n}, \vartheta_{q_n+1}) = \varepsilon, \text{ by 2.3 and 2.4}. \tag{2.5}$$

Similarly, we have

$$\lim_{n\to+\infty} d(\vartheta_{p_n+1}, \vartheta_{q_n}) = \varepsilon. \tag{2.6}$$

Using the following inequalities

$$d(\vartheta_{p_n+1}, \vartheta_{q_n+1}) \le d(\vartheta_{p_n+1}, \vartheta_{q_n}) + d(\vartheta_{p_n}, \vartheta_{q_n}) + d(\vartheta_{q_n}, \vartheta_{q_n+1}),$$

$$\varepsilon \le d(\vartheta_{p_n}, \vartheta_{q_n}) \le d(\vartheta_{p_n}, \vartheta_{p_n+1}) + d(\vartheta_{p_n+1}, \vartheta_{q_n+1}) + d(\vartheta_{q_n}, \vartheta_{q_n+1}),$$

and 2.3, 2.4, 2.5, 2.6 we have

$$\lim_{n\to+\infty} d(\vartheta_{p_n+1}, \vartheta_{q_n+1}) = \varepsilon. \tag{2.7}$$

Using $\alpha^+(\theta, \phi)$-proximality of H and substituting $\vartheta_1 = \vartheta_{p_n+1}, \vartheta_2 = \vartheta_{q_n+1}$ in 2.1, we have

$$\theta(d(\vartheta_{p_n+1}, \vartheta_{q_n+1})) \le \phi(\theta(M(\vartheta_{p_n}, \vartheta_{q_n}))), \tag{2.8}$$

where

$$\begin{aligned}
M(\vartheta_{p_n},\vartheta_{q_n}) &= \zeta_1 d(\vartheta_{p_n},\vartheta_{q_n}) + \zeta_2\Big\{d(\vartheta_{p_n},H\vartheta_{p_n}) - d(P,Q)\Big\}\\
&\quad +\zeta_3\Big\{d(\vartheta_{q_n},H\vartheta_{q_n}) - d(P,Q)\Big\} + \zeta_4\Big\{d(\vartheta_{p_n},H\vartheta_{q_n})\\
&\quad -d(P,Q)\Big\} + \zeta_5\Big\{d(\vartheta_{q_n},H\vartheta_{p_n}) - d(P,Q)\Big\}\\
&\leq \zeta_1 d(\vartheta_{p_n},\vartheta_{q_n}) + \zeta_2\Big\{\Big(d(\vartheta_{p_n},\vartheta_{p_n+1}) + d(\vartheta_{p_n+1},H\vartheta_{p_n})\Big)\\
&\quad -d(P,Q)\Big\}\\
&\quad +\zeta_3\Big\{\Big(d(\vartheta_{q_n},\vartheta_{q_n+1}) + d(\vartheta_{q_n+1},H\vartheta_{q_n})\Big) - d(P,Q)\Big\}\\
&\quad +\zeta_4\Big\{\Big(d(\vartheta_{p_n},\vartheta_{q_n+1}) + d(\vartheta_{q_n+1},H\vartheta_{q_n})\Big) - d(P,Q)\Big\}\\
&\quad +\zeta_5\Big\{\Big(d(\vartheta_{q_n},\vartheta_{p_n+1}) + d(\vartheta_{p_n+1},H\vartheta_{p_n})\Big) - d(P,Q)\Big\}\\
&= \zeta_1 d(\vartheta_{p_n},\vartheta_{q_n}) + \zeta_2\Big\{d(\vartheta_{p_n},\vartheta_{p_n+1}) + \Big(d(\vartheta_{p_n+1},H\vartheta_{p_n})\\
&\quad -d(P,Q)\Big)\Big\}\\
&\quad +\zeta_3\Big\{d(\vartheta_{q_n},\vartheta_{q_n+1}) + \Big(d(\vartheta_{q_n+1},H\vartheta_{q_n}) - d(P,Q)\Big)\Big\}\\
&\quad +\zeta_4\Big\{d(\vartheta_{p_n},\vartheta_{q_n+1}) + \Big(d(\vartheta_{q_n+1},H\vartheta_{q_n}) - d(P,Q)\Big)\Big\}\\
&\quad +\zeta_5\Big\{d(\vartheta_{q_n},\vartheta_{p_n+1}) + \Big(d(\vartheta_{p_n+1},H\vartheta_{p_n}) - d(P,Q)\Big)\Big\}\\
&= \zeta_1 d(\vartheta_{p_n},\vartheta_{q_n}) + \zeta_2 d(\vartheta_{p_n},\vartheta_{p_n+1}) + \zeta_3 d(\vartheta_{q_n},\vartheta_{q_n+1})\\
&\quad +\zeta_4 d(\vartheta_{p_n},\vartheta_{q_n+1}) + \zeta_5 d(\vartheta_{q_n},\vartheta_{p_n+1}),\ \text{by } 2.2.
\end{aligned}$$

Using 2.3, 2.4, 2.5, 2.6, 2.7 and $\zeta_1+\zeta_2+\zeta_3+\zeta_4+\zeta_5 < 1$ implies

$$\lim_{n\to+\infty} M(\vartheta_{p_n},\vartheta_{q_n}) \leq \varepsilon. \tag{2.9}$$

Again, using properties of θ,ϕ, 2.8, 2.9 and limiting we have

$$\theta(\varepsilon) \leq \phi(\theta(\varepsilon)) < \theta(\varepsilon),\ \text{since } \theta(\varepsilon) > 1$$

$$\Rightarrow \varepsilon < \varepsilon,\ \text{a contradiction.}$$

Which proves $\lim_{n\to+\infty} d(\vartheta_n,\vartheta_m) = 0$, that is $\{\vartheta_n\}$ is a Cauchy in $P_0 \subseteq V$. Being (V,d) is complete, there must be $\vartheta^* \in V$, in fact, being P is closed in V and $\{\vartheta_n\}$ is Cauchy in $P_0 \subseteq P$, ϑ^* must be in P, with

$$\lim_{n\to+\infty} d(\vartheta_n,\vartheta^*) = 0$$

$$\Rightarrow \lim_{n\to+\infty} d(H\vartheta_n,H\vartheta^*) = 0,\ \text{since } H \text{ is continuous.}$$

Again

$$\begin{aligned}
d(\vartheta_{n+1},H\vartheta_n) &= d(P,Q)\\
\Rightarrow \lim_{n\to+\infty} d(\vartheta_{n+1},H\vartheta_n) &= \lim_{n\to+\infty} d(P,Q)\\
\Rightarrow d(\vartheta^*,H\vartheta^*) &= d(P,Q).
\end{aligned}$$

This shows that ϑ^* is a best proximity point of H in P. To prove uniqueness, suppose $\vartheta^*,\vartheta^{**}$ are two distinct best proximity points of H in $P_0 \neq \emptyset$. Then $d(\vartheta^*,\vartheta^{**}) \geq 0$ and

$$d(\vartheta^*,H\vartheta^*) = d(P,Q)$$

$$d(\vartheta^{**}, H\vartheta^{**}) = d(P,Q).$$

Since the pair (P,Q) satisfies weak p-property we have

$$d(\vartheta^{*}, \vartheta^{**}) \leq d(H\vartheta^{*}, H\vartheta^{**}).$$

Using $\alpha^{+}(\theta,\phi)$-proximality of H and substituting $\vartheta^{*}, \vartheta_2 = \vartheta^{**}$ in 2.1, we have

$$\theta(d(\vartheta^{*}, \vartheta^{**})) \leq \theta(d(H\vartheta^{*}, H\vartheta^{**})) \leq \phi(\theta(M(\vartheta^{*}, \vartheta^{**}))) < \theta(M(\vartheta^{*}, \vartheta^{**})), \tag{2.10}$$

where

$$\begin{aligned}
M(\vartheta^{*}, \vartheta^{**}) &= \zeta_1 d(\vartheta^{*}, \vartheta^{**}) + \zeta_2\Big\{d(\vartheta^{*}, H\vartheta^{*}) - d(P,Q)\Big\} \\
&\quad +\zeta_3\Big\{d(\vartheta^{**}, H\vartheta^{**}) - d(P,Q)\Big\} + \zeta_4\Big\{d(\vartheta^{*}, H\vartheta^{**}) \\
&\quad -d(P,Q)\Big\} + \zeta_5\Big\{d(\vartheta^{**}, H\vartheta^{*}) - d(P,Q)\Big\} \\
&\leq \zeta_1 d(\vartheta^{*}, \vartheta^{**}) + \zeta_4\Big\{\Big(d(\vartheta^{*}, \vartheta^{**}) + d(\vartheta^{**}, H\vartheta^{**})\Big) \\
&\quad -d(P,Q)\Big\}\zeta_5\Big\{\Big(d(\vartheta^{**}, \vartheta^{*}) + d(\vartheta^{*}, H\vartheta^{*})\Big) - d(P,Q)\Big\} \\
&= \zeta_1 d(\vartheta^{*}, \vartheta^{**}) + \zeta_4\Big\{d(\vartheta^{*}, \vartheta^{**}) + \Big(d(\vartheta^{**}, H\vartheta^{**}) \\
&\quad -d(P,Q)\Big)\Big\}\zeta_5\Big\{d(\vartheta^{**}, \vartheta^{*}) + \Big(d(\vartheta^{*}, H\vartheta^{*}) - d(P,Q)\Big)\Big\} \\
&= \Big(\zeta_1 + \zeta_4 + \zeta_5\Big) d(\vartheta^{*}, \vartheta^{**}) \\
\Rightarrow \theta(M(\vartheta^{*}, \vartheta^{**})) &\leq \theta\Big(\Big(\zeta_1 + \zeta_4 + \zeta_5\Big) d(\vartheta^{*}, \vartheta^{**})\Big), \text{ by Definition 1.3.}
\end{aligned}$$

Which gives

$$\theta(d(\vartheta^{*}, \vartheta^{**})) < \theta(M(\vartheta^{*}, \vartheta^{**})) \leq \theta\Big(\Big(\zeta_1 + \zeta_4 + \zeta_5\Big) d(\vartheta^{*}, \vartheta^{**})\Big)$$

$$\Rightarrow d(\vartheta^{*}, \vartheta^{**}) < \Big(\zeta_1 + \zeta_4 + \zeta_5\Big) d(\vartheta^{*}, \vartheta^{**}), \text{ by Definition 1.3}$$

$$\Rightarrow \Big\{1 - \Big(\zeta_1 + \zeta_4 + \zeta_5\Big)\Big\} d(\vartheta^{*}, \vartheta^{**}) < 0, \text{ a contradiction,}$$

since $\zeta_1 + \zeta_4 + \zeta_5 < 1$ and $d(\vartheta^{*}, \vartheta^{**}) \geq 0$.

Hence the proximity point is unique. □

The example below illustrates Theorem 2.3.

Example 2.4. Consider the Ćirić $\alpha^{+}(\theta,\phi)$-proximal contraction defined in Example 2.2. Where $(\mathbb{R}^2, d_u)$ is a complete metric space, $P_0 = P \neq \emptyset$ and $Q_0 = Q$. Also, the pair (P,Q) meets weak p-property. That is, H satisfies all the conditions of Theorem 2.3. Clearly $(1,0)$ is a unique best proximity point of H.

Definition 2.5. A mapping $H : P \longrightarrow Q$ is known as a Chatterjea $\alpha^{+}(\theta,\phi)$-proximal contraction when it applies to non-empty subsets P,Q of any metric space (V,d) if for some $\theta \in \Theta, \phi \in \Phi$ we have

$$d(\vartheta_1, \vartheta_2) > 0 \Rightarrow \theta(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) \leq \phi(\theta(M(\vartheta_1, \vartheta_2))),$$

for all $\vartheta_1, \vartheta_2 \in P$. Where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$M(\vartheta_1, \vartheta_2) = \zeta\Big\{d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_1) - 2d(P,Q)\Big\},$$

with $0 \leq \zeta < \frac{1}{2}$.

The following is an example of a Chatterjea $\alpha^+(\theta,\phi)$-proximal contraction.

Example 2.6. Consider two non-empty subsets

$$P = \{(\vartheta_1,\vartheta_2) \in \mathbb{R}^2 : \vartheta_1 \geq 0, \vartheta_2 = 1\},$$
$$Q = \{(\vartheta_1,\vartheta_2) \in \mathbb{R}^2 : \vartheta_1 \geq 0, \vartheta_2 = 0\}$$

of the metric space $(\mathbb{R}^2, d_u)$, where d_u is usual metric of $\mathbb{R}^2$. Suppose $H : P \longrightarrow Q$ is given by

$$H(\vartheta_1,\vartheta_2) = \Big(\frac{\vartheta_1}{6}, 0\Big), \forall(\vartheta_1,\vartheta_2) \in P$$

and $\alpha^+ : P \times P \longrightarrow (-\infty,+\infty)$ is given by $\alpha(r_1,r_2) = \frac{3}{2}$. Then for $\theta(r) = \ln r \in \Theta$ and $\phi(r) = \frac{1+r}{2} \in \Phi$, H is a Chatterjea $\alpha^+(\theta,\phi)$-proximal contraction for $\zeta = \frac{1}{3}$.

Theorem 2.7. *Consider two non-empty subsets P, Q satisfying weak p-property of any complete metric space (V,d) with $P_0 \neq \emptyset$. Assume that a continuous mapping $H : P \longrightarrow Q$ is a Chatterjea $\alpha^+(\theta,\phi)$-proximal contraction so that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P,Q)$ and $\alpha(\vartheta_0,\vartheta_1) \geq 0$ for some $\vartheta_0, \vartheta_1 \in P_0$. Then unique best proximity point ϑ^* of H exists in P so that $d(\vartheta^*, H\vartheta^*) = d(P,Q)$.*

Proof. Putting $\zeta_1 = \zeta_2 = \zeta_3 = 0, \zeta_4 = \zeta_5 = \zeta$ we get the proof from the Theorem 2.3. □

The example below illustrates Theorem 2.7.

Example 2.8. Consider the Chatterjea $\alpha^+(\theta,\phi)$-proximal contraction defined in Example 2.6. Where $(\mathbb{R}^2, d_u)$ is a complete metric space, $P_0 = P \neq \emptyset$ and $Q_0 = Q$. Also, the pair (P,Q) meets weak p-property. That is, H satisfies all the conditions of Theorem 2.7. Clearly $(0,1)$ is a unique best proximity point of H.

Like Chatterjea $\alpha^+(\theta,\phi)$-proximal contraction, we now present definitions and subsequent corollaries that are analogous to Reich, Kannan, Banach, and many more.

Definition 2.9. A mapping $H : P \longrightarrow Q$ is known as a Reich $\alpha^+(\theta,\phi)$-proximal contraction when it applies to non-empty subsets P, Q of any metric space (V,d) if for some $\theta \in \Theta, \phi \in \Phi$ we have

$$d(\vartheta_1,\vartheta_2) > 0 \Rightarrow \theta(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1,\vartheta_2) \leq \phi(\theta(M(\vartheta_1,\vartheta_2)),$$

for all $\vartheta_1, \vartheta_2 \in P$. Where $\alpha : P \times P \longrightarrow (-\infty,+\infty)$ and

$$M(\vartheta_1,\vartheta_2) = \zeta_1 d(\vartheta_1,\vartheta_2) + \zeta_2\Big\{d(\vartheta_1, H\vartheta_1) - d(P,Q)\Big\} + \zeta_3\Big\{d(\vartheta_2, H\vartheta_2) - d(P,Q)\Big\},$$

with $\zeta_1, \zeta_2, \zeta_3 \geq 0, \zeta_1 + \zeta_2 + \zeta_3 < 1, \zeta_3 \neq 1$.

Corollary 2.10. *Consider two non-empty subsets P, Q satisfying weak p-property of any complete metric space (V,d) with $P_0 \neq \emptyset$. Assume that a continuous mapping $H : P \longrightarrow Q$ is a Reich $\alpha^+(\theta,\phi)$-proximal contraction so that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P,Q)$ and $\alpha(\vartheta_0,\vartheta_1) \geq 0$ for some $\vartheta_0, \vartheta_1 \in P_0$. Then unique best proximity point ϑ^* of H exists in P so that $d(\vartheta^*, H\vartheta^*) = d(P,Q)$.*

Proof. Putting $\zeta_4 = \zeta_5 = 0$ in Theorem 2.3, we get the proof. □

Definition 2.11. A mapping $H : P \longrightarrow Q$ is known as a Kannan $\alpha^+(\theta,\phi)$-proximal contraction when it applies to non-empty subsets P, Q of any metric space (V,d) if for some $\theta \in \Theta, \phi \in \Phi$ we have

$$d(\vartheta_1,\vartheta_2) > 0 \Rightarrow \theta(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1,\vartheta_2) \leq \phi(\theta(M(\vartheta_1,\vartheta_2))),$$

for all $\vartheta_1, \vartheta_2 \in P$. Where $\alpha : P \times P \longrightarrow (-\infty,+\infty)$ and

$$M(\vartheta_1,\vartheta_2) = \zeta\Big\{d(\vartheta_1, H\vartheta_1) + d(\vartheta_2, H\vartheta_2) - 2d(P,Q)\Big\},$$

with $\zeta \in [0, \frac{1}{2})$.

Corollary 2.12. *Consider two non-empty subsets P, Q satisfying weak p-property of any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that a continuous mapping $H : P \longrightarrow Q$ is a Kannan $\alpha^+(\theta,\phi)$-proximal contraction so that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P,Q)$ and $\alpha(\vartheta_0,\vartheta_1) \geq 0$ for some $\vartheta_0, \vartheta_1 \in P_0$. Then unique best proximity point ϑ^* of H exists in P so that $d(\vartheta^*, H\vartheta^*) = d(P,Q)$.*

Proof. Putting $\zeta_1 = \zeta_4 = \zeta_5 = 0, \zeta_2 = \zeta_3$ in Theorem 2.3, we get the proof. □

Definition 2.13. A mapping $H : P \longrightarrow Q$ is known as a Banach $\alpha^+(\theta,\phi)$-proximal contraction when it applies to non-empty subsets P, Q of any metric space (V, d) if for some $\theta \in \Theta, \phi \in \Phi$ we have

$$d(\vartheta_1,\vartheta_2) > 0 \Rightarrow \theta(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1,\vartheta_2) \leq \phi(\theta(M(\vartheta_1,\vartheta_2))),$$

for all $\vartheta_1,\vartheta_2 \in P$. Where $\alpha : P \times P \longrightarrow (-\infty,+\infty)$ and

$$M(\vartheta_1,\vartheta_2) = \zeta d(\vartheta_1,\vartheta_2),$$

with $0 \leq \zeta < 1$.

Corollary 2.14. *Consider two non-empty subsets P, Q satisfying weak p-property of any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that a continuous mapping $H : P \longrightarrow Q$ is a Banach $\alpha^+(\theta,\phi)$-proximal contraction so that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P,Q)$ and $\alpha(\vartheta_0,\vartheta_1) \geq 0$ for some $\vartheta_0, \vartheta_1 \in P_0$. Then unique best proximity point ϑ^* of H exists in P so that $d(\vartheta^*, H\vartheta^*) = d(P,Q)$.*

Proof. Putting $\zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = 0$, in Theorem 2.3, we get the proof. □

In Theorem 2.3, if $\alpha = 0$ on P, then we get a new result.

Corollary 2.15. *Consider two non-empty subsets P, Q satisfying weak p-property of any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that a continuous mapping $H : P \longrightarrow Q$ is so that, for some $\theta \in \Theta, \phi \in \Phi$ we have*

$$d(\vartheta_1,\vartheta_2) > 0 \Rightarrow \theta(d(H\vartheta_1, H\vartheta_2)) \leq \phi(\theta(M(\vartheta_1,\vartheta_2))),$$

for all $\vartheta_1,\vartheta_2 \in P$ with $H(P_0) \in Q_0$, $d(\vartheta_3, H\vartheta_4) = d(P,Q)$ for some $\vartheta_3, \vartheta_4 \in P_0$. Where

$$\begin{aligned} M(\vartheta_1,\vartheta_2) \;=\; & \zeta_1 d(\vartheta_1,\vartheta_2) + \zeta_2\Big\{d(\vartheta_1,H\vartheta_1) - d(P,Q)\Big\} + \zeta_3\Big\{d(\vartheta_2,H\vartheta_2) \\ & -d(P,Q)\Big\} + \zeta_4\Big\{d(\vartheta_1,H\vartheta_2) - d(P,Q)\Big\} + \zeta_5\Big\{d(\vartheta_2,H\vartheta_1) \\ & -d(P,Q)\Big\}, \end{aligned}$$

with $\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5 \geq 0, \zeta_1+\zeta_2+\zeta_3+2\zeta_4 < 1, \zeta_4 = \zeta_5, \zeta_3+\zeta_4 \neq 1$. Then best proximity point ϑ^* of H exists in P so that $d(\vartheta^*, H\vartheta^*) = d(P,Q)$. In fact, the best proximity point is unique in P.

Remark 2.16. Following the Corollary 2.15 if $\alpha = 0$ on P, we can get more corollaries from the Theorems 2.7 and Corollaries 2.10, 2.12, 2.14.

3 Application

Integral equations are typically essential when solving differential equations in mathematics. Many writers have used methods from proximity point theory to tackle integral problems. In order to show the usefulness of our findings, we explore that there is a solution to the following integral equation over the space of all real-valued continuous functions on $V = [0, r]$

$$\vartheta(v) = f(v) + \int_0^r g(v,u)\gamma(u,\vartheta(u))du, \tag{3.1}$$

where $f : V \longrightarrow \mathbb{R}, g : V \times V \longrightarrow [0, +\infty]$ and $\gamma : V \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

Let $(C(V), ||\cdot||)$ be the complete normed linear space of all real-valued continuous functions on V with respect to usual "sup" norm and define $H : C(V) \longrightarrow C(V)$ by

$$H\vartheta(v) = f(v) + \int_0^r g(v, u)\gamma(u, \vartheta(u))du.$$

Being f, g, γ continuous, the function H is well defined. Then to find a solution of 3.1 is same as to find a best proximity point of H i.e., a solution to the equation $d(H\vartheta(v), \vartheta(v)) = d(C(V), C(V))$, where metric d is given by $d(\vartheta_1, \vartheta_2) = ||\vartheta_1 - \vartheta_2||$.

Now we state our result.

Theorem 3.1. *The problem 3.1 will have unique solution if for some* $0 < \zeta < 1$*, the mappings* $g : V \times V \longrightarrow [0, +\infty], \gamma : V \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies*

(a) $\int_0^r g(v, u)du \leq 1,$

(b) $|\gamma(u, \vartheta_1(v)) - \gamma(u, \vartheta_2(v)))| \leq \zeta|\vartheta_1(v) - \vartheta_2(v)| - \tau$*, for* τ *with* $e^\tau = 2$*,*

for all $u \in V; \vartheta_1(v), \vartheta_2(v) \in C(V)$*.*

Proof. From the definition of H, for $\vartheta_1(v), \vartheta_2(v) \in C(V)$ with $d(\vartheta_1, \vartheta_2) > 0$, we have

$$\begin{aligned} |H\vartheta_1(v) - H\vartheta_2(v)| &= \Big|\int_0^r g(v, u)(\gamma(u, \vartheta_1(u)) - \gamma(u, \vartheta_2(u)))du\Big| \\ &\leq \int_0^r g(v, u)|\gamma(u, \vartheta_1(u)) - \gamma(u, \vartheta_2(u)))|du \\ &\leq \int_0^r g(v, u)(\zeta|\vartheta_1(u) - \vartheta_2(u)| - \tau)du, \text{ by (b)}. \end{aligned}$$

Taking supremum both sides and using condition (a), we have

$$\begin{aligned} d(H\vartheta_1, H\vartheta_2) &\leq \zeta d(\vartheta_1, \vartheta_2) - \tau \\ \Rightarrow e^{d(H\vartheta_1, H\vartheta_2)} + \frac{1}{2} &\leq 1 + \frac{e^{\zeta d(\vartheta_1, \vartheta_2)} - 1}{2} \end{aligned}$$

Now if $\theta(r) = e^r, \phi(r) = 1 + \frac{r-1}{2}$, for $\zeta = \zeta_1$, and any choice of $\zeta_2, \zeta_3, \zeta_4, \zeta_4 \geq 0$, H satisfies the condition

$$\theta(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) \leq \phi(\theta(M(\vartheta_1, \vartheta_2)))$$

of the Theorem 2.3 for $\alpha(\vartheta_1, \vartheta_2) = \frac{1}{2}, \forall(\vartheta_1(v), \vartheta_2(v)) \in C(V) \times C(V)$. Thus there is a unique best proximity point of H i.e., a solution of 3.1. □

4 Conclusion

Starting with the introduction of Ćirić $\alpha^+(\theta, \phi)$-proximal contraction, a new class of non-self generalized proximal contraction mapping on a non-empty closed subset of any metric space, we have proved that such contractions satisfying some conditions must have unique best proximity point if we take the base space as complete. We arrived at several distinct forms of proximal contractions for certain values of the constants that we utilized to generalize the proximal contraction. We substantiated the deduced findings with examples. There is an application discussed to demonstrate the solution to an integral equation over the space of all real-valued continuous functions in the context of proximity point results. We hope that by showcasing these findings, we can inspire a new generation of academics to continue investigating this fascinating field with so many potential uses.

5 Authors' contributions

Conceptualization, K.H.A. and Y.R. Formal analysis, Y.R. Investigation, Y.R. and K.H.A. Writing-original draft preparation, K.H.A. Writing-review and editing, Y.R. and K.H.A. All authors have read and agreed to the published version of the manuscript.

References

[1] K. H. Alam, Y. Rohen and N. Saleem, Fixed points of (α, β, F^*) and (α, β, F^{**})-weak Geraghty contractions with an application, Symmetry **15(1)**, 243, 1–17, (2023).

[2] A. Eldred, W. Kirk and P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Mathematica, **3(171)**, 283–293, (2005).

[3] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae, **3(1)**, 133–181, (1922).

[4] S. Chandok, T. D. Narang and M. A. Taoudi, *Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type*, Palestine Journal of Mathematics, **4(2)**, 360–366, (2015).

[5] H. Isik, A. Ali, F. Uddin, A. Asif and M Arshad, *Best proximity points for multivalued mappings satisfying z_σ-proximal contractions with applications*, In Metric Fixed Point Theory, Springer, 227–245, (2021).

[6] A. Tomar, R. Sharma and A. H. Ansari, *Strict coincidence and common strict fixed point of a faintly compatible hybrid pair of maps via $C-$class function and applications*, Palestine Journal of Mathematics, **9(1)**, 274–288, (2020).

[7] M. B. Devi, N. Priyobarta and Y. Rohen, *Some common best proximity points theorems for generalised rational $\alpha - \varphi$-geraghaty proximal contractions*, J. Math. Comput. Sci., **10(3)**, 713–727, (2020).

[8] N. Saleem, H. Isik, S. Khaleeq and C. Park, *Interpolative ćirić-reich-rus-type best proximity point results with applications*, AIMS Mathematics, **7(6)**, 9731–9747, (2022).

[9] N. Hussain, M. Hezarjaribi, M. A. Kutbi and P. Salimi, *Best proximity results for Suzuki and convex type contractions*, Fixed Point Theory and Applications, **2016(1)**, 1–20, (2016).

[10] M. P. Singh, Y. Rohen, N. Saleem, K. H. Alam, K. A. Singh and A. Razzaque, On fixed-point equations involving Geraghty-type contractions with solution to integral equation. Mathematics, **11(24)**, 4882, 1–18, (2023).

[11] J. Zhang, Y. Su and Q. Cheng, *A note on a best proximity point theorem for Geraghty-contractions*, Fixed Point Theory and Applications, **2013(1)**, 1–4, (2013).

[12] M. Jleli and B. Samet, *A new generalization of the banach contraction principle*, Journal of Inequalities and Applications, **2014(1)**, 1–8, (2014).

[13] D. Zheng, Z. Cai and P. Wang, *New fixed point theorems for $\theta - \varphi$-contraction in complete metric spaces*, Journal of Nonlinear Sciences & Applications, **10(5)**, (2017).

[14] S. Petwal, A. Tomar and M. Joshi, *Best proximant for set-valued maps via proximal relations*, Int. J. Nonlinear Anal. Appl., **14(1)**, 1237–1247, (2023).

[15] V. Berinde and F. Takens, *Iterative approximation of fixed points*, **1912**, Berlin: Springer, (2007).

[16] J. Ahmad, K. Ullah and M. Arshad, *Approximating fixed points of mappings satisfying conditions (CC) in Banach spaces*, Palestine Journal of Mathematics, **11(3)**, 127–132, (2022).

Author information

Khairul Habib Alam, Department of Mathematics, National Institute of Technology Manipur, Langol, Imphal, 795004, India.
E-mail: `almkrlhb@gmail.com/alamkh@gmx.co.uk`

Yumnam Rohen, Department of Mathematics, Manipur University, Canchipur, Imphal, 795003, India.
E-mail: `ymnehor2008@yahoo.com`

Received: 2023-01-20
Accepted: 2024-01-02