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MJP-INJECTIVE RINGS AND MJGP-INJECTIVE RINGS

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Abstract A ring R is called right monomorphism JP-injective (or MJP-injective for short), if, for any $a \in J(R)$, every right R-monomorphism from aR to R extends to an endomorphism of R. A ring R is called right monomorphism JGP-injective (or MJGP-injective for short), if, for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-monomorphism from a^nR to R extends to an endomorphism of R. In this paper, several properties of the two classes rings are given. Moreover, some new characterizations of quasi-Frobenius rings are obtained.

1 Introduction

Throughout this paper, R is an associative ring with identity, and all modules are unitary. As usual, $J(R)$ (or J), Z_r and S_r denote respectively the Jacobson radical, the right singular ideal and the right socle of R. The left (respectively, right) annihilators of a subset X of R is denoted by $\mathbf{l}(X)$ (respectively, $\mathbf{r}(X)$).

Recall that R is called a *right principally injective (briefly right P-injective) ring* [7] if every R-homomorphism from a principal right ideal of R into R extends to R; In [11], the concept of right P-injective rings is generalized to *right JP-injective rings*. Following [11], a ring R is called a *right JP-injective ring* if every R-homomorphism from a principal right ideal in $J(R)$ into R extends to R. We recall also that a ring R is called a *right JGP-injective) ring* [11] , if, for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. In this paper, we shall generalize the concepts of right JP-injective rings and right JGP-injective rings to *right MJP-injective rings* and right *MJGP-injective rings* respectively, and give some properties of these rings. Moreover, right MJGP-injective left noethrian rings will be investigated, and quasi-Frobenius rings will be characterized by right MJGP-injective rings. Concepts which have not been explained can be found in [9].

2 MJP-injective rings

We start with the following definition.

Definition 2.1. Let R be a ring. A right R-module N is called JP-injective if for any $a \in J(R)$, every homomorphism from aR to N extends to a homomorphism of R to N. A right R-module N is called MJP-injective if for any $a \in J(R)$, every monomorphism from aR to N extends to a homomorphism of R to N. R is called right MJP-injective if R_R is MJP-injective.

Theorem 2.2. *The following conditions are equivalent for a ring R.*
(1) R is right MJP-injective.
(2) $\mathbf{r}(a) = \mathbf{r}(b), a \in J(R), b \in R$, implies that $Ra = Rb$.

Proof. (1) $\Rightarrow$ (2). If $\mathbf{r}(a) = \mathbf{r}(b), a \in J(R)$, then the mapping $f : aR \to bR$, $ar \mapsto br$ is a monomorphism. Since R is right MJP-injective, $f = c\cdot$ for some $c \in R$, and so $b = ca$. This implies that $Rb \subseteq Ra$. Observing that $b \in J(R)$, we have that $Ra \subseteq Rb$ by a similar way as the above proof.

(2) $\Rightarrow$ (1). Let $f : aR \to R$ be monic, where $a \in J(R)$. Then $\mathbf{r}(a) = \mathbf{r}(f(a))$. By (2), $Ra = Rf(a)$, so $f(a) = ca$ for some $c \in R$. Hence f is left multiplication by c, as required. $\square$

Theorem 2.3. *Let R be a right MJP-injective ring, and T be a right ideal of R contained in J(R). If T is isomorphic to a direct summand of R_R, then $T = 0$.*

Proof. If $T \cong eR$, where $e^2 = e \in R$, then $T = aR$ for some $a \in J(R)$ and T is projective. Hence $\mathbf{r}(a) \subseteq^{\oplus} R_R$. Write $\mathbf{r}(a) = fR$, where $f^2 = f \in R$. Then $\mathbf{r}(a) = \mathbf{r}(1 - f)$. By Theorem 2.2, $Ra = R(1 - f) \subseteq^{\oplus} {}_RR$, and so $a = 0$. Thus, $T = aR = 0$. $\square$

Corollary 2.4. *If R is a right MJP-injective ring, then the following are equivalent for an element $a \in J(R)$:*
(1) aR is projective.
(2) $a = 0$.
(3) aR is an MJP-injective module.

Theorem 2.5. *Let R be right MJP-injective, and assume that the sum $\sum_{i=1}^{n} Rb_i$ is direct, $b_i \in J(R)$. Then any monomorphism $\alpha : \sum_{i=1}^{n} b_iR \to R$ can be extended to R.*

Proof. For each i, $\alpha(b_i) = a_ib_i$ for some $a_i \in R$ by hypothesis, and similarly $\alpha(b_1 + \cdots + b_n) = a(b_1 + \cdots + b_n)$ for some $a \in R$. Thus $a_1b_1 + a_2b_2 + \cdots + a_nb_n = ab_1 + ab_2 + \cdots + ab_n$. Since $\sum_{i=1}^{n} Rb_i$ is direct , so $a_ib_i = ab_i$ for each i. Hence $\alpha = a\cdot$. $\square$

Proposition 2.6. *If R is right MJP-injective and $e^2 = e \in R$ with $ReR = R$, then eRe is right MJP-injective.*

Proof. Write $S = eRe$, then $J(S) = eJ(R)e$. Let $\mathbf{r}_S(a) = \mathbf{r}_S(b)$, where $a \in J(S), b \in S$. Then $a \in J(R)$. Let $bx = 0, x \in R$, and write $1 = \sum_{i=1}^{n} p_ieq_i$, where $p_i, q_i \in R$. Then $b(exp_ie) = bxp_ie = 0$ for each i and so $a(exp_ie) = 0$ by hypothesis. Hence $ax = \sum_{i=1}^{n} axp_ieq_i = 0$. So $\mathbf{r}_R(b) \subseteq \mathbf{r}_R(a)$. Similarly, $\mathbf{r}_R(a) \subseteq \mathbf{r}_R(b)$. Hence, $\mathbf{r}_R(a) = \mathbf{r}_R(b)$. Since R is right MJP-injective , $Ra = Rb$, so $a = ea \in eRb = Sb$, and $b = eb \in eRa = Sa$, as required. $\square$

Lemma 2.7. *Let M be a right R-module such that $R \oplus M$ is MJP-injective. Then M is JP-injective.*

Proof. Let $a \in J(R)$ and $f : aR \to M$ be a right R-homomorphism. Define $g : aR \to R \oplus M$ by $g(x) = (x, f(x))$, then g is a monomorphism. Since $R \oplus M$ is MJP-injective, g can be extended to a homomorphism h of R to $R \oplus M$. Write $h(1) = (b, m)$, then for any $x \in aR$, we have $(x, f(x)) = g(x) = h(x) = h(1)x = (b, m)x = (bx, mx)$, and so $f(x) = mx$. Thus f can be extended to a homomorphism from R to M, and hence M is JP-injective. $\square$

It is easy to see that every direct sum of JP-injective modules is JP-injective. So, by Lemma 2.7, we have immediately the following corollaries.

Corollary 2.8. *Let R be a right MJP-injective ring. Then the following statements are equivalent:*
(1) Every direct sum of MJP-injective right R-modules is MJP-injective;
(2) Every MJP-injective right R-module is JP-injective.

Corollary 2.9. *A ring R is right JP-injective if and only if the right R-module R^2 is MJP-injective.*

Lemma 2.10. *If the full matrix ring $\mathbb{M}_n(R)$ is right MJP-injective, then the right R-module R^n is MJP-injective.*

Proof. Let $a \in J(R)$ and $f : aR \to R^n$ be a right R-monomorphism. Write $f(a) = (a_1, a_2, \cdots, a_n)$, and define $g : (\sum_{i=1}^{n} E_{i1}a)\mathbb{M}_n(R) \to \mathbb{M}_n(R); (\sum_{i=1}^{n} E_{i1}a)X \mapsto (\sum_{i=1}^{n} E_{i1}a_i)X$, where $X \in \mathbb{M}_n(R)$, then g is a right $\mathbb{M}_n(R)$-monomorphism. Since $\mathbb{M}_n(R)$ is MJP-injective and $\sum_{i=1}^{n} E_{i1}a \in J(\mathbb{M}_n(R))$, there exists $B = (b_{ij})_{nn} \in \mathbb{M}_n(R)$ such that $g = B\cdot$. Thus $\sum_{i=1}^{n} E_{i1}a_i = B(\sum_{i=1}^{n} E_{i1}a)$, and so $a_i = b_ia$, where $b_i = \sum_{j=1}^{n} b_{ij}, i = 1, 2, \cdots, n$. Hence, for every $r \in R$, we have $f(ar) = f(a)r = (a_1, a_2, \cdots, a_n)r = (b_1, b_2, \cdots, b_n)ar$, so that f extends to a homomorphism of R to R^n. Therefore, as a right R-module, R^n is MJP-injective. $\square$

Theorem 2.11. *If there exists a positive integer $n \geq 2$ such that the full matrix ring $\mathbb{M}_n(R)$ is right MJP-injective, then R is right JP-injective.*

Proof. Since the full matrix ring $\mathbb{M}_n(R)$ is right MJP-injective, by Lemma 2.10, as a right R-module, R^n is MJP-injective. So, by Lemma 2.7, the right R-module R^{n-1} is JP-injective, and hence R is right JP-injective. □

Recall that a ring R is called *right MP-injective* [13] if, for any $a \in R$,any right R-monomorphism from aR to R extends to an endomorphism of R; a ring R is called *J-regular* [10] if $R/J(R)$ is regular; a submodule N of M has a *weak supplement* L in M if $N + L = M$ and $N \cap M << L$ [5].

Theorem 2.12. *If R is J-regular, then R is right MP-injective if and only if R is right MJP-injective.*

Proof. The necessity is obvious. For the sufficient part, assume f is a monomorphism from aR to R. Since R is J-regular, by [5, Proposition 3.18], aR has a weak supplement in R_R. That is, there exists a right ideal K of R such that $aR + K = R$ and $aR \cap K \subseteq J$. Thus, there are $r \in R$ and $b \in K$ such that $ar + b = 1$ and $aR \cap bR \subseteq aR \cap K \subseteq J$. By [5, Lemma 3.4], $aR \cap bR = baR$. Let $f_1 = f\mid_{baR}$, then f_1 is a monomorphism from baR to R. Since R is right MJP-injective, there exists an endomorphism g of R such that $g\mid_{baR} = f_1$. Note that $aR + bR = R$, for each $x \in R$, there exist $x_1 \in aR$ and $x_2 \in bR$ such that $x = x_1 + x_2$. Now we define $\varphi : R_R \to R_R; x \mapsto f(x_1) + g(x_2)$, then it is easy to see that φ is a well-defined endomorphism of R_R which extends f. □

3 MJGP-injective rings

Recall that a ring R is called *right GP-injective* [1, 3, 6] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R. The concept of GP-injective rings has been generalized in several ways. For example, a ring R is called *right JGP-injective* [11] if, for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R; a ring R is called *right MGP-injective* [13] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-monomomorphism from a^nR to R extends to an endomorphism of R.

Definition 3.1. Let R be a ring. A right R-module N is called right MJGP-injective if for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any R-monomorphism from a^nR to N extends to a homomorphism of R to N. The ring R is called right MJGP-injective if R_R is MJGP-injective.

Theorem 3.2. *The following conditions are equivalent for a ring R:*
(1) R is right MJGP-injective.
(2) For any $0 \neq a \in J(R)$, there exists $n > 0$ such that $a^n \neq 0$ and $b \in Ra^n$ for every $b \in R$ with $\mathbf{r}(a^n) = \mathbf{r}(b)$.

Proof. (1)⇒ (2). Let $0 \neq a \in J(R)$. Since R is right MJGP-injective, there exists a positive integer n, such that $a^n \neq 0$ and every monomorphism from a^nR to R extends to R. Suppose that $\mathbf{r}(a^n) = \mathbf{r}(b)$. Then $f : a^nR \to R, a^nr \mapsto br$, is a monomorphism, which extends to an endomorphism g of R. So $b = f(a^n) = g(a^n) = g(1)a^n \in Ra^n$.

(2)⇒ (1). Let $0 \neq a \in J(R)$. By (2), there exists $n > 0$ such that $a^n \neq 0$ and $b \in Ra^n$ for any $b \in R$ with $\mathbf{r}(a^n) = \mathbf{r}(b)$. Let $f : a^nR \to R$ be monic. Then $\mathbf{r}(a^n) = \mathbf{r}(f(a^n))$, and so $f(a^n) = ca^n$ for some $c \in R$. It follows that $f = c\cdot$, as requied. □

Example 3.3. Let $M = \oplus_{i=1}^{\infty}\mathbb{Z}_{p_i}$, where p_i is the ith prime number, and let

$$R = \left\{ \left[\begin{array}{cc} n & x \\ 0 & n \end{array} \right] \middle| n \in \mathbb{Z}, x \in M \right\}.$$

Then R is right MJGP-injective , but R is not right MGP-injective.

Proof. By [11, Example 3.1], R is right *JP-injective*, so it is right MJGP-injective . But R is not right MGP-injective by [13, Example 3.3]. □

Theorem 3.4. *Let R be right MJGP-injective. Then*
(1) If $0 \neq a \in J(R)$, then $\mathbf{r}(a) \neq 0$.
(2) $J(R) \subseteq Z_r$.

Proof. (1) If $\mathbf{r}(a) = 0$. Then since R is right MJGP-injective, there exists a positive integer n such that $a^n \neq 0$ and every monomorphism from a^nR to R extends to R. Define $f : a^nR \to R, a^nx \mapsto x$. Then f is a monomorphism, and hence it extends to an endomorphism g of R. Thus $1 = f(a^n) = g(a^n) = g(1)a^n \in J(R)$, a contradiction.

(2) Let $a \in J(R)$, then we will show that $a \in Z_r$. If not, then there exists $0 \neq b \in R$ such that $\mathbf{r}(a) \cap bR = 0$. Clearly $ab \neq 0$. Since R is right MJGP-injective, there exists a positive integer n such that $(ab)^n \neq 0$ and $c \in R(ab)^n$ for any $c \in R$ with $\mathbf{r}((ab)^n) = \mathbf{r}(c)$. Now let $c = b(ab)^{n-1}$. Then $\mathbf{r}((ab)^n) = \mathbf{r}(c)$, and so $c = d(ab)^n$ for some $d \in R$. Thus $(1 - da)c = 0$. Since $a \in J(R)$, $1 - da$ is invertible, and so $c = 0$. Hence $(ab)^n = ac = 0$, a contradiction. □

Recall that a ring R is called *right C_2* [11, 12] if every right ideal that is isomorphic to a direct summand of R is itself a direct summand of R; a ring R is called *right GC_2* [11, 12] if every right ideal that is isomorphic to R is itself a direct summand of R: a ring R is called *left (resp., right) Kasch* if every simple left (resp., right) R-module can be embedded in R_R (resp., ${}_RR$).

Corollary 3.5. *If R is a right MJGP-injective and right GC2 ring, then $J(R) = Z_r$. In particular, if R is a right MJGP-injective left Kasch ring, then $J(R) = Z_r$.*

Proof. Since R is right GC2, by [11, Proposition 2.6], $J(R) \supseteq Z_r$. Since R is right MJGP-injective, by Theorem 3.4, $J(R) \subseteq Z_r$. So $J(R) = Z_r$. If R is a left Kasch ring, then by [9, Proposition 1.46], R is right C2, and hence right GC2, so the last assertion follows. □

Recall that a ring R is called *right mininjective* [8] if every R-homomorphism from a minimal right ideal of R into R extends to R.

Theorem 3.6. *Let R be a right MJGP-injective ring. Then R is right mininjective.*

Proof. Let aR be simple. If $(aR)^2 \neq 0$, then $aR = eR$ for an idempotent $e \in R$. Thus, every R-homomorphism from aR to R extends to R. If $(aR)^2 = 0$, then $a \in J(R)$. Since R is right MJGP-injective, there exists a positive integer n, such that $a^n \neq 0$ and every right R-monomomorphism from a^nR to R extends to an endomorphism of R. Noting that $a^nR = aR$ because aR is simple, so every right R-homomorphism from aR to R extends to R. □

A ring R is called *right CF* [9] if every cyclic right R-module embeds in a free module; a ring R is called *semiregular* if $R/J(R)$ is regular and idempotents can lifted modulo $J(R)$ [9]. Our next result improves [2, Corollary 2.10].

Corollary 3.7. *Let R be a right CF, semiregular, right MJGP-injective ring. Then it is QF.*

Proof. Since R is right MJGP-injective, by Theorem 3.4 and Theorem 3.6, R is right mininjective and $J(R) \subseteq Z_r$. By [2, Corollary 2.9], every right CF, semiregular ring with $J(R) \subseteq Z_r$ is right artinian, so R is right artinian. Note that right CF ring is left P-injective by [9, Lemma 7.2 (1)], so R is left and right mininjective right artinian ring, and hence R is QF by [8, Corollary 4.8]. □

Lemma 3.8. *Let M be a right R-module with a left noetherian endomorphism ring $S = \mathrm{End}(M_R)$. If ${}_SM$ is finitely generated , I is a right ideal of S and $\mathbf{r}_M(I) \subseteq^{ess} M_R$, then I is nilpotent.*

Proof. Since I is a right ideal of S, $\mathbf{r}_M(I^i)$ is a submodule of ${}_SM$ for each positive integer i. Since S is left noetherian and ${}_SM$ is finitely generated , ${}_SM$ is a noetherian module, and so there exists $k \geq 1$ such that $\mathbf{r}_M(I^k) = \mathbf{r}_M(I^{k+1}) = \cdots$. If I is not nilpotent, choose $\mathbf{l}_S(x)$ maximal in $\{\mathbf{l}_S(y) \mid I^ky \neq 0\}$. Then $I^{2k}x \neq 0$ because $\mathbf{r}_M(I^{2k}) = \mathbf{r}_M(I^k)$, so there exists $a \in I^k$ such that $I^kax \neq 0$. Observing that $\mathbf{r}_M(I) \subseteq \mathbf{r}_M(I^k)$ and $\mathbf{r}_M(I) \subseteq^{ess} M_R$, we have that $\mathbf{r}_M(I^k) \subseteq^{ess} M_R$. Thus $axR \cap \mathbf{r}_M(I^k) \neq 0$, say $0 \neq axb \in \mathbf{r}_M(I^k)$ for some $b \in R$, then, $I^k xb \neq 0$ as $0 \neq axb \in I^k xb$, and $I^k a \subseteq \mathbf{l}_S(xb)$ but $I^k a \nsubseteq \mathbf{l}_S(x)$, which contradicts the maximality of $\mathbf{l}_S(x)$. Therefore I is nilpotent. □

Theorem 3.9. *Let R be a left noetherian right MJGP-injective ring. Then:*
(1) $\mathbf{r}(J) \subseteq^{ess} R_R$.
(2) J is nilpotent .
(3) $\mathbf{r}(J) \subseteq^{ess} {}_RR$.

Proof. (1). Let $0 \neq x \in R$. Since R is left noetherian, the non-empty set $F = \{\mathbf{l}(xa) \mid a \in R$ such that $xa \neq 0\}$ has a maximal element, say $\mathbf{l}(xy)$.

We claim that $Jxy = 0$. If not, then there exists $t \in J$ such that $txy \neq 0$. Since R is right MJGP-injective, there exists a positive integer n such that $(txy)^n \neq 0$ and $b \in R(txy)^n$ for every $b \in R$ with $\mathbf{r}((txy)^n) = \mathbf{r}(b)$. Write $s = (txy)^{n-1}t$, then $s \in J$ and $(txy)^n = sxy$. We proceed with the following two cases.

Case 1. $\mathbf{r}(xy) = \mathbf{r}(sxy)$. Then $xy = csxy$ for some $c \in R$, i.e., $(1 - cs)xy = 0$. Since $s \in J$, $1 - cs$ is invertible. So we have $xy = 0$. This is a contradiction.

Case 2. $\mathbf{r}(xy) \neq \mathbf{r}(sxy)$. Then there exists $u \in \mathbf{r}(sxy)$ but $u \notin \mathbf{r}(xy)$. Thus, $sxyu = 0$ and $xyu \neq 0$. This shows that $s \in \mathbf{l}(xyu)$ and $\mathbf{l}(xyu) \in F$. Noting that $s \notin \mathbf{l}(xy)$, the inclusion $\mathbf{l}(xy) \subset \mathbf{l}(xyu)$ is strict. This contracts the maximality of $\mathbf{l}(xy)$ in F.

Thus, $Jxy = 0$, and so $0 \neq xy \in xR \cap \mathbf{r}(J)$, proving (1).

(2). By Lemma 3.8.

(3). If $0 \neq c \in R$, we must show that $Rc \cap \mathbf{r}(J) \neq 0$. In fact, if $Jc = 0$, then $0 \neq c \in Rc \cap \mathbf{r}(J)$. If $Jc \neq 0$. Then since J is nilpotent, there exists $m \geq 1$ such that $J^m c \neq 0$ but $J^{m+1}c = 0$, and so $0 \neq J^m c \subseteq Rc \cap \mathbf{r}(J)$, as required. □

Recall that a ring R is *right minfull* [8] if it is semiperfect, right mininjective and $Soc(eR) \neq 0$ for each local idempotent $e \in R$; a ring R is called *left Johns* [4] if it is left noetherian and every left ideal is a left annihilator.

Theorem 3.10. *Let R be a left noetherian right MJGP-injective ring. Then the following statements are equivalent:*
(1) R is right Kasch.
(2) R is left C_2.
(3) R is left GC_2.
(4) R is semilocal.
(5) R is left artinian.

Proof. (1)⇒ (2). By [9, Proposition 1.46].

(2)⇒ (3) is obvious.

(3)⇒ (4). Since left noetherian ring is left finite dimensional, and left finite dimensional left GC_2 ring is semilocal [12, Lemma 1.1], so (4) follows from (3).

(4)⇒ (5). Since R is left noetherian right MJGP-injective, By Theorem 3.9(2), J is nilpotent. Thus R is a left noetherian semiprimary by ring, i.e., it is left artinian.

(5)⇒ (1). Assume (5). Then R is semiperfect right mininjective ring and $S_r \subseteq^{ess} R_R$. So that R is a right minfull ring. By [8, Theorem 3.7(1)], R is right Kasch. □.

Theorem 3.11. *Let R be a left noetherian left mininjective right MJGP-injective ring. Then the following statements are equivalent:*
(1) R is a quasi-Frobenius ring.
(2) R is right Kasch.
(3) R is left C_2.
(4) R is left GC_2.
(5) R is semilocal.
(6) R is left artinian.

Proof. The equivalence of (2), (3), (4), (5), (6) follows from Theorem 3.10.

$(1) \Rightarrow (6)$ is obvious.

(6)$\Rightarrow$ (1). Since R is right MJGP-injective, by Theorem 3.6, R is right mininjective. Thus R is a two-sided mininjective left artinian ring, and so it is a quasi-Frobenius ring by [8, Corollary 4.8]. □

Lemma 3.12. *Let R be a left perfect right mininjective ring. Then R is right Kasch.*

Proof. By hypothesis, R is a semiperfect right mininjective ring with essential right socle, so it is a right minfull ring. Hence, by [8, Theorem 3.7], R is right Kasch. □

Theorem 3.13. *The following statements are equivalent for a ring R:*
(1) R is a quasi-Frobenius ring.
(2) R is left Johns and left MGP-injective.
(3) $M_2(R)$ is left Johns and R is left GC_2.
(4) R is left artinian and right 2-injective.
(5) R is left noetherian right finite dimensional and right 2-injective.
(6) R is left noetherian left mininjective and right MGP-injective.

Proof. $(1) \Rightarrow (2), (3), (5), (6)$ are obvious.

$(2) \Rightarrow (1)$. Since R is left Johns, it is left noetherian and right P-injective, and so it is left noetherian and right MJGP-injective. Since R is left MGP-injective, by Theorem 3.6, R is left mininjective, and by [13, Theorem 3.4], R is left GC_2. And so, by Theorem 3.11, R is a quasi-Frobenius ring.

$(3) \Rightarrow (4)$. Since $M_2(R)$ is left Johns, R is left noetherian and right 2-injective. Note that R is left GC_2, by Theorem 3.10, R is left artinian.

$(4) \Rightarrow (1)$. By Lemma 3.12, R is right Kasch. It follows from [7, Lemma 2.2] that R is left P-injective. Note R has ACC on left annihilators, by [9, Proposition 5.15], R is also right artinian. Thus, R is two-sided mininjective and two-sided artinian, and therefore it is a quasi-Frobenius ring by Ikeda's Theorem (see [9, Theorem 2.30]).

$(5) \Rightarrow (4)$. Since right 2-injective ring is right P-injective, and by [7, Theorem 3.3(2)], right P-injective right finite dimensional ring is semilocal, so R is semilocal. Thus, by Theorem 3.10, R is left artinian.

$(6) \Rightarrow (1)$. Since R is right MGP-injective, it is right MJGP-injective. Moreover, it is also GC_2 by [13, Theorem 3.4(1)]. Thus, by Theorem 3.11, R is a quasi-Frobenius ring. □

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