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RESULTS ON THE LINEARLY EQUIVALENT IDEAL TOPOLOGIES

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Abstract Let R be a commutative Noetherian ring and let N be a non-zero finitely generated R-module. In this note, the main result asserts that for any N-proper ideal $\mathfrak{a}$ of R, the $\mathfrak{a}$-symbolic topology on N is linearly equivalent to the $\mathfrak{a}$-adic topology on N if and only if, for every $\mathfrak{p} \in \operatorname{Supp}(N)$, $\operatorname{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ consists of a single prime ideal and $\dim N \leq 1$.

1 Introduction

Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and let N be a non-zero finitely generated R-module. For a non-negative integer n, the nth *symbolic power* of $\mathfrak{a}$ w.r.t. N, denoted by $(\mathfrak{a}N)^{(n)}$, is defined to be the intersection of those primary components of $\mathfrak{a}^nN$ which correspond to the minimal elements of $\operatorname{Ass}_R N/\mathfrak{a}N$. Then the $\mathfrak{a}$-adic filtration $\{\mathfrak{a}^nN\}_{n\geq 0}$ and the $\mathfrak{a}$-symbolic filtration $\{(\mathfrak{a}N)^{(n)}\}_{n\geq 0}$ induce topologies on N which are called the $\mathfrak{a}$-adic topology and $\mathfrak{a}$-symbolic topology, respectively. These two topologies are said to be *linearly equivalent* if, there is an integer $k \geq 0$ such that $(\mathfrak{a}N)^{(n+k)} \subseteq \mathfrak{a}^nN$ for all integers $n \geq 0$. For a prime ideal $\mathfrak{p}$ of R, the linearly equivalence of $\mathfrak{p}$-adic topology and the $\mathfrak{p}$-symbolic topology were first studied by Schenzel in [15] and Huckaba in [4], in the case $N = R$.

Our main point of the present paper concerns an investigation of the linearly equivalent of the $\mathfrak{a}$-symbolic and the $\mathfrak{a}$-adic topology topologies on N. More precisely we shall show that:

Theorem 1.1. *Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R-module. Then for any N-proper ideal $\mathfrak{a}$ of R, the $\mathfrak{a}$-symbolic topology on N is linearly equivalent to the $\mathfrak{a}$-adic topology on N if and only if, for every $\mathfrak{p} \in \operatorname{Supp}(N)$, $\operatorname{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ consists of a single prime ideal and* $\dim N \leq 1$.

The result in Theorem 1.1 is proved in Theorem 2.4. Our method is based on the theory of the asymptotic and essential primes of $\mathfrak{a}$ w.r.t. N which were introduced by McAdam [7], and in [1], Ahn extended these concepts to a finitely generated R-module N.

Recall that a prime ideal $\mathfrak{p}$ of R is called a *quintessential* (resp. *quintasymptotic*) *prime ideal of* $\mathfrak{a}$ w.r.t. N precisely when there exists $\mathfrak{q} \in \operatorname{Ass}_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$ (resp. $\mathfrak{q} \in \operatorname{mAss}_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$) such that $\operatorname{Rad}(\mathfrak{a}R^*_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}R^*_\mathfrak{p}$. The set of quintessential (resp. quintasymptotic) prime ideals of $\mathfrak{a}$ w.r.t. N is denoted by $Q(\mathfrak{a}, N)$ (resp. $\bar{Q}^*(\mathfrak{a}, N)$) which is a finite set. One of our tools for proving Theorem 1.1 is the following, which plays a key role in this paper.

Proposition 1.2. *Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal of R. Let N be a non-zero finitely generated R-module such that* $\dim N > 0$, *and let* $\mathfrak{p} \in \operatorname{Supp}(N) \cap V(\mathfrak{a})$. *Then the following conditions are equivalent:*

(i) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$.

(ii) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, *for any N-proper ideal $\mathfrak{b}$ of R with* $\operatorname{height}_N \mathfrak{b} > 0$.

(iii) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, *for any N-proper element x of R with* $x \notin \bigcup_{\mathfrak{p} \in \operatorname{mAss}_R N} \mathfrak{p}$.

We denote by $\mathscr{R}$ the *graded Rees ring* $R[u, \mathfrak{a}t] := \oplus_{n\in\mathbb{Z}} \mathfrak{a}^n t^n$ of R w.r.t. $\mathfrak{a}$, where t is an indeterminate and $u = t^{-1}$. Also, the *graded Rees module* $N[u, \mathfrak{a}t] := \oplus_{n\in\mathbb{Z}} \mathfrak{a}^n N$ over $\mathscr{R}$ is denoted by $\mathscr{N}$, which is a finitely generated graded $\mathscr{R}$-module. Then we say that a prime ideal $\mathfrak{p}$ of R is an *essential prime ideal of* $\mathfrak{a}$ w.r.t. N, if $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in Q(u\mathscr{R}, \mathscr{N})$. The set of essential prime ideals of $\mathfrak{a}$ w.r.t. N will be denoted by $E(\mathfrak{a}, N)$.

Also, the *asymptotic prime ideals of* $\mathfrak{a}$ w.r.t. N, denoted by $\bar{A}^*(\mathfrak{a}, N)$, is defined to be the set $\{\mathfrak{q} \cap R \mid \mathfrak{q} \in \bar{Q}^*(u\mathscr{R}, \mathscr{N})\}$.

In [16], Sharp et al. introduced the concept of integral closure of $\mathfrak{a}$ relative to N, and they showed that this concept have properties which reflect some of those of the usual concept of integral closure introduced by Northcott and Rees in [13]. The integral closure of $\mathfrak{a}$ relative to N is denoted by $\mathfrak{a}^{-(N)}$. In [12], it is shown that the sequence $\{\mathrm{Ass}_R\, R/(\mathfrak{a}^n)^{-(N)}\}_{n \geq 1}$, of associated prime ideals, is increasing and ultimately constant; we denote its ultimate constant value by $\hat{A}^*(\mathfrak{a}, N)$. In the case $N = R$, $\hat{A}^*(\mathfrak{a}, N)$ is the asymptotic primes $\hat{A}^*(\mathfrak{a})$ of $\mathfrak{a}$ introduced by Ratliff in [14]. Also, it is shown in [11, Proposition 3.2] that $\hat{A}^*(\mathfrak{a}, N) = \bar{A}^*(\mathfrak{a}, N)$.

If $(R, \mathfrak{m})$ is local, then R^* (resp. N^*) denotes the completion of R (resp. N) w.r.t. the $\mathfrak{m}$-adic topology. In particular, for every prime ideal $\mathfrak{p}$ of R, we denote $R^*_{\mathfrak{p}}$ and $N^*_{\mathfrak{p}}$ the $\mathfrak{p}R_{\mathfrak{p}}$-adic completion of $R_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$, respectively. For any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. Finally, for each R-module L, we denote by $\mathrm{mAss}_R\, L$ the set of minimal prime ideals of $\mathrm{Ass}_R\, L$.

Recall that an ideal $\mathfrak{b}$ of R is called N-proper if $N/\mathfrak{b}N \neq 0$, and, when this the case, we define the N-*height* of $\mathfrak{b}$ (written $\mathrm{height}_N\, \mathfrak{b}$) to be

$$\inf\{\mathrm{height}_N\, \mathfrak{p} : \mathfrak{p} \in \mathrm{Supp}\, N \cap V(\mathfrak{b})\},$$

where $\mathrm{height}_N\, \mathfrak{p}$ is defined to be the supremum of lengths of chains of prime ideals of $\mathrm{Supp}(N)$ terminating with $\mathfrak{p}$. Also, we say that an element x of R is an N-proper element if $N/xN \neq 0$.

For any unexplained notation and terminology we refer the reader to [3] or [8].

2 The main result

Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R-module. The purpose of the present paper is to give an investigation of the linearly equivalent of the $\mathfrak{a}$-symbolic and the $\mathfrak{a}$-adic topologies on N. The main goal of this section is Theorem 2.4. The following proposition plays a key role in the proof of the main theorem.

Proposition 2.1. *Let* $\mathfrak{a}$ *be an ideal of* R *and let* N *be a non-zero finitely generated* R*-module with* $\dim N > 0$. *Let* $\mathfrak{p} \in \mathrm{Supp}(N) \cap V(\mathfrak{a})$. *Then the following conditions are equivalent:*

(i) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$.

(ii) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, *for any* N*-proper ideal* $\mathfrak{b}$ *of* R *with* $\mathrm{height}_N\, \mathfrak{b} > 0$.

(iii) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, *for any* N*-proper element* x *of* R *with* $x \notin \bigcup_{\mathfrak{p} \in \mathrm{mAss}_R N} \mathfrak{p}$.

(iv) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, *for some* N*-proper element* x *of* R *with* $x \notin \bigcup_{\mathfrak{p} \in \mathrm{mAss}_R N} \mathfrak{p}$.

Proof. (i)$\Longrightarrow$(ii): Let $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$ and let $\mathfrak{b}$ be an N-proper ideal of R such that $\mathrm{height}_N\, \mathfrak{b} > 0$. Then, in view of [11, Remark 2.4],

$$\mathfrak{p}/\mathrm{Ann}_R\, N \in \hat{A}^*(\mathfrak{a} + \mathrm{Ann}_R\, N/\mathrm{Ann}_R\, N).$$

Hence, as by [9, Theorem 2.1],

$$\mathrm{height}_N\, \mathfrak{b} = \mathrm{height}(\mathfrak{b} + \mathrm{Ann}_R\, N/\mathrm{Ann}_R\, N) > 0,$$

it follows from [6, Proposition 3.26] that

$$\mathfrak{p}/\mathrm{Ann}_R\, N \in \hat{A}^*(\mathfrak{a}\mathfrak{b} + \mathrm{Ann}_R\, N/\mathrm{Ann}_R\, N).$$

Therefore by using [11, Remark 2.4], we obtain that $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, as required.

(ii)$\Longrightarrow$(iii): Let (ii) hold and let x be an N-proper element of R such that $x \notin \bigcup_{\mathfrak{p} \in \mathrm{mAss}_R N} \mathfrak{p}$. Then it is easy to see that $\mathrm{height}_N\, xR > 0$, and so according to the assumption (ii), we have $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$.

(iii)$\Longrightarrow$(iv): Since $\dim N > 0$, there exists $\mathfrak{q} \in \mathrm{Supp}\, N$ such that $\mathrm{height}_N\, \mathfrak{q} > 0$. Hence $\mathfrak{q} \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{mAss}_R N} \mathfrak{p}$, and so there is $x \in \mathfrak{q}$ such that $x \notin \bigcup_{\mathfrak{p} \in \mathrm{mAss}_R N} \mathfrak{p}$. Consequently, it follows from the hypothesis (iii) that $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$.

(iv)$\Longrightarrow$(i): Let x be an N-proper element of R such that $x \notin \bigcup_{\mathfrak{p}\in \mathrm{mAss}_R N}\mathfrak{p}$ and let $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$. Then

$$\mathfrak{p}/\operatorname{Ann}_R N \in \hat{A}^*(x\mathfrak{a} + \operatorname{Ann}_R N/\operatorname{Ann}_R N),$$

by [11, Remark 2.4]. Now, since $x \notin \bigcup_{\mathfrak{p}\in \mathrm{mAss}_R N}\mathfrak{p}$, it is easy to see that $x + \operatorname{Ann}_R N$ is not in any minimal prime $R/\operatorname{Ann}_R N$. Therefore, it follows from [6, Proposition 3.26] that

$$\mathfrak{p}/\operatorname{Ann}_R N \in \hat{A}^*(\mathfrak{a} + \operatorname{Ann}_R N/\operatorname{Ann}_R N).$$

Consequently, in view of [11, Remark 2.4], $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$, and this completes the proof. □

Before we state Theorem 2.4 which is our main result, we give a couple of lemmas that will be used in the proof of Theorem 2.4.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a local ring and let N be a non-zero finitely generated R-module such that* $\dim N > 0$ *and that* $\mathrm{Ass}_R N$ *has at least two elements. Then there is an ideal $\mathfrak{a}$ of R such that* $\mathfrak{m} \in Q(\mathfrak{a}, N) \setminus \mathrm{mAss}\, N/\mathfrak{a}N$.

Proof. See [2, Proposition 4.2]. □

Lemma 2.3. *Let N be a non-zero finitely generated R-module and let $\mathfrak{a}$ be an N-proper ideal of R. Then* $E(\mathfrak{a}, N) = \mathrm{mAss}_R N/\mathfrak{a}N$ *if and only if the $\mathfrak{a}$-symbolic topology is linearly equivalent to the $\mathfrak{a}$-adic topology.*

Proof. The assertion follows easily from [10, Theorem 4.1]. □

We are now ready to state and prove the main theorem of this paper which is a characterization of the certain modules in terms of the linear equivalence of certain topologies induced by families of submodules of a finitely generated module N over a commutative Noetherian ring R. We denote by $Z_R(N)$ the set of zero divisors on N, i.e.,

$$Z_R(N) := \{r \in R \,|\, rx = 0 \text{ for some } x(\neq 0) \in N\}.$$

Theorem 2.4. *Let N be a non-zero finitely generated R-module. Then the following conditions are equivalent:*

(i) *For every N-proper ideal $\mathfrak{b}$ of R, the $\mathfrak{b}$-symbolic topology is linearly equivalent to the $\mathfrak{b}$-adic topology.*

(ii) $\dim N \leq 1$ *and* $\mathrm{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ *consists of a single prime ideal, for all* $\mathfrak{p} \in \mathrm{Supp}(N)$.

Proof. Suppose that (i) holds. Firstly, we show that $\dim N \leq 1$. To achieve this, suppose the contrary is true. That is $\dim N > 1$. Then there exists $\mathfrak{p} \in \mathrm{Supp}(N)$ such that $\mathrm{height}_N \mathfrak{p} > 1$. Hence $\mathfrak{p} \nsubseteq \bigcup_{\mathfrak{q}\in \mathrm{mAss}_R N}\mathfrak{q}$, and so there exists $x \in \mathfrak{p}$ such that $x \notin \bigcup_{\mathfrak{q}\in \mathrm{mAss}_R N}\mathfrak{q}$. Now, since $\mathfrak{p} \in \bar{A}^*(\mathfrak{p}, N)$ and $xN \neq N$, it follows from Proposition 2.1 that $\mathfrak{p} \in \bar{A}^*(x\mathfrak{p}, N)$. Therefore, in view of [1, Theorem 3.17] we have $\mathfrak{p} \in E(x\mathfrak{p}, N)$.

On other hand, since $x \notin \bigcup_{\mathfrak{q}\in \mathrm{mAss}_R N}\mathfrak{q}$, it is easily seen that $\mathfrak{p} \notin \mathrm{mAss}_R N/x\mathfrak{p}N$, and so by the assumption (i) and Lemma 2.3 we have $\mathfrak{p} \notin E(x\mathfrak{p}, N)$, which is a contradiction. Hence, $\dim N \leq 1$. Now, we show that $\mathrm{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ consists of a single prime ideal, for all $\mathfrak{p} \in \mathrm{Supp}(N)$. To do this, if $\dim N = 0$, then $\dim N_\mathfrak{p} = 0$. Hence $\mathrm{Ass}_{R_\mathfrak{p}} N_\mathfrak{p} = \{\mathfrak{p}R_\mathfrak{p}\}$, as required. Consequently, we have $\dim N_\mathfrak{p} = 1$. Now, if $\mathrm{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ has at least two elements, then in view of Lemma 2.2 there exists an ideal $\mathfrak{a}R_\mathfrak{p}$ of $R_\mathfrak{p}$ such that $\mathfrak{p}R_\mathfrak{p} \in Q(\mathfrak{a}R_\mathfrak{p}, N_\mathfrak{p})$ but $\mathfrak{p}R_\mathfrak{p} \notin \mathrm{mAss}_{R_\mathfrak{p}} N_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}$. Therefore, in view of [1, Lemma 3.2 and Theorem 3.17], $\mathfrak{p} \in E(\mathfrak{a}, N) \setminus \mathrm{mAss}\, N/\mathfrak{a}N$, which is a contradiction.

In order to show the implication (ii)$\Longrightarrow$(i), in view of Lemma 2.3 it is enough for us to show that $E(\mathfrak{b}, N) = \mathrm{mAss}\, N/\mathfrak{b}N$. To this end, let $\mathfrak{p} \in E(\mathfrak{b}, N)$. By virtue of [1, Lemma 3.2], we may assume that $(R, \mathfrak{p})$ is local.

Firstly, suppose $\dim N = 0$. Then it readily follows that $\mathfrak{p} \in \mathrm{mAss}\, N/\mathfrak{b}N$, as required. So we may assume that $\dim N = 1$. There are two cases to consider:

Case 1. $\mathfrak{b} \nsubseteq Z_R(N)$. Then $\mathrm{grade}(\mathfrak{b}, N) > 0$. Since $\dim N = 1$, it follows that $\mathrm{height}_N \mathfrak{b} = 1$, and so $\mathfrak{b} + \operatorname{Ann}_R N$ is $\mathfrak{p}$-primary. Hence $\mathfrak{p} \in \mathrm{mAss}\, N/\mathfrak{b}N$, as required.

Case 2. Now, suppose that $\mathfrak{b} \subseteq Z_R(N)$. Then there exists $z \in \operatorname{Ass}_R N$ such that $\mathfrak{b} \subseteq z$. Since $\operatorname{Ass}_R N$ consists of a single prime ideal, so $\operatorname{Ass}_R N = \{z\}$. Hence in view of [1, Proposition 3.6], $\mathfrak{p}/z \in E(\mathfrak{b} + z/z, R/z)$. Since $\mathfrak{b} \subseteq z$, it follows from [5, Remark 2.3] that $\mathfrak{p} = z$, which is a contradiction, because dim $N = 1$. Consequently, $\mathfrak{b} \nsubseteq Z_R(N)$ and the claim holds. □

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