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REAL HYPERSURFACE OF A NEARLY KAEHLER MANIFOLD

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Abstract We prove that real hypersurface of a nearly Kaehler manifold is a Hopf hypersurface if it has trans-Sasakian or almost Kenmotsu or nearly Kenmotsu structure. Further we prove that the principal curvature of Hopf vector field is a constant.

1 Introduction

Nearly Kaehler manifolds, which like Kaehler manifolds have rich geometric properties. This class of manifolds have been extensively studied by Gray in ([9],[2], [10]). The submanifold theory and in particular the study of real hypersurfaces in a Kaehler manifold, has been of great interest to geometers for the last decades. In 1987, Hussain and Deshmukh [16] studied CR-submanifolds of nearly Kaehler manifolds. Hui and Matsuyama studied [14, 15] real hypersurfaces of a complex projective space. Ganchev and Hristov[6] obtained four basic classes of real hypersurfaces of Kaehler manifolds, which generate sixteen classes of real hypersurfaces. These classes of hypersurfaces are characterised using the classification of almost contact metric manifolds obtained by Alexiev and Ganchev[1]. Goldberg[7] obtained conditions for a smooth orientable hypersurface of a Kaehler manifold to be cosymplectic, and the same author in [8] showed that if a hypersurface of a Kaehler manifold is immersed as an invariant hypersurface of an orientable hypersurface of the ambient space such that if the unit normal vector field of the immersion is Killing then the hypersurface is a totally geodesic submanifold. Further Ramesh Sharma[21], by assuming the second fundamental form to be Codazzi, characterized and classified contact hypersurfaces of a Kaehler manifold. Deshmukh and Al Solamy [5] obtained a characterization for a compact Hopf hypersurface in the nearly Kaehler sphere S^6. Okumara [20] gave a classification of hypersurfaces of non flat complex projective space satisfying $A\phi = \phi A$. In 1982 Richard Hamilton [11] introduced the concept of Ricci flow which smooths out the geometry of manifold M, that is if there are singular points, these can be minimized under Ricci flow. Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric, that is they are stationary points of the Ricci flow $\frac{\partial}{\partial t}g(t) = -2Ric(t)$, in the space of metrics on M. These facts motivate the study of real hypersurfaces of nearly Kaehler manifolds admitting Ricci solitons. In this paper we aim at proving that these hypersurfaces with different contact structures are Hopf hypersurfaces which satisfy $A\phi = \phi A$.

2 Preliminaries

A nearly Kaehler manifold $(\bar{M}, J, \bar{g})$ is an almost Hermitian manifold $\bar{M}$ with Riemannian metric $\bar{g}$, almost complex structure J satisfying

$$J^2 = -I, \;\; g(JX, JY) = g(X, Y), \tag{2.1}$$

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0, \tag{2.2}$$

where $\bar{\nabla}$ denote the operator of covariant differentiation with respect to $\bar{g}$, X and Y are any vector fields on $\bar{M}$. Let $(\bar{M}, J, \bar{g})$ be an n-dimensional nearly Kaehler manifold and (M, g) be a hypersurface of $\bar{M}$ with the induced Riemannian metric g oriented by unit normal vector field N on M. Let ∇ and $\bar{\nabla}$ denote the Levi-Civita connections with respect to the Riemannian metric g and $\bar{g}$ of M and $\bar{M}$ respectively. The Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{2.3}$$

$$\bar{\nabla}_X N = -AX, \tag{2.4}$$

where X and Y are vector fields on M and A is the shape operator of M in the direction of N. If R and $\bar{R}$ denote the curvature tensors of M and $\bar{M}$ respectively, then the Gauss and Codazzi equations are given by

$$\bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) + g(AX, Z)g(AY, W) - g(AY, Z)g(AX, W), \tag{2.5}$$

$$\bar{g}(\bar{R}(X, Y)Z, N) = g((\nabla_X A)Y - (\nabla_Y A)X, Z), \tag{2.6}$$

for any vector fields X, Y and Z on M.
The Hermitian structure of a nearly Kaehler manifold $\bar{M}$ induces an almost contact metric structure on a real hypersurface of $\bar{M}$. We set

$$\phi X = JX + \eta(X)N, \tag{2.7}$$

$$JN = -\xi, \tag{2.8}$$

for any vector field X on M.
Then by properties of the almost complex structure J we see that the set (ϕ, ξ, η, g) defines an almost contact metric structure on M satisfying

$$\begin{aligned} &\phi^2 = -Id + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \end{aligned} \tag{2.9}$$

Definition 2.1. A complex structure of the ambient space yields in each of their oriented hypersurfaces a special tangent vector field which is obtained by applying the complex structure of the ambient space to a unit normal vector field defined on the hypersurface. This special vector field is called as the Hopf vector field of the hypersurface. A hypersurface is said to be a Hopf hypersurface when the foliation given by its Hopf vector field is geodesic, in other words, when the integral curves of its Hopf vector field are geodesics of the hypersurface. Equivalently a real hypersurface M of a nearly Kaehler manifold $\bar{M}$ is a Hopf hypersurface if the characteristic vector field ξ of M is an eigen vector of the shape operator A. i.e. $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$ called as the Hopf principal curvature.

The fundamental 2-form Φ on M is defined by

$$\Phi(X, Y) = g(X, \phi Y),$$

for any vector fields X and Y.
If there are smooth functions α, β on hypersurface M with the induced almost contact metric structure (ϕ, ξ, η, g) satisfying

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.10}$$

for any vector fields X and Y on M then M is called a trans-Sasakian manifold of type (α, β). A trans-Sasakian manifold of type $(1, 0)$, $(0, 1)$ and $(0, 0)$ are respectively Sasakian, Kenmotsu and Co-symplectic manifolds. It follows from (2.9) and (2.10) that in a trans-Sasakian manifold the following holds:

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.11}$$

An almost contact metric manifold with $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is called an almost Kenmotsu manifold. In an almost Kenmotsu manifold, the following hold ([3]).

$$h\xi = 0, \quad l\xi = 0, \quad tr\ h = 0, \quad tr\ (h\phi) = 0, \quad h\phi + \phi h = 0, \tag{2.12}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X, \tag{2.13}$$

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.14}$$

for any vector fields X, Y on M, where $l = R(\cdot, \xi)\xi$ and $h = \frac{1}{2}L_\xi \phi$ are symmetric operators on TM.

An almost contact metric manifold is called a nearly Kenmotsu manifold [19] if the following relations hold:

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(Y)\phi X - \eta(X)\phi Y, \tag{2.15}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.16}$$

$$\nabla_\xi \phi = 0. \tag{2.17}$$

Definition 2.2. A Ricci soliton (g, V, λ) on a Riemannian manifold M is defined by

$$L_V g + 2S + 2\lambda g = 0, \tag{2.18}$$

it is said to be shrinking, steady, or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

We have the following result due to Hu et al[13](Remark 2.1) and Martins[18](Theorem 2):

Theorem 2.3. *The hypersurfaces of the homogeneous nearly Kaehler S^6 satisfy $A\phi = \phi A$ if and only if they are geodesic hyper spheres.*

Throughout this paper the characteristic vector field ξ refers to both Reeb vector field and Hopf vector field.

3 Real hypersurface of a nearly Kaehler manifold

Let $(\bar{M}, J, g)$ be an n-dimensional nearly Kaehler manifold and M be a hypersurface of $\bar{M}$ with almost contact metric structure (ϕ, ξ, η, g) induced from structure $(J, \bar{g})$. Throught this section $\{e_i\}, i = 1, \ldots, n-1$ denote an orthonormal basis of the tangent space T_pM at each point p of the hypersurface M.

Differentiating $JX = \phi X + \eta(X)N$ along M, we obtain

$$\bar{\nabla}_X JY = \bar{\nabla}_X \phi Y + (\bar{\nabla}_X \eta(Y))N + \eta(Y)\bar{\nabla}_X N. \tag{3.1}$$

Using (2.3), (2.4), (2.7) in (3.1), we get

$$(\bar{\nabla}_X J)Y = (\nabla_X \phi)Y + ((\nabla_X \eta)Y + g(AX, \phi Y))N - \eta(Y)AX + g(AX, Y)\xi. \tag{3.2}$$

Interchanging X and Y in (3.2) and adding the two equations, by using (2.2), we obtain

$$\begin{aligned}(\nabla_X \phi)Y + (\nabla_Y \phi)X =& \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi \\ & - ((\nabla_X \eta)Y + (\nabla_Y \eta)X + g((\phi A - A\phi)X, Y))N.\end{aligned} \tag{3.3}$$

Taking tangential and normal parts in (3.3), we obtain

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi. \tag{3.4}$$

$$(\nabla_X \eta)Y + (\nabla_Y \eta)X + g((A\phi - \phi A)X, Y) = 0. \tag{3.5}$$

The conformal curvature tensor $\bar{C}$ in $\bar{M}$ is defined in the usual way as follows.

$$\begin{aligned}\bar{C}(X, Y)Z = \bar{R}(X, Y)Z &+ \frac{1}{n-2}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y] \\ &+ \frac{\bar{r}}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y].\end{aligned} \tag{3.6}$$

It is well known that in a conformally flat space the conformal curvature tensor vanishes. Let us suppose that nearly Kaehler manifold $\bar{M}$ is conformally flat. Then from (3.6), we get

$$\begin{aligned}\bar{R}(X,Y)Z =& \frac{1}{n-2}[\bar{S}(X,Z)Y - \bar{S}(Y,Z)X + g(X,Z)\bar{Q}Y - g(Y,Z)\bar{Q}X] \\ &+ \frac{\bar{r}}{(n-1)(n-2)}[g(X,Z)Y - g(Y,Z)X].\end{aligned} \tag{3.7}$$

Contraction of (3.7) with $Z = N$ yield the following:

$$\begin{aligned}\bar{R}(X,Y)N =& \frac{1}{n-2}[\bar{S}(X,N)Y - \bar{S}(Y,N)X + g(X,N)\bar{Q}Y - g(Y,N)\bar{Q}X] \\ &+ \frac{\bar{r}}{(n-1)(n-2)}[g(X,N)Y - g(Y,N)X].\end{aligned} \tag{3.8}$$

$$\bar{S}(Y,N) = -\frac{\bar{r}}{n-2}g(Y,N). \tag{3.9}$$

$$\bar{S}(N,N) = -\frac{\bar{r}}{n-2}. \tag{3.10}$$

From the equations (3.8) and (3.9), it is easy to see that in a conformally flat $\bar{M}$,

$$\bar{R}(X,Y)N = 0. \tag{3.11}$$

In the Gauss equation (2.5) taking $X = W = e_i$, and summing over $i = 1, 2, \ldots, n-1$, we get

$$\bar{S}(Y,Z) - \bar{R}(N,Y,Z,N) = S(Y,Z) + g(AY,AZ) - (trA)g(AY,Z). \tag{3.12}$$

But from (3.11), $\bar{R}(N,X,Y,N) = 0$. Therefore (3.12) takes the form

$$\bar{S}(Y,Z) = S(Y,Z) + g(AY,AZ) - (trA)g(AY,Z). \tag{3.13}$$

Taking $X = Y = e_i$, and summing over $i = 1, 2, \ldots, n-1$ in (3.13), we get

$$\bar{r} = r + tr(A)^2 - (trA)^2. \tag{3.14}$$

Contraction of (3.7) twice leads to $\bar{r} = 0$. i.e. in a conformally flat nearly Kaehler manifold $\bar{r} = 0$. Hence the equation (3.14) reduces to

$$r = (trA)^2 - tr(A^2). \tag{3.15}$$

4 Trans-Sasakian hypersurface of a nearly Kaehler manifold

Let the induced almost contact metric structure on the hypersurface M of $\bar{M}$ be trans-Sasakian. We prove the following theorem for a trans-Sasakian hypersurface of a nearly Kaehler manifold.

Theorem 4.1. *A real hypersurface with trans-Sasakian structure of a nearly Kaehler manifold is a Hopf hypersurface.*

Proof. Putting (2.10) in (3.4), we obtain

$$\begin{aligned}\alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y) \\ = \eta(Y)AX + \eta(X)AY - 2g(AX,Y)\xi.\end{aligned} \tag{4.1}$$

Taking $Y = \xi$ in (4.1), we have

$$AX = 2\eta(AX)\xi - \eta(X)A\xi + \alpha(\eta(X)\xi - X) - \beta\phi X. \tag{4.2}$$

Setting $X = \xi$ in (4.2), we get

$$A\xi = \eta(A\xi)\xi, \tag{4.3}$$

and then M is a Hopf hypersurface. □

It is well known that

Remark 4.2. [5] Every non-Kaehler nearly Kaehler manifold of constant curvature is nothing but S^6 .

4.1 α-Sasakian hypersurface of a nearly Kaehler manifold.

Let M be an α-Sasakian hypersurface of a nearly Kaehler manifold $\bar{M}$. Then by Theorem (4.1), M is a Hopf hypersurface of $\bar{M}$.

Proposition 4.3. *Given an α-Sasakian hypersurface M of a nearly Kaehler manifold $\bar{M}$, if $\alpha = -\eta(A\xi)$, then every vector X is an eigen vector of the shape operator A corresponding to principal curvature $-\alpha$ as a constant with positive constant scalar curvature. Further if $\bar{M}$ is non-Kaehler then M is a geodesic hypersphere.*

Proof. Contracting (4.2) with ξ, we have

$$\eta(AX) = \eta(A\xi)\eta(X). \tag{4.4}$$

Substituting (4.4) back in (4.2), we get

$$AX = \eta(A\xi)\eta(X)\xi + \alpha(\eta(X)\xi - X) - \beta\phi X. \tag{4.5}$$

Setting $\beta = 0$ and $\alpha = -\eta(A\xi)$ in the above equation we get

$$AX = \eta(A\xi)X. \tag{4.6}$$

Now from Codazzi equation (2.6) , (3.11) and (4.6), we get

$$(Y\alpha)X - (X\alpha)Y = 0, \tag{4.7}$$

for all $X, Y \in TM$.
Contracting (4.7) with ξ and taking $Y = \xi$ in the resulting equation, we get

$$(X\alpha) = (\xi\alpha)\eta(X). \tag{4.8}$$

Substituting this back in (4.7), we obtain

$$(\xi\alpha) = 0. \tag{4.9}$$

Putting this in (4.8), we get

$$X\alpha = 0. \tag{4.10}$$

i.e., $\alpha = \eta(A\xi)$ is a constant.

From (4.6), the trace of A is given by

$$trA = -(n-1)\alpha. \tag{4.11}$$

Using (3.15), (4.11) reduces to

$$r = (n-1)(n-2)\alpha^2. \tag{4.12}$$

Using (2.11) in (3.5), we obtain

$$g((A\phi - \phi A)X, Y) = 2\beta(g(X,Y) - \eta(X)\eta(Y)). \tag{4.13}$$

Since $\beta = 0$ for an α-Sasakian hypersurface, the result follows from equation (4.13), Remark(4.2) and Theorem(2.3). □

Proposition 4.4. *In a conformaly flat nearly Kaehler manifold $(\bar{M}, J, \bar{g})$, where $\bar{g}$ is a Ricci soliton, if M is an α-Sasakian hypersurface with $\alpha = -\eta(A\xi)$ of $\bar{M}$ then the Ricci soliton is expanding or shrinking according as $\alpha > 0$ or $\alpha < 0$.*

Proof. If $\bar{g}$ is a Ricci soliton then taking $V = N$ in (2.18), we get

$$\bar{g}(\bar{\nabla}_X N, Y) + \bar{g}(\bar{\nabla}_Y N, X) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) = 0. \tag{4.14}$$

Using (2.4) and (3.13) in (4.14), we obtain

$$S(X,Y) = (trA)g(AX,Y) - g(AX,AY) + g(AX,Y) - \lambda g(X,Y). \tag{4.15}$$

In view of (4.6), (4.15) leads to

$$S(X,Y) = ((n-2)\alpha^2 + \alpha - \lambda)g(X,Y). \tag{4.16}$$

Taking $X = Y = e_i$ in (4.16) and summing over $i = 1, \ldots, n-1$, we get

$$r = (n-1)((n-2)\alpha^2 + \alpha - \lambda). \tag{4.17}$$

Comparing this equation with (4.12), we get $\lambda = \alpha$ and the lemma is proved. □

4.2 β-Kenmotsu hypersurface of a nearly Kaehler manifold.

Let M be a β -Kenmotsu hypersurface of a conformally flat nearly Kaehler manifold $\bar{M}$. Then by Theorem (4.1), M is a Hopf hypersurface of $\bar{M}$.

Theorem 4.5. *The only β-Kenmotsu hypersurfaces M of a conformally flat nearly Kaehler manifold $\bar{M}$ are co-symplectic ones.*

Proof. If we proceed as in section 4.1, by contracting (4.2) with ξ, we get

$$\eta(AX) = \alpha\eta(X). \tag{4.18}$$

Substituting (4.18) back in (4.2), and taking $\alpha = 0$, we get

$$AX = \alpha\eta(X)\xi - \beta\phi X. \tag{4.19}$$

Now from Codazzi equation (2.6) and equations (3.11) and (4.19), we obtain

$$\begin{aligned} ((Y\alpha)X - (X\alpha)Y)\xi + \beta\alpha(\eta(Y)X - \eta(X)Y) - ((X\beta)\phi Y - (Y\beta)\phi X) \\ - 2\beta^2(g(\phi X, Y)\xi - \eta(X)\phi Y) = 0, \end{aligned} \tag{4.20}$$

for all $X, Y \in TM$.
Contracting (4.20) with ξ , we get

$$(Y\alpha)\eta(X) - (X\alpha)\eta(Y) = 2\beta^2 g(\phi X, Y). \tag{4.21}$$

Taking $Y = \xi$ in the above equation we have

$$(X\alpha) = (\xi\alpha)\eta(X). \tag{4.22}$$

Substituting this back in (4.21), we obtain

$$\beta = 0. \tag{4.23}$$

i.e. the β -Kenmotsu hypersurface of $\bar{M}$ becomes co-symplectic. □

Remark 2: (Theorem 4.2) It follows that a hypersurface M of conformally flat nearly Kaehler manifold, $(\bar{M}, J, \bar{g})$, where $\bar{g}$ is a Ricci soliton is co-symplectic. Further the Ricci soliton $\bar{g}$ is steady.

5 Almost Kenmotsu and nearly Kenmotsu hypersurfaces of a nearly Kaehler manifold

Let the hypersurface M of nearly Kaehler manifold with induced almost contact structure (ϕ, ξ, η, g) be an almost Kenmotsu manifold.
First we prove the following.

Lemma 5.1. *A real hypersurface with almost Kenmotsu structure of a nearly Kaehlerian manifold is a Hopf hypersurface satisfying $A\phi = \phi A$. And the principal curvature $\eta(A\xi)$ is a constant if it is constant along the Reeb vector field ξ.*

Proof. Using (2.13) in (3.5) we get
$g((\phi A - A\phi)X, Y) = (\nabla_X \eta)Y + (\nabla_Y \eta)X = 0$ or $\phi A = A\phi$.
From (2.14) and (3.4), we obtain

$$-\eta(Y)\phi X - \eta(X)\phi Y = \eta(Y)AX + \eta(X)AY - 2g(AX, Y)\xi. \tag{5.1}$$

Taking $Y = \xi$ in the above equation we get

$$AX = -\phi X + \eta(AX)\xi. \tag{5.2}$$

Putting $X = \xi$ in above equation it leads to

$$A\xi = \eta(A\xi)\xi. \tag{5.3}$$

i.e., M is a Hopf hypersurface. Contracting (5.1) with ξ , we get

$$\eta(Y)\eta(AX) + \eta(X)\eta(AY) - 2g(AX,Y) = 0.$$

Taking $Y = \xi$ in the above equation gives

$$\eta(AX) = \eta(A\xi)\eta(X).$$

Substituting this in (5.2), we obtain

$$AX = -\phi X + \eta(A\xi)\eta(X)\xi. \tag{5.4}$$

Interchanging X by Y in (5.4) then differentiating, we get

$$\begin{aligned}(\nabla_X A)Y =& - g(\phi X,Y)\xi + \eta(Y)\phi X + (X\alpha)\eta(Y)\xi \\ &- \alpha(g(\phi^2 X,Y) + g(\phi hX,Y) + (\phi^2 X)\eta(Y) + (\phi hX)\eta(Y)).\end{aligned} \tag{5.5}$$

Substituting this in the Codazzi equation (2.6), we get

$$\begin{aligned}\bar{R}(X,Y)N =& - 2g(\phi X,Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y + (X\alpha)\eta(Y)\xi - (Y\alpha)\eta(X)\xi \\ &- \alpha(\eta(Y)\phi^2 X - \eta(X)\phi^2 Y + \eta(Y)\phi hX - \eta(X)\phi hY).\end{aligned} \tag{5.6}$$

Contracting (5.6) with ξ, in view of (3.11), we get

$$(Y\alpha)\eta(X) - (X\alpha)\eta(Y) - 2g(\phi X,Y) = 0. \tag{5.7}$$

Taking $Y = \xi$ in the above equation leads to $X\alpha = (\xi\alpha)\eta(X)$.
i.e., $\eta(A\xi)$ is a constant provided it is constant along ξ.

□

Lemma 5.2. *A nearly Kenmotsu hypersurface of a conformally flat nearly Kaehler manifold is a Hopf hypersurface and the principal curvature $\alpha = \eta(A\xi)$ is a constant if it is constant along ξ.*

Proof. If M has nearly Kenmotsu manifold structure then from (2.15) and (3.4), it follows that

$$-\eta(Y)\phi X - \eta(X)\phi Y = \eta(X)AY - \eta(Y)AX - 2g(AX,Y)\xi. \tag{5.8}$$

Taking $Y = \xi$ in (5.8), we get

$$AX = -\phi X - \eta(X)A\xi + 2\eta(AX)\xi. \tag{5.9}$$

Putting $X = \xi$ in the above equation, we get

$$A\xi = \eta(A\xi)\xi. \tag{5.10}$$

Substituting this in (5.10) gives

$$AX = -\phi X + \eta(X)\eta(A\xi)\xi. \tag{5.11}$$

Interchanging X by Y in (5.11) then differentiating covariantly with respect to X and using (5.10), we have

$$(\nabla_X A)Y = -(\nabla_X \phi)Y + g(X,Y)\eta(A\xi)\xi - 2\eta(X)\eta(Y)\eta(A\xi)\xi + \eta(Y)(X\alpha)\xi + \eta(Y)\eta(A\xi)X. \tag{5.12}$$

Substituting (5.12) in (2.6), we get

$$\bar{R}(X,Y)N = (\nabla_Y \phi)X - (\nabla_X \phi)Y + (X\alpha)\eta(Y)\xi - (Y\alpha)\eta(X)\xi + \eta(Y)\eta(A\xi)X - \eta(X)\eta(A\xi)Y. \tag{5.13}$$

Putting (3.11) in (5.13), we have

$$(\nabla_Y\phi)X-(\nabla_X\phi)Y+(X\alpha)\eta(Y)\xi-(Y\alpha)\eta(X)\xi+\eta(Y)\eta(A\xi)X-\eta(X)\eta(A\xi)Y=0. \quad (5.14)$$

Using (2.15) in (5.14), we get

$$g((\nabla_Y\phi)X,\xi)-g((\nabla_X\phi)Y,\xi)+(X\alpha)\eta(Y)-(Y\alpha)\eta(X)=0. \quad (5.15)$$

$$2(\nabla_X\phi)Y=-\eta(X)\phi Y-\eta(Y)\phi X+(X\alpha)\eta(Y)\xi-(Y\alpha)\eta(X)\xi+\eta(Y)\eta(A\xi)X-\eta(X)\eta(A\xi)Y. \quad (5.16)$$

Taking $X=\xi$ in the above equation, from (2.17), we have

$$-\phi Y+(\xi\alpha)\eta(Y)\xi-(Y\alpha)\xi+\eta(Y)\eta(A\xi)\xi-\eta(A\xi)Y=0. \quad (5.17)$$

Contracting the above equation with respect to ξ, we obtain

$$(Y\alpha)=(\xi\alpha)\eta(Y). \quad (5.18)$$

□

Combining lemmas (5.1) and (5.2), we state the following theorem:

Theorem 5.3. *An almost Kenmotsu or a nearly Kenmotsu hypersurface of a nearly Kaehlerian manifold is a Hopf hypersurface with constant principal curvature provided the principal curvature is constant along the Reeb vector field ξ.*

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