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F-TRINOMIAL NUMBERS

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Abstract *Generalization of Fibonomial coefficients, a confluence have accelerated interest of mathematicians for a long time now. In this article, we introduce F-trinomial numbers, a three-dimensional extension of Fibonomial coefficients. We show that these numbers always possess integer values and obtain some of its interesting properties. Diophantine equations involving these numbers are also discussed.*

1 Introduction

In the theory of numbers, Fibonacci sequence has always fertile the ground for mathematicians. This sequence follows the recurrence relation $F_n = F_{n-1} + F_{n-2}; F_0 = 0$ *and* $F_1 = 1$. *There are numerous results available in the literature involving this sequence and its generalization. For more details, one can see [1, 2, 3]. On the other hand, in mathematics, binomial coefficients* $\binom{n}{k}$ *always been one of the most significant tools and there are different definitions available in literature regarding them. Normally binomial coefficient is defined as* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$*; where* $n! = n \times (n-1) \times \cdots \times 1$. *In this definition,* n *is partitioned into two parts* k *and* $(n-k)$. *In one of the generalizations of binomial coefficient, it is suggested to divide* n *into three parts, viz.* r, s *and* t. *The numbers defined using this idea is called trinomial numbers and for* $n = r + s + t$, *it is defined to be* $\begin{bmatrix} n \\ r, s, t \end{bmatrix} = \frac{n!}{r!s!t!}$.

In 1915, Fontené published a one-page note [4] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence A_n *of real or complex numbers. In this paper, we generalize the concept of trinomial numbers by replacing the natural numbers by means of the terms of the sequence of Fibonacci numbers* F_n *and we name as F-trinomial numbers.*

2 F-trinomial Numbers:

For any positive integer n, *if we consider* $n = r + s + t$, *then F-trinomial number is defined as* $\begin{bmatrix} n \\ r, s, t \end{bmatrix}_F = \frac{F_n^*}{F_r^* F_s^* F_t^*}$; *where* $F_n^* = F_n \times F_{n-1} \times \cdots \times F_1$.
Following results follows right away from this definition.

Lemma 2.1. $\begin{bmatrix} n \\ 0, s, t \end{bmatrix}_F = \binom{n}{s}_F$, *the regular Fibonomial coefficient.*

Lemma 2.2. $\begin{bmatrix} n \\ r,0,0 \end{bmatrix}_F = 1.$

Lemma 2.3. $\begin{bmatrix} n \\ 1,s,t \end{bmatrix}_F = F_n\binom{n-1}{s}_F$. *In this case, F-trinomial number is a product of a Fibonacci number and a Fibonomial coefficient.*

The following result gives the recurrence relation connecting the F-trinomial numbers.

Lemma 2.4. $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{s+t+1}\begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_F + F_{r-1}F_{t+1}\begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_F + F_{r-1}F_{s-1}\begin{bmatrix} n-1 \\ r,s,t-1 \end{bmatrix}_F.$

Proof. For the Fibonacci numbers F_n, it is known that $F_{r+s} = F_rF_{s+1} + F_{r-1}F_s$. Using this result along with the definition of F-trinomial numbers, we get the required result easily. □

Following result uses the above lemma to give recurrence relation of F-trinomial numbers in the series form.

Corollary 2.5. $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = \sum_{j=1}^{t}\Big\{F_{r-1}^{j-1}F_{s-1}^{j-1}F_{s+t-j+2}\begin{bmatrix} n-j \\ r-1,s,t-j+1 \end{bmatrix}_F + F_{r-1}^{j-1}F_{s-1}^{j-1}F_{s+t-j+1}\Big[\,r,s-$

$F_{r-1}^{t}F_{s-1}^{t}\begin{bmatrix} n-t \\ r,s,0 \end{bmatrix}_F.$

Proof. Using lemma 2.4 iteratively, we can easily get the desired result. □

It is not evident from the definition of F-trinomial numbers that they always possess integer values or not. In the following theorem, we show that they indeed always have integer values.

Theorem 2.6. *F-trinomial always holds integer values.*

Proof. We have $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = \frac{F_n^*}{F_r^*F_s^*F_t^*} = \frac{F_n\times F_{n-1}\times\cdots\times F_{s+t+1}\times F_{s+t}\times\cdots\times F_{t+1}\times F_t\times\cdots\times F_1}{F_r^*F_s^*F_t^*}$. This fraction contains r, s and t consecutive Fibonacci numbers in the numerator as well as in denominator. Since multiplication of any $'m'$ consecutive Fibonacci numbers is always divisible by the multiplication of first $'m'$ consecutive Fibonacci numbers, it is now evident that $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F$ is always integer. □

In [5], Gould gave an interesting result known as Star of David theorem for binomial coefficients, which states that the greatest common divisors of the binomial coefficients forming each of the two triangles in the Star of David shape in Pascal's triangle are equal. This can be restated as $\binom{n-a}{r-a}\binom{n}{r+a}\binom{n+a}{r} = \binom{n-a}{r}\binom{n+a}{r+a}\binom{n}{r-a}$. *In case of F-trinomial numbers too we can discover similar result, which will be the 3-dimensional version of star of David theorem.*

Theorem 2.7. $\begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_F \begin{bmatrix} n \\ r-1,s,t+1 \end{bmatrix}_F \begin{bmatrix} n \\ r+1,s,t-1 \end{bmatrix}_F \begin{bmatrix} n+1 \\ r,s+1,t \end{bmatrix}_F$

$= \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_F \begin{bmatrix} n \\ r,s-1,t+1 \end{bmatrix}_F \begin{bmatrix} n \\ r,s+1,t-1 \end{bmatrix}_F \begin{bmatrix} n+1 \\ r+1,s,t \end{bmatrix}_F$

$= \begin{bmatrix} n-1 \\ r,s,t-1 \end{bmatrix}_F \begin{bmatrix} n \\ r-1,s+1,t \end{bmatrix}_F \begin{bmatrix} n \\ r+1,s-1,t \end{bmatrix}_F \begin{bmatrix} n+1 \\ r,s,t+1 \end{bmatrix}_F$

The result follows easily from the definition of F-trinomial numbers.

2.1 Bounds of $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F$ in terms of α

In this section, we discuss about the bounds of F-trinomial numbers in terms of α*, where* $\alpha = \frac{1+\sqrt{5}}{2}$ *is the root of characteristic equation* $1 - x - x^2 = 0$ *of* F_n*.*

Theorem 2.8. $\alpha^{\frac{2n-1}{2}(r+s)-\left(r^2+s^2+rs-3\right)} \leq \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F \leq \alpha^{n(r+s)-\left(r^2+s^2+rs\right)}$

Proof. It is well known that $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$; for all $n \geq 1$. Also, from the definition of F-trinomial numbers, we have $\frac{\alpha^{(n-2)+(n-3)+\cdots+(n-r-s-1)}}{\left(\alpha^{(r-1)+(r-2)+\cdots+1}\right)\left(\alpha^{(s-1)+(s-2)+\cdots+1}\right)} \leq \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F \leq \frac{\alpha^{(n-1)+(n-2)+\cdots+(n-r-s)}}{\left(\alpha^{(r-2)+(r-3)+\cdots+(-1)}\right)\left(\alpha^{(s-2)+(s-3)+\cdots+(\right.}$
This gives $\alpha^{\frac{2n-1}{2}(r+s)-\left(r^2+s^2+rs-3\right)} \leq \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F \leq \alpha^{n(r+s)-\left(r^2+s^2+rs\right)}$, as required. □

Pascal-like triangle for Fibonomial numbers have many interesting properties analogous to the Pascal's triangle of binomial coefficients. In the following article, we obtain 3-dimensional Pascal-like Pyramid for the F-trinomial numbers, which possess interesting properties, some of which match with Pascal's pyramid of trinomial coefficients.

3 Pascal-like pyramid:

Pascal's pyramid is a three-dimensional arrangement of the trinomial numbers. Similarly, if we place F-trinomial numbers in to three-dimensional structure, we get Pascal-like pyramid, as seen in the figure 1. Due to the limitations of the three-dimensional structure on a paper, we divide this pyramid into layers. The top layer is "Layer 0" and other layers can be assumed of as overhead views with the previous layers removed. The first six layers are shown in the figure 2. While going through Figure 2, the following observations can be made.

- *Every layer has three-way symmetry.*
- *Every number in any layer is a simple whole number ratio of the adjacent numbers in the same layer. Since the pyramid has three-way symmetry, the ratio relation also holds for diagonal pairs in both directions.*
- *The number along the three edges of the nth layer are the numbers of the nth line of Pascal-like triangle of Fibonomial numbers.*
- *Multiplying the numbers of each line of Pascal-like triangle down to the* n^{th} *line by the number of the* n^{th} *line generates the* n^{th} *layer of this pyramid.*

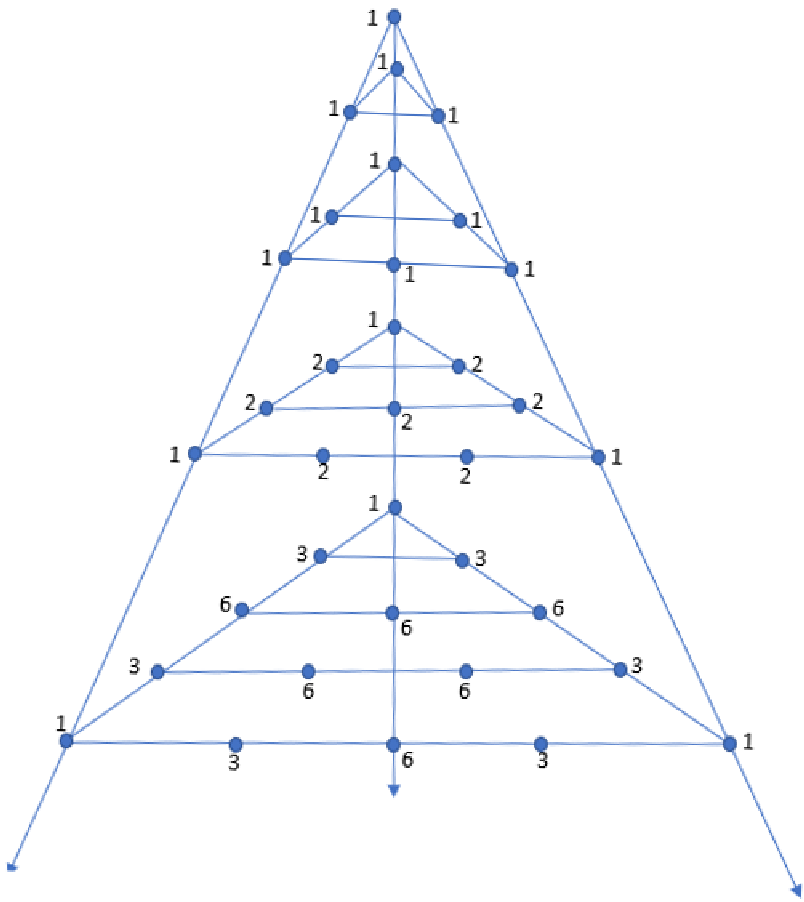

Figure 1

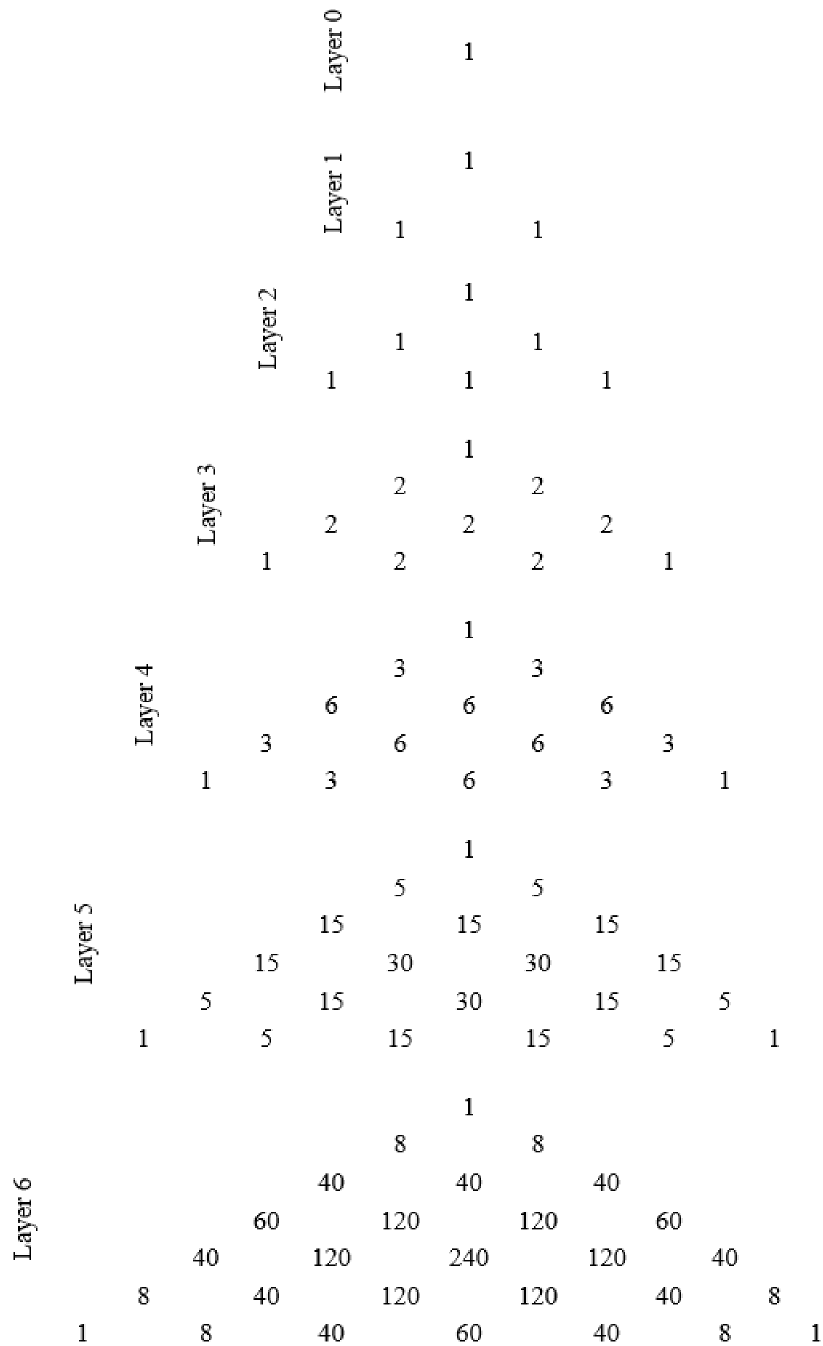

Figure 2

3.1 Fractal structure of Pascal-like pyramid:

To obtain another interesting property of this pyramid, if we apply modulo 2 to its values., we get fractal structure as shown for first six layers below.

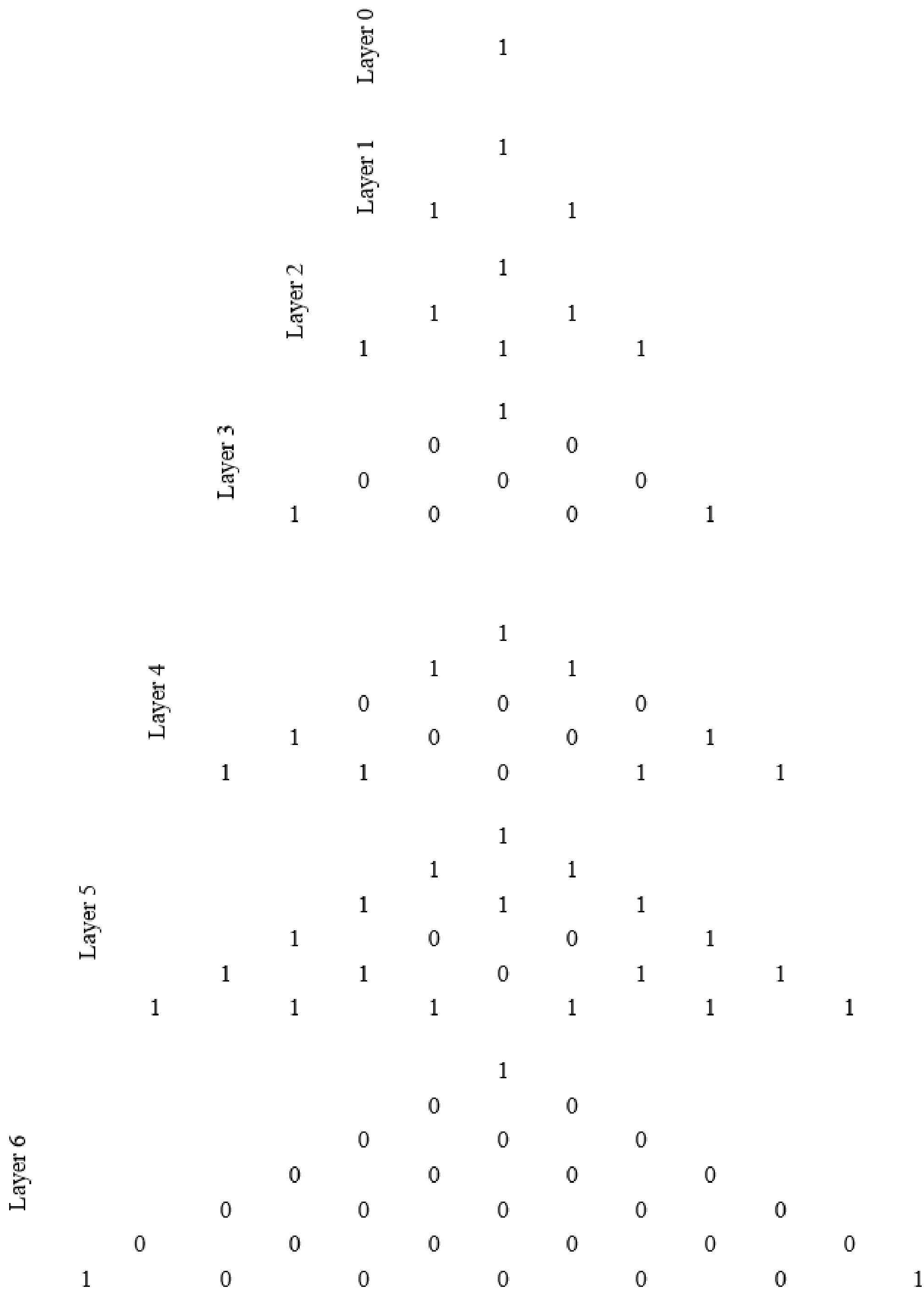

Figure 3

It can be observed that the structures for layers 0,1 and 2 repeats themselves in the three angles of the layers 3,4 and 5 respectively and all the remaining values of these layers becomes 0. Also, this structural behavior repeats itself. In other words, the structure of layer 0 appears in the angles of layers 3,6,9,... and all the remaining values becomes 0. Similarly, layer 1 appears in the angles of layers 4,7,10,... and layer 2 appears in the angles of layers 5, 8,11,... .
To prove this, we know that for Fibonacci numbers F_n, [6]

$$F_n \equiv_2 \begin{cases} 0; if\ n \equiv_3 0 \\ 1; if\ n \equiv_3 1\ or\ 2 \end{cases} \tag{3.1}$$

Now if we consider $n \equiv_3 0$, *the right most angle will have values* $r = s = 0$ *and* $t = n$. *Thus* $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = 1$ *and the symmetric structure will guarantee the values of other angles as 1 as well. But because of (1), the remaining values of these layers will vanish.*
Also, for $n \equiv_3 1$, *because of the symmetric structure, we need to check the angular values, when* $r = s = 0, t = n$ *and* $r = 1, s = 0, t = n - 1$. *In both the cases, (1) guarantees the value of* $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = 1$. *And from (1), we can claim that all the non-angular values vanish.*
The similar argument will justify the case when $n \equiv_3 2$.

4 F-trinomial numbers and Fibonacci numbers:

The Diophantine equation containing Fibonacci numbers has always been an interesting subject for the enthusiastic. In [7, 8, 9], one finds the solution of Diophantine equations containing Fibonacci numbers along with generalized Fibonomial numbers.
From the definition of F-trinomial numbers, it is clear that the Diophantine equation $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_m$ *has the trivial solution* $(n,r,s,t,m) = (n,0,1,n-1,n), (n,0,n-1,1,n), (n,1,0,n-1,n), (n,1,n-1$ *and* $(n, n-1, 1, 0, n)$. *Following result claims that there no other possible solution for the considered Diophantine equation.*

Theorem 4.1. *The Diophantine equation* $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_m$ *has no non-trivial solution.*

Proof. By [10], it is known that a primitive divisor of a Fibonacci number F_n is any prime integer p such that $p \mid F_n$ but $p \dagger F_m$; where $m < n$. Also, primitive divisor theorem says that for $n \geq 13$, every F_n has a primitive divisor.
Using the definition of F-trinomial numbers, the Diophantine equation $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_m$ implies

$$\frac{F_n^*}{F_r^* F_s^* F_t^*} = F_m \tag{4.1}$$

If we consider $n \geq 13$ and $n > m$, then by the primitive divisor theorem, there exists a prime p such that $p \mid F_n$ but $p \dagger F_m$. Thus (2) has no solution in this case. Similarly, for $m \geq 13$ and $m > n$, primitive divisor theorem again implies that (2) has no solution. Thus, we can narrow down the range of m and n as $max\,(m,n) < 13$. A quick look in this interval reveals that the Diophantine equation $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_m$ has no non-trivial solution. □

As F-trinomial numbers have 3-way symmetry, without loss of generality, we can assume that $r \geq s \geq t$. *From Lemma 2.2, we have* $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = 1$ *for* $s = t = 0$. *Also, from the Pascal-like Pyramid for the F-trinomial numbers, it is clear that every F-trinomial number* $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F$ *for* $n \leq 4$ *satisfies the Diophantine equation* $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F \pm 1 = F_m$. *We call these two cases the trivial solution of the given Diophantine equation. Following theorem proves that there no non-trivial solution available.*

Theorem 4.2. *The Diophantine equation* $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F \pm 1 = F_m$ *has no non-trivial solution.*

Proof. For the non-trivial solutions of the Diophantine equation considered, we let $n > 4$. We know that a primitive divisor p of F_n is a prime factor of F_n which does not divide $\prod_{j=1}^{n-1} F_j$ and Primitive Divisor Theorem states that a primitive divisor p of F_n exists whenever $n \geq 13$. The sequence of the Lucas numbers is defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$, with $L_0 = 2$ and $L_1 = 1$. We may note that both the Fibonacci and Lucas sequences can be extrapolated backwards using $F_n = F_{n+2} - F_{n+1}$ and $L_n = L_{n+2} - L_{n+1}$. Thus, for example, $F_{-1} = 1, F_{-2} = -1$ and so on. If $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, then the following results are true:

a $F_a L_b = F_{a+b} + (-1)^b F_{a-b}$; for any integers a and b

b $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$.

As a consequence of the above results, a straight forward calculation gives a different factorization for $F_n \mp 1$ depending on the class of n modulo 4.

$$\begin{aligned} F_{4l} + 1 &= F_{2l-1}L_{2l+1} & F_{4l} - 1 &= F_{2l+1}L_{2l-1} \\ F_{4l+1} + 1 &= F_{2l+1}L_{2l} & F_{4l+1} - 1 &= F_{2l}L_{2l+1} \\ F_{4l+2} + 1 &= F_{2l+2}L_{2l} & F_{4l+2} - 1 &= F_{2l}L_{2l+2} \\ F_{4l+3} + 1 &= F_{2l+1}L_{2l+2} & F_{4l+3} - 1 &= F_{2l+2}L_{2l+1} \end{aligned}$$

The Diophantine equation $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F \pm 1 = F_m$ can be rewritten as $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_m \mp 1$. From the above relation, one gets eight possibilities for this Diophantine equation (again depending on the class of n modulo 4). For the (+) case:

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l+1}L_{2l-1} \quad \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l}L_{2l+1} \quad \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l}L_{2l+2} \quad \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l+2}L_{2l+1}$$

For the (-) case:

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l-1}L_{2l+1} \quad \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l+1}L_{2l} \quad \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l+2}L_{2l} \quad \begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l+1}L_{2l+2}$$

To begin with, let us consider $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = F_{2l-1}L_{2l+1}$. That is $\frac{F_n \times F_{n-1} \times \cdots \times F_{m-t+1}}{(F_r \times F_{r-1} \times \cdots \times F_1)(F_s \times F_{s-1} \times \cdots \times F_1)} = F_{2l-1}L_{2l+1}$. Let us assume that $n \geq max\{14, r+1\}$. Thus,

$$F_n \times F_{n-1} \times \cdots \times F_{m-t+1} = F_{2l-1}L_{2l+1} \times (F_r \times F_{r-1} \times \cdots \times F_1)(F_s \times F_{s-1} \times \cdots \times F_1).$$

Since $L_{2l+1} = \frac{F_{4l+2}}{F_{2l+1}}$, we can write

$$\begin{gathered} F_n \times F_{n-1} \times \cdots \times F_{m-t+1} \times F_{2l+1} = \\ F_{2l-1}F_{4l+2} \times (F_r \times F_{r-1} \times \cdots \times F_1)(F_s \times F_{s-1} \times \cdots \times F_1). \end{gathered}$$

But since $l = \lfloor n/4 \rfloor > 2$, we have $4l + 2 > 2l - 1$. Thus, Primitive Divisor Theorem gives $n = 4l + 2$. That is

$$F_{n-1} \times \cdots \times F_{m-t+1} \times F_{2l+1} = F_{2l-1} \times (F_r \times F_{r-1} \times \cdots \times F_1)(F_s \times F_{s-1} \times \cdots \times F_1).$$

As $r \geq s$ and $n - 1 \geq 13$, Primitive Divisor Theorem again gives $n - 1 = max\{2l - 1, r\}$. However, $n - 1 = 4l + 1 > 2l - 1$. That is $n - 1 = r$, which is absurd. So, we only need to consider the range $4 < n \leq 14$ and $0 \leq k \leq 13$. A simple calculation shows that there is no possible solution of the given Diophantine equation in this range. □

5 Some more properties of F-trinomial numbers:

In this subsection, we first find the number of F-trinomial numbers in each layer.

Lemma 5.1. *The number of different F-trinomial numbers in layer u is given by* $\left\lfloor \frac{u^2+6u}{12} \right\rfloor$; $u > 1$.

Proof. By the definition of F-trinomial numbers $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F$, we have $n = r+s+t$. Therefore, number of different F-trinomial numbers in each layer is dependent on the number of partitions of n into at most 3 parts.
If $P_k(n)$ denotes the number of partitions of n into at most k parts then $P_3(n) = 1 + \left\lfloor \frac{n^2+6n}{12} \right\rfloor$.
Thus, the number of different F-trinomial numbers in each layer u should be $1 + \left\lfloor \frac{u^2+6u}{12} \right\rfloor$. But, if we take the partitions of $u > 1$ as $(r,s,t) = (0,2,n-2)$ and $(1,1,n-2)$ in the definition of F-trinomial number, then respective Fibonorial values are same. Hence, the number of different F-trinomial numbers in each layer u is given by $\left\lfloor \frac{u^2+6u}{12} \right\rfloor$. □

Lastly, we find the smallest and largest element in each layer.

Lemma 5.2. *The smallest and largest F-trinomial number in layer u is given by* $\begin{bmatrix} u \\ 0,1,u-1 \end{bmatrix}_F$ *and*

$$\begin{cases} \begin{bmatrix} u \\ \frac{u}{3},\frac{u}{3},\frac{u}{3} \end{bmatrix}_F; & u \equiv 0(mod3) \\ \begin{bmatrix} u \\ \frac{u-1}{3},\frac{u-1}{3},\frac{u+2}{3} \end{bmatrix}_F; & u \equiv 1(mod3) \\ \begin{bmatrix} u \\ \frac{u-2}{3},\frac{u+1}{3},\frac{u+1}{3} \end{bmatrix}_F; & u \equiv 2(mod3) \end{cases}$$ *respectively.*

Proof. From the definition of the F-trinomial numbers, we have $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = \frac{F_n^*}{F_r^* F_s^* F_t^*}$. Therefore, it is obvious that to find the smallest F-trinomial number in u^{th} layer, we need to consider the partition in such a way that we can cancel out maximum number of factors from the numerator and denominator. Here, we neglect the trivial case $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_F = 1$. Therefore, to cancel out maximum number of factors from the numerator, we must take one of the values of the partition of u as $u-1$. Thus, the smallest F-trinomial number in the u^{th} layer is given by $\begin{bmatrix} u \\ 0,1,u-1 \end{bmatrix}_F$.
Also, to find the largest F-trinomial number in each layer, we must find a partition such that the minimum number of factors get canceled out from the numerator and denominator. For that, we must take a partition of p such that the difference between each part is minimum. Therefore, we have three cases.
Case 1: $u \equiv 0(mod3)$. In this case, the partition of u occurs such that the difference between each part is minimum when $(r,s,t) = \left(\frac{u}{3},\frac{u}{3},\frac{u}{3}\right)$. Thus, the largest F-trinomial number in this layer is $\begin{bmatrix} u \\ \frac{u}{3},\frac{u}{3},\frac{u}{3} \end{bmatrix}$.
Case 2: $u \equiv 1(mod3)$. In this case, the partition of u occurs such that the difference between each part is minimum when $(r,s,t) = \left(\frac{u-1}{3},\frac{u-1}{3},\frac{u+2}{3}\right)$. Thus, the largest F-trinomial number in this layer is $\begin{bmatrix} u \\ \frac{u-1}{3},\frac{u-1}{3},\frac{u+2}{3} \end{bmatrix}$.
Case 3: $u \equiv 2(mod3)$. In this case, the partition of u occurs such that the difference between each part is minimum when $(r,s,t) = \left(\frac{u-2}{3},\frac{u+1}{3},\frac{u+1}{3}\right)$. Thus, the largest F-trinomial number in this layer is $\begin{bmatrix} u \\ \frac{u-2}{3},\frac{u+1}{3},\frac{u+1}{3} \end{bmatrix}$. □

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