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Simpson type Tensorial Inequalities for Continuous functions of Selfadjoint operators in Hilbert Spaces

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Abstract *In this paper several tensorial norm inequalities for continuous functions of selfadjoint operators in Hilbert spaces have been obtained. Multiple inequalities are obtained with variations due to the convexity properties of the mapping f*

$$\left\| \frac{1}{6}\left[\mathfrak{f}(\mathfrak{A}) \otimes 1 + 4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1 + 1 \otimes \mathfrak{B}}{2}\right) + 1 \otimes \mathfrak{f}(\mathfrak{B})\right] - \int_0^1 \mathfrak{f}((1-k)\mathfrak{A}\otimes 1 + k1 \otimes \mathfrak{B})dk \right\|$$

$$\leqslant \frac{5}{36} \|1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1\| \, \|\mathfrak{f}'\|_{I,+\infty} .$$

1 Introduction and preliminaries

When Gibbs in 19th century originally came up with the concept of a tensor, he used the term "dyadic" instead of the now formal name "tensor." It is known by its modern name, the mathematical explanation of the tensor definition's genesis. Because of the widespread usage of mathematical inequalities, tensors have also been found as a tool which can also benefit from their use. Inequalities have a significant impact on mathematics and other scientific disciplines. There are many different kinds of inequalities, but those involving Jensen, Ostrowski, Hermite-Hadamard, and Minkowski are of particular importance. Interested readers can learn more about inequalities and their history in these books [18, 24, 25]. Regarding the generalizations of the aforementioned inequalities, numerous studies have been published; for additional information, check the following and the references therein [34, 11, 12, 13, 14, 15, 27, 7, 28, 29, 30, 31, 32, 33, 1, 2, 3, 4, 5, 8, 9, 10].
Since our paper is about tensorial Simpson type inequalities, we give the brief introduction to the topic. The following inequality is well known in the literature as the Simpson inequality:

Theorem 1.1. *Let* $\mathfrak{f} : [a_1, a_2] \to \mathbb{R}$ *be a four times continuously differentiable function on* (a_1, a_2) *and* $\|\mathfrak{f}^4\|_\infty = \sup_{x\in(a,b)} |\mathfrak{f}^4(x)| < +\infty$, *then*

$$\left| \frac{1}{3}\left[\frac{\mathfrak{f}(a_1) + \mathfrak{f}(a_2)}{2} + 2\mathfrak{f}\left(\frac{a_1 + a_2}{2}\right)\right] - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \mathfrak{f}(x)dx \right| \tag{1.1}$$

$$\leq \frac{1}{2880} \left\|\mathfrak{f}^4\right\|_\infty (a_2 - a_1)^4.$$

Recent advances concerning the theory of inequalities in Hilbert spaces will be shown to supplement the presentation of this work. Dragomir [19] gave the following Mond-Pecarić type inequality.

Theorem 1.2. *Let $\mathfrak{A}$ be a selfadjoint operator on the Hilbert space H and assume that $Sp(\mathfrak{A}) \subset [m, M]$ for some scalars m, M with $m < M$. If $\mathfrak{f}$ is a convex function on $[m, M]$, then*

$$\frac{\mathfrak{f}(m)+\mathfrak{f}(M)}{2} \geq \langle \frac{\mathfrak{f}(\mathfrak{A})+\mathfrak{f}((m+M)1_H-\mathfrak{A})}{2} x, x\rangle$$

$$\geq \frac{\mathfrak{f}(\langle \mathfrak{A}x,x\rangle)+\mathfrak{f}(m+M-\langle \mathfrak{A}x,x\rangle)}{2} \geq \mathfrak{f}\left(\frac{m+M}{2}\right),$$

for each $x \in H$ with $||x|| = 1$.
In addition, if $x \in H$ with $||x|| = 1$ and $\langle \mathfrak{A}x, x\rangle \neq \frac{m+M}{2}$, then also

$$\frac{\mathfrak{f}(\langle \mathfrak{A}x,x\rangle)+\mathfrak{f}(m+M-\langle \mathfrak{A}x,x\rangle)}{2}$$

$$\geq \frac{1}{m+M-2\langle \mathfrak{A}x,x\rangle}\int_{\langle \mathfrak{A}x,x\rangle}^{m+M-\langle \mathfrak{A}x,x\rangle} \mathfrak{f}(u)du \geq \mathfrak{f}\left(\frac{m+M}{2}\right).$$

Another interesting result is the Hermite-Hadamard inequality in the selfadjoint operator sense given by Dragomir [20].

Theorem 1.3. *Let $\mathfrak{f} : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\mathfrak{f}\left(\frac{\mathfrak{A}+\mathfrak{B}}{2}\right) \leq \left[\mathfrak{f}\left(\frac{3\mathfrak{A}+\mathfrak{B}}{4}\right)+\mathfrak{f}\left(\frac{\mathfrak{A}+3\mathfrak{B}}{4}\right)\right]$$

$$\leq \int_0^1 \mathfrak{f}((1-t)\mathfrak{A}+t\mathfrak{B})dt$$

$$\leq \frac{1}{2}\left[\mathfrak{f}\left(\frac{\mathfrak{A}+\mathfrak{B}}{2}\right)+\frac{\mathfrak{f}(\mathfrak{A})+\mathfrak{f}(\mathfrak{B})}{2}\right] \leq \frac{\mathfrak{f}(\mathfrak{A})+\mathfrak{f}(\mathfrak{B})}{2}.$$

The first paper related to tensorial inequalities in Hilbert space was written by Dragomir [17]. In the paper, he proved the tensorial version of the Ostrowski type inequality given by the following.
Assume that $\mathfrak{f}$ is continuously differentiable on I with $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}'(t)| < +\infty$ and A, B are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$, then

$$\left\|\mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1+\lambda 1\otimes \mathfrak{B}) - \int_0^1 \mathfrak{f}((1-u)\mathfrak{A}\otimes 1+u1\otimes \mathfrak{B})du\right\|$$

$$\leq \|\mathfrak{f}'\|_{I,+\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes \mathfrak{B}-\mathfrak{A}\otimes 1\|$$

for $\lambda \in [0, 1]$.
Recently, various inequalities in the same tensorial surrounding have been obtained. The following result of Simpson type was obtained by Stojiljković [33].
Assume that $\mathfrak{f}$ is continuously differentiable on I and $|\mathfrak{f}''|$ is convex and A, B are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$, then

$$\left\|\frac{1}{6}\left(\mathfrak{f}(\mathfrak{A})\otimes 1+4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes \mathfrak{B}}{2}\right)+1\otimes \mathfrak{f}(\mathfrak{B})\right)\right.$$

$$-\frac{1}{2}\alpha\Bigg(\int_0^1 \mathfrak{f}\left(\left(\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\left(\frac{1+k}{2}\right)1\otimes \mathfrak{B}\right)k^{\alpha-1}dk$$

$$+\int_0^1 \mathfrak{f}\left(\left(1-\frac{k}{2}\right)\mathfrak{A}\otimes 1+\frac{k}{2}1\otimes\mathfrak{B}\right)(1-k)^{\alpha-1}dk\right)\Bigg\|$$

$$\leq \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|^2\,\frac{\left(\|\mathfrak{f}''(\mathfrak{A})\|+\|\mathfrak{f}''(\mathfrak{B})\|\right)\left(3\alpha^2+8\alpha+7\right)}{(\alpha+2)(24\alpha+24)}$$

for $\alpha \geq 0$. The following inequality has been recently obtained by the same author [32].
Assume that $\mathfrak{f}$ is continuously differentiable on I with $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}''(t)| < +\infty$ and A, B are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$, then

$$\left\|\int_0^1 \mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1+\lambda 1\otimes\mathfrak{B})d\lambda-\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}}{2}\right)\right\|$$

$$\leqslant \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|^2\,\frac{\|\mathfrak{f}'\|_{I,+\infty}}{24}.$$

In order to derive similar inequalities of the tensorial type, we need the following introduction and preliminaries.
Let $\mathrm{I}_1,...,\mathrm{I}_k$ be intervals from $\mathbb{R}$ and let $\mathfrak{f}: I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $\mathfrak{A}=(\mathfrak{A}_1,...,\mathfrak{A}_k)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of $\mathfrak{A}_i$ is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of $\mathfrak{f}$. If

$$\mathfrak{A}_i=\int_{I_i}\lambda_i dE_i(\lambda_i) \tag{1.2}$$

is the spectral resolution of $\mathfrak{A}_i$ for $i=1,...,k$ by following , we define

$$\mathfrak{f}(\mathfrak{A}_1,...,\mathfrak{A}_k):=\int_{I_1}...\int_{I_k} f(\lambda_1,...,\lambda_k)dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k) \tag{1.3}$$

as bounded selfadjoint operator on the tensorial product $H_1\otimes...H_k$.
If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [6] extends the definition of Koranyi [23] for functions of two variables and have the property that

$$\mathfrak{f}(\mathfrak{A}_1,...\mathfrak{A}_k)=\mathfrak{f}_1(\mathfrak{A}_1)\otimes...\otimes\mathfrak{f}_k(\mathfrak{A}_k),$$

whenever $\mathfrak{f}$ can be separated as a product $\mathfrak{f}(t_1,...,t_k)=\mathfrak{f}_1(t_1)...\mathfrak{f}_k(t_k)$ of k functions each depending on only one variable.

Recall the following property of the tensorial product [22],

$$(\mathfrak{A}\mathfrak{C})\otimes(\mathfrak{B}\otimes\mathfrak{D})=(\mathfrak{A}\otimes\mathfrak{B})(\mathfrak{C}\otimes\mathfrak{D})$$

that holds for any $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}\in B(H)$.
From the property we can deduce easily the following consequences

$$\mathfrak{A}^n\otimes\mathfrak{B}^n=(\mathfrak{A}\otimes\mathfrak{B})^n, n\geqslant 0,$$

$$(\mathfrak{A}\otimes 1)(1\otimes\mathfrak{B})=(1\otimes\mathfrak{B})(\mathfrak{A}\otimes 1)=\mathfrak{A}\otimes\mathfrak{B},$$

which can be extended, for two natural numbers m,n we have

$$(\mathfrak{A}\otimes 1)^n(1\otimes\mathfrak{B})^m=(1\otimes\mathfrak{B})^n(\mathfrak{A}\otimes 1)^m=\mathfrak{A}^n\otimes\mathfrak{B}^m.$$

The properties given above can be found in the book [22].
The following Lemma which we require can be found in a paper of Dragomir [16].

Lemma 1.4. *Assume A and B are selfadjoint operators with $Sp(\mathfrak{A})\subset I, Sp(\mathfrak{B})\subset J$ and having the spectral resolutions . Let $f;h$ be continuous on I, g,k continuous on J and ϕ and ψ continuous on an interval K that contains the sum of the intervals $f(I)+g(J); h(I)+k(J)$,then*

$$\phi(\mathfrak{f}(\mathfrak{A})\otimes 1+1\otimes g(\mathfrak{B}))\psi(h(\mathfrak{A})\otimes 1+1\otimes k(\mathfrak{B})) \tag{1.4}$$

$$=\int_I\int_J\phi(\mathfrak{f}(t)+g(s))\psi(h(t)+k(s))dE_t\otimes dF_s.$$

The previously given results showcase how tensorial inequalities can arise according to the inequalities given in the standard analysis.

In [26], Sarikaya et al. obtained inequalities for differentiable convex mappings which are connected with Simpson's inequality, and they used the following lemma to prove this.

Lemma 1.5. *Let $\mathfrak{f} : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I^0 such that $\mathfrak{f}' \in L_1[a,b]$, where $a, b \in I$ with $a < b$, then the following equality holds:*

$$\frac{1}{6}\left[\mathfrak{f}(a) + 4\mathfrak{f}\left(\frac{a+b}{2}\right) + \mathfrak{f}(b)\right] - \frac{1}{b-a}\int_a^b \mathfrak{f}(x)dx \tag{1.5}$$

$$= \frac{b-a}{2}\int_0^1 \left[\left[\frac{t}{2} - \frac{1}{3}\right]\mathfrak{f}'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t}{2}\right)\mathfrak{f}'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right]dt.$$

Novel aspects in this work can be seen in the development of the inequalities of the Simpson type for the differentiable functions in the Hilbert space of tensorial type. This field is relatively new, therefore obtaining new bounds for various convex combinations of the functions is instrumental to the development of it. The rest of the paper is structured as follows, main results is the section in which results concerning the novelty of the work will be given. The following section , some examples and consequences will feature examples of the obtained results by using the known fact about the exponential operator and its integral, therefore by utilizing it and choosing $\mathfrak{f}$ to be a specific convex function, we obtain numerous examples and bounds of the Simpson type in the tensorial sense. In the conclusion section we conclude what has been done in the paper. In the following Theorem, we give a fundamental result which we will use in our paper to produce inequalities

2 Main results

Lemma 2.1. *Assume that $\mathfrak{f}$ is continuously differentiable on I, A and B are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$, then*

$$\frac{1}{6}\left[\mathfrak{f}(\mathfrak{A}) \otimes 1 + 4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1 + 1 \otimes \mathfrak{B}}{2}\right) + 1 \otimes \mathfrak{f}(\mathfrak{B})\right] - \int_0^1 \mathfrak{f}((1-k)\mathfrak{A}\otimes 1 + k1\otimes\mathfrak{B})dk \tag{2.1}$$

$$= \frac{1\otimes\mathfrak{B} - \mathfrak{A}\otimes 1}{2}\int_0^1 \left[\left(\frac{\lambda}{2} - \frac{1}{3}\right)\mathfrak{f}'\left(\frac{1+\lambda}{2}1\otimes\mathfrak{B} + \frac{1-\lambda}{2}\mathfrak{A}\otimes 1\right) + \left(\frac{1}{3} - \frac{\lambda}{2}\right)\mathfrak{f}'\left(\frac{1+\lambda}{2}\mathfrak{A}\otimes 1 + \frac{1-\lambda}{2}1\otimes\mathfrak{B}\right)\right]d\lambda.$$

Proof. We will start the proof with Lemma (1.5). Introducing a substitution on the left hand side, namely $x = \lambda b + (1-\lambda)a$, then assuming that $\mathfrak{A}$ and $\mathfrak{B}$ have the spectral resolutions (1.2)

$$\mathfrak{A} = \int_I tdE(t) \text{ and } \mathfrak{B} = \int_I sdF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, we get

$$\int_I\int_I \left(\frac{1}{6}\left[\mathfrak{f}(t) + 4\mathfrak{f}\left(\frac{t+s}{2}\right) + \mathfrak{f}(s)\right] - \int_0^1 \mathfrak{f}(\lambda s + (1-\lambda)t)d\lambda\right)dE_t \otimes dF_s$$

$$= \int_I\int_I \left(\frac{s-t}{2}\int_0^1 \left[\left[\frac{\lambda}{2} - \frac{1}{3}\right]\mathfrak{f}'\left(\frac{1+\lambda}{2}s + \frac{1-\lambda}{2}t\right) + \left(\frac{1}{3} - \frac{\lambda}{2}\right)\mathfrak{f}'\left(\frac{1+\lambda}{2}t + \frac{1-\lambda}{2}s\right)\right]d\lambda\right)dE_t \otimes dF_s.$$

Considering the left hand side

$$\int_I\int_I\left(\frac{1}{6}\left[\mathfrak{f}(t)+4\mathfrak{f}\left(\frac{t+s}{2}\right)+\mathfrak{f}(s)\right]-\int_0^1\mathfrak{f}(\lambda s+(1-\lambda)t)d\lambda\right)dE_t\otimes dF_s,$$

we provide the proof for the part which involves a function alone and the part which consists of three integrals, the remaining factors are obtained in an analogous way.

By utilizing the Fubinis Theorem on the integrals and (1.4) for appropriate choices of the functions involved, we have successively

$$\int_I\int_I\mathfrak{f}(t)dE_t\otimes dF_s=\mathfrak{f}(\mathfrak{A})\otimes 1,$$

$$\int_I\int_I\int_0^1\mathfrak{f}((1-k)t+ks)dkdE_t\otimes dF_s=\int_0^1\int_I\int_I\mathfrak{f}((1-k)t+ks)dE_t\otimes dF_sdk$$
$$=\int_0^1\mathfrak{f}((1-k)\mathfrak{A}\otimes 1+k1\otimes\mathfrak{B})dk.$$

Considering the right hand side, we provide a proof for the product of the integral and the factor, we obtain

$$\int_I\int_I\int_0^1\left[\frac{\lambda}{2}-\frac{1}{3}\right]\frac{(s-t)}{2}\mathfrak{f}'\left(\frac{1+\lambda}{2}s+\frac{1-\lambda}{2}t\right)d\lambda dE_t\otimes dF_s$$
$$=\int_0^1\left[\frac{\lambda}{2}-\frac{1}{3}\right]\int_I\int_I\frac{(s-t)}{2}\mathfrak{f}'\left(\frac{1+\lambda}{2}s+\frac{1-\lambda}{2}t\right)dE_t\otimes dF_sd\lambda$$
$$=\int_0^1\frac{(1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1)}{2}\mathfrak{f}'\left(\frac{1+\lambda}{2}1\otimes\mathfrak{B}+\frac{1-\lambda}{2}\mathfrak{A}\otimes 1\right)\left[\frac{\lambda}{2}-\frac{1}{3}\right]d\lambda.$$

□

Theorem 2.2. *Assume that $\mathfrak{f}$ is continuously differentiable on I with $\|\mathfrak{f}'\|_{I,+\infty}:=\sup_{t\in I}|\mathfrak{f}'(t)|<+\infty$ and A,B are selfadjoint operators with $Sp(\mathfrak{A}),Sp(\mathfrak{B})\subset I$, then*

$$\left\|\frac{1}{6}\left[\mathfrak{f}(\mathfrak{A})\otimes 1+4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}}{2}\right)+1\otimes\mathfrak{f}(\mathfrak{B})\right]-\int_0^1\mathfrak{f}((1-k)\mathfrak{A}\otimes 1+k1\otimes\mathfrak{B})dk\right\|$$
$$\leqslant\frac{5}{36}\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\,\|\mathfrak{f}'\|_{I,+\infty}. \tag{2.2}$$

Proof. If we take the operator norm of the previously obtained Lemma (2.1) and use the triangle inequality, we get

$$\left\|\frac{1}{6}\left[\mathfrak{f}(\mathfrak{A})\otimes 1+4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}}{2}\right)+1\otimes\mathfrak{f}(\mathfrak{B})\right]-\int_0^1\mathfrak{f}((1-k)\mathfrak{A}\otimes 1+k1\otimes\mathfrak{B})dk\right\|$$
$$\leq\frac{\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|}{2}\left\|\int_0^1\left[\left(\frac{\lambda}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\frac{1+\lambda}{2}1\otimes\mathfrak{B}+\frac{1-\lambda}{2}\mathfrak{A}\otimes 1\right)\right.\right.$$
$$\left.\left.+\left(\frac{1}{3}-\frac{\lambda}{2}\right)\mathfrak{f}'\left(\frac{1+\lambda}{2}\mathfrak{A}\otimes 1+\frac{1-\lambda}{2}1\otimes\mathfrak{B}\right)\right]d\lambda\right\|.$$

Realize here that by (1.4),

$$\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right|=\int_I\int_I\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}t+\frac{1+\lambda}{2}s\right)\right|dE_t\otimes dF_s.$$

Since

$$\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}t+\frac{1+\lambda}{2}s\right)\right|\leqslant\|\mathfrak{f}'\|_{I,+\infty},$$

for all $t, s \in I$. If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get by (1.4)

$$\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right| = \int_I\int_I \left|\mathfrak{f}'\left(\frac{1-\lambda}{2}t+\frac{1+\lambda}{2}s\right)\right| dE_t\otimes dF_s$$
$$\leqslant \|\mathfrak{f}'\|_{I,+\infty}\int_I\int_I dE_t\otimes dF_s = \|\mathfrak{f}'\|_{I,+\infty}.$$

From which we get the following,

$$\int_0^1 \left\|\frac{\lambda}{2}-\frac{1}{3}\right\| \left\|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right\| d\lambda \leqslant \|\mathfrak{f}'\|_{I,+\infty}\int_0^1\left\|\frac{\lambda}{2}-\frac{1}{3}\right\|d\lambda = \frac{5}{36}\|\mathfrak{f}'\|_{I,+\infty}.$$

The calculation has been shown for the multiple of $\left\|\frac{\lambda}{2}-\frac{1}{3}\right\|$, other multiple with $\left\|\frac{1}{3}-\frac{\lambda}{2}\right\|$ uses the same technique. Summing everything, we obtain the desired result. □

Theorem 2.3. *Assume that $\mathfrak{f}$ is continuously differentiable on I and $|\mathfrak{f}'|$ is convex and A, B are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$, then*

$$\left\|\frac{1}{6}\left[\mathfrak{f}(\mathfrak{A})\otimes 1+4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}}{2}\right)+1\otimes\mathfrak{f}(\mathfrak{B})\right]-\int_0^1\mathfrak{f}((1-k)\mathfrak{A}\otimes 1+k1\otimes\mathfrak{B})dk\right\|$$
$$\leqslant \frac{5}{72}\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\left(\|\mathfrak{f}'(\mathfrak{A})\|+\|\mathfrak{f}'(\mathfrak{B})\|\right). \tag{2.3}$$

Proof. Since $|\mathfrak{f}'|$ is convex on I, then we get

$$\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}t+\frac{1+\lambda}{2}s\right)\right| \leqslant \frac{1-\lambda}{2}|\mathfrak{f}'(t)|+\frac{1+\lambda}{2}|\mathfrak{f}'(s)|$$

for all $\lambda \in [0,1]$ and $t, s \in I$.
If we take the integral $\int_I\int_I$ over $dE_t \otimes dF_s$, then we get by (1.4)

$$\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right| = \int_I\int_I \left|\mathfrak{f}'\left(\frac{1-\lambda}{2}t+\frac{1+\lambda}{2}s\right)\right| dE_t\otimes dF_s$$
$$\leqslant \int_I\int_I\left[\frac{1-\lambda}{2}|\mathfrak{f}'(t)|+\frac{1+\lambda}{2}|\mathfrak{f}'(s)|\right]dE_t\otimes dF_s$$
$$= \frac{1-\lambda}{2}|\mathfrak{f}'(\mathfrak{A})|\otimes 1+\frac{1+\lambda}{2}1\otimes|\mathfrak{f}'(\mathfrak{B})|$$

for all $\lambda \in [0,1]$.
If we take the norm in the inequality, we get the following

$$\left\|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right\| \leqslant \left\|\frac{1-\lambda}{2}|\mathfrak{f}'(\mathfrak{A})|\otimes 1+\frac{1+\lambda}{2}1\otimes|\mathfrak{f}'(\mathfrak{B})|\right\|$$
$$\leqslant \frac{1-\lambda}{2}\||\mathfrak{f}'(\mathfrak{A})|\otimes 1\|+\frac{1+\lambda}{2}\|1\otimes|\mathfrak{f}'(\mathfrak{B})|\|$$
$$= \frac{1-\lambda}{2}\|\mathfrak{f}'(\mathfrak{A})\|+\frac{1+\lambda}{2}\|\mathfrak{f}'(\mathfrak{B})\|.$$

Therefore, we obtain

$$\int_0^1\left\|\frac{\lambda}{2}-\frac{1}{3}\right\|\left\|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right\|d\lambda$$
$$\leqslant \int_0^1\left\|\frac{\lambda}{2}-\frac{1}{3}\right\|\left(\frac{1-\lambda}{2}\|\mathfrak{f}'(\mathfrak{A})\|+\frac{1+\lambda}{2}\|\mathfrak{f}'(\mathfrak{B})\|\right)d\lambda.$$
$$= \frac{29\|\mathfrak{f}'(\mathfrak{A})\|+61\|\mathfrak{f}'(\mathfrak{B})\|}{648}.$$

The procedure is analogous for the other part, namely for $f'\left(\frac{1+\lambda}{2}t+\frac{1-\lambda}{2}s\right)$. Adding everything up yields the desired result. □

We recall that the function $\mathfrak{f} : I \to \mathbb{R}$ *is quasi-convex [21], if*

$$\mathfrak{f}((1-\lambda)t+\lambda s) \leqslant \max(\mathfrak{f}(t),\mathfrak{f}(s)) = \frac{1}{2}(\mathfrak{f}(t)+\mathfrak{f}(s)+|\mathfrak{f}(s)-\mathfrak{f}(t)|)$$

holds for all $t, s \in I$ *and* $\lambda \in [0,1]$.

Theorem 2.4. *Assume that* $\mathfrak{f}$ *is continuously differentiable on I with* $|\mathfrak{f}'|$ *is quasi-convex on* I*,* A *and* B *are selfadjoint operators with* $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$*, then*

$$\left\|\frac{1}{6}\left[\mathfrak{f}(\mathfrak{A})\otimes 1+4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}}{2}\right)+1\otimes\mathfrak{f}(\mathfrak{B})\right]-\int_0^1 \mathfrak{f}((1-k)\mathfrak{A}\otimes 1+k1\otimes\mathfrak{B})dk\right\| \leqslant \frac{\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|}{72}(\||\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|\|+\||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})|\|). \tag{2.4}$$

Proof. Since $|\mathfrak{f}'|$ is quasi-convex on I, then we get

$$\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}t+\frac{1+\lambda}{2}s\right)\right| \leqslant \frac{1}{2}(|\mathfrak{f}'(t)|+|\mathfrak{f}'(s)|+||\mathfrak{f}'(t)|-|\mathfrak{f}'(s)||)$$

for all $\lambda \in [0,1]$ and $t, s \in I$. If we take the integral $\int_I\int_I$ over $dE_t \otimes dF_s$, then we get by (1.4)

$$\begin{aligned}
&\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right| \\
&= \int_I\int_I\left|\mathfrak{f}'\left(\frac{1-\lambda}{2}t+\frac{1+\lambda}{2}s\right)\right|dE_t\otimes dF_s \\
&\leqslant \frac{1}{2}\int_I\int_I(|\mathfrak{f}'(t)|+|\mathfrak{f}'(s)|+||\mathfrak{f}'(t)|-|\mathfrak{f}'(s)||)dE_t\otimes dF_s \\
&= \frac{1}{2}(|\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|+||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})||)
\end{aligned}$$

for all $\lambda \in [0,1]$.
If we take the norm, then we get

$$\begin{aligned}
&\left\|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right\| \\
&\leqslant \left\|\frac{1}{2}(|\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|+||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})||)\right\| \\
&\leqslant \frac{1}{2}\left(\||\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|\|+\||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})|\|\right)
\end{aligned}$$

Which when applied in our case, we get

$$\begin{aligned}
&\int_0^1\left\|\frac{\lambda}{2}-\frac{1}{3}\right\|\left\|\mathfrak{f}'\left(\frac{1-\lambda}{2}\mathfrak{A}\otimes 1+\frac{1+\lambda}{2}1\otimes\mathfrak{B}\right)\right\|d\lambda \\
&\leqslant \int_0^1\left\|\frac{\lambda}{2}-\frac{1}{3}\right\|\left(\frac{1}{2}\left(\||\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|\|+\||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})|\|\right)\right)d\lambda.
\end{aligned}$$

The procedure is analogous for the other part, namely for $f'\left(\frac{1+\lambda}{2}t+\frac{1-\lambda}{2}s\right)$. Adding everything up yields the desired result. □

3 Some examples and consequences

It is known that if U and V are commuting, that is $UV = VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U + V).$$

Also, if U is invertible and $a, b \in \mathbb{R}$ and $a < b$ then

$$\int_a^b \exp(tU)dt = U^{-1}[\exp(bU) - \exp(aU)].$$

Moreover, if U and V are commuting and $V - U$ is invertible, then

$$\int_0^1 \exp((1-k)U + kV)dk = \int_0^1 \exp(k(V-U))\exp(U)dk$$

$$= \left(\int_0^1 \exp(k(V-U))dk\right)\exp(U)$$

$$= (V-U)^{-1}[\exp(V-U) - I]\exp(U) = (V-U)^{-1}[\exp(V) - \exp(U)].$$

Since the operators $U = \mathfrak{A} \otimes 1$ and $V = 1 \otimes \mathfrak{B}$ are commutative and if $1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1$ is invertible, then

$$\int_0^1 \exp((1-k)\mathfrak{A} \otimes 1 + k 1 \otimes \mathfrak{B})dk$$

$$= (1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1)^{-1}[\exp(1 \otimes \mathfrak{B}) - \exp(\mathfrak{A} \otimes 1)].$$

Corollary 3.1. *If A, B are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset [m, M]$ and $1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1$ is invertible, then by Theorem 2.1 (2.2), we get*

$$\Bigg\| \frac{1}{6}\left[\exp(\mathfrak{A}) \otimes 1 + 4\exp\left(\frac{\mathfrak{A} \otimes 1 + 1 \otimes \mathfrak{B}}{2}\right) + 1 \otimes \exp(\mathfrak{B})\right] \tag{3.1}$$

$$-(1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1)^{-1}\left[\exp(1 \otimes \mathfrak{B}) - \exp(\mathfrak{A} \otimes 1)\right] \Bigg\|$$

$$\leqslant \frac{5}{36}\|1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1\| \exp(M).$$

Corollary 3.2. *Since for $\mathfrak{f}(t) = \exp(t), t \in \mathbb{R}$, $|\mathfrak{f}'|$ is convex, then by Theorem 2.2 (2.3)*

$$\Bigg\| \frac{1}{6}\left[\exp(\mathfrak{A}) \otimes 1 + 4\exp\left(\frac{\mathfrak{A} \otimes 1 + 1 \otimes \mathfrak{B}}{2}\right) + 1 \otimes \exp(\mathfrak{B})\right] \tag{3.2}$$

$$-(1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1)^{-1}\left[\exp(1 \otimes \mathfrak{B}) - \exp(\mathfrak{A} \otimes 1)\right] \Bigg\|$$

$$\leqslant \frac{5}{72}\|1 \otimes \mathfrak{B} - \mathfrak{A} \otimes 1\| \left(\|\exp(\mathfrak{A})\| + \|\exp(\mathfrak{B})\|\right).$$

4 Conclusion

Tensors have become important in various fields such as physics as they provide a concise mathematical framework for the formulation and solution of physical problems in fields such as mechanics, electromagnetism and quantum theory and many others. As such inequalities are useful in numerical aspects. This work reflects the tensorial version of Sarikaya's lemma, allowing us to obtain Simpson type inequalities in Hilbert spaces. New inequalities of Simpson type are given, along with examples of specific convex functions and their inequalities based on our results are presented in the "Some examples and consequences" section.

Plans for future research can be reflected in the fact that the obtained inequalities in this work can be sharpened or generalized by using other methods. An interesting perspective can be seen in incorporating other techniques for Hilbert space inequalities with the techniques shown in this paper. One direction is the technique of the Mond-Pecaric inequality, on which we will work on. Motivation in further research can be seen in obtaining sharper inequalities of the Trapezoid,Ostrowski,Midpoint type, as inequalities arise in various applications as tensors are a useful tool in physics.

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