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A Note on Fibonacci Numbers and the Golden Ratio of Order k

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Abstract We define and study the notion of the golden ratio of order $k \geq 0$, denoted ϕ_k, as a generalized form of the golden ratio ϕ for any real number $k \geq 0$. We show that similar to the special case of ϕ and its conjugate ψ, ϕ_k and ψ_k are the two distinct roots of a quadratic polynomial for any fixed real $k \geq 0$. We express some numerical and algebraic properties of ϕ_k and ψ_k and write their relations to ϕ and ψ, respectively, with some examples for some special values of k. In particular, it is shown that $\phi_k = \phi$ and $\psi_k = \psi$ if and only if $k = 0$. We show that $\mathbb{Z}[(k+1)\phi_k]$ is a subring of the ring $\mathbb{Z}[\phi]$ for any nonnegative integer k. We will define the golden rectangle of order k (or k-golden rectangle for short) with a class of examples for all $k \geq 0$. We also discuss some cases of two Fibonacci numbers in connection to the golden ratio. We will show that the ratio of height to width of the pages of the Gutenberg Bible is the golden ratio of order $k \neq 0$. Actually, some erroneous ideas and examples of disputed observations related to the golden ratio are good reasons to apply ϕ_k to improve the measurements regarding ϕ for some $k \neq 0$. Finally, we end the paper by posing a question related to the Penrose tiling and quasicrystals in connection to the golden ratio of order $k > 0$.

1 Introduction

The main goal of this paper is to generalize the notion of the (classic) *golden ratio* ϕ to the *golden ratio of order* k, denoted ϕ_k, for any fixed real number $k \geq 0$ (Definition 3.1).

In this section and the last part of Section 2, we recall some definitions and results related to *the Fibonacci numbers*. In the next section, we mainly focus on the golden ratio ϕ and compare those results with our results for our generalized ϕ_k and ψ_k. We will discuss the main results in the third section.

• In Section 3, we study some *numerical and algebraic properties* of ϕ_k and ψ_k and their relations to ϕ and ψ, respectively, (Theorems 3.4, 3.6, Remarks 3.2, and 3.9) with some examples for some special values of k (Example 3.3). In particular, it is shown that $\phi_k = \phi$ and $\psi_k = \psi$ if and only if $k = 0$. We show that $\mathbb{Z}[(k+1)\phi_k]$ is a *subring* of the *ring* $\mathbb{Z}[\phi_0 = \phi]$ for any nonnegative integer k (Remark 3.9). Similar to the case of ϕ^n and ψ^n, it is also shown that ϕ_k^n [resp. ψ_k^n] can be *decomposed* into a *linear combination* of ϕ_k [resp. ψ_k] and a *constant* for any integer $n \geq 2$ (Remark 3.8). It is shown that ϕ_k and ψ_k are *bijective maps* from nonnegative reals to $(1, \phi]$ and $[\psi, 1)$, respectively, (Propositions 3.5 and 3.7). We also define *the golden rectangle of order* k (or k-golden rectangle for short), as a general form of *the classic golden rectangle*, with a class of examples for any real $k \geq 0$. We will show that the ratio of height to width of the pages of *the Gutenberg Bible* is the golden ratio of order $k \neq 0$ (Example 3.11 and Remark 3.10). Finally, we end the paper by posing a question related to *the Penrose tiling and quasicrystals* in connection to the golden ratio of order $k > 0$.

The Fibonacci numbers, commonly denoted by F_n, form a *sequence*, the *Fibonacci sequence*, in which each term is the sum of the two preceding ones. For the sake of convenience, f_n will denote the nth Fibonacci number in the sequel. The sequence commonly starts from 0 and 1, although some authors omit the initial terms and start the sequence from 1 and 1 or from 1 and 2. Starting from 0 and 1, the next few values in the sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots.$$

The Fibonacci numbers are also an example of a *complete sequence*. This means that every positive integer can be written as a sum of Fibonacci numbers, where any one number is used once at most. Moreover, every positive integer can be written *in a unique way* as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This is known as *Zeckendorf's theorem*, and a sum of Fibonacci numbers that satisfies these conditions is called a *Zeckendorf representation*.

Note that The authors, in [1], introduced the notion of *the Fibonacci representation*, as a general form of the Zeckendorf representation and provided some *upper bounds* for it by using a combination of *the elementary arithmetic and the center-of-mass technique*. Also, for a detailed study of the center-of-mass technique applying to *elementary arithmetic problems* (e.g. inequalities), see [2], which is an elementary approach to a class of the Diophantine equations using center of mass.

• The authors assume that the reader is familiar with the results in number theory that we use in this article. Actually, the internet search will provide all necessary sources that are required and mentioned in this paper without any direct reference.

2 The Golden Ratio and some Comparative Cases of Two Fibonacci Numbers

In this section we mainly focus on the golden ratio and discuss a few simple properties of Fibonacci numbers (Proposition 2.2, Corollary 2.3, and Remark 2.4) and conclude the section with a result related to *the limits of the (consecutive) quotients of Fibonacci numbers in general*. We recall some definitions and properties related to the golden ratio ϕ and its conjugate ψ for the sake of comparison with the generalized forms given in *the main section of the paper* (Section 3).

We now start with the definition of the golden ratio, denoted by ϕ.

• Two quantities, real numbers $a > b > 0$, are in the golden ratio if their ratio is the same as the *ratio of their sum to the larger of the two quantities*. That is,

$$\frac{a+b}{a} = \frac{a}{b} = \phi,$$

where *the Greek letter "phi"* (ϕ) denotes the golden ratio and from the above identity, we get *the quadratic equation*

$$x^2 - x - 1 = 0$$

by assuming $x = \frac{a}{b}$.
Thus,

$$\phi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

are *two distinct roots* of the above quadratic equation and they are *algebraic numbers* (i.e. a number is algebraic if it is the root of a *polynomial equation with integer coefficients*).

Clearly, the *constant* ϕ satisfies the quadratic equation $\phi^2 = \phi + 1$, and is an *irrational number* with a value of

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618033988749 \cdots.$$

and

$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.618033\cdots.$$

Because ϕ is *a ratio between positive quantities*, ϕ is necessarily the *positive root*. The *negative root* is in fact the *negative inverse* $-\frac{1}{\phi}$, which *shares many properties* with the golden ratio and we have

$$\psi = \frac{1-\sqrt{5}}{2} = 1-\phi = -\frac{1}{\phi} \approx -0.6180339887\cdots.$$

Historically, there is a rectangle whose proportions are found most pleasing to the eye. It is neither too fat nor too skinny, neither too long nor too short. As a result of the definition of ϕ, we can define a *golden rectangle* as follows:

• A rectangle is called a golden rectangle if the ratio of the longer side to the shorter side is the golden ratio. For example, a rectangle with sides equal to $1+\sqrt{5}$ and 2 is a golden rectangle. In the next section, we will define the notion of the *golden rectangle of order* $k \geq 0$ for any nonnegative real number k.

Note that, from the above quadratic equation, *the sum and the product* of ϕ and ψ, respectively, are

$$\phi + \psi = 1 \text{ and } \phi\psi = -1.$$

Clearly, from the above identity, we have $\psi = -\phi^{-1}$.

• As a result of the equation $x^2 = x + 1$, one can easily *decompose any positive integer power* of ϕ into a *linear combination* of ϕ and a constant as follows.

Remark 2.1. Since the golden ratio satisfies the equation $\phi^2 = \phi + 1$, this expression can be used to *decompose higher powers* ϕ^n as a function of lower powers, which in turn can be decomposed all the way down to a *linear combination* of ϕ and a constant using induction on $n \geq 2$, an integer.

• Note that similar to the above remark, we can also *decompose* ψ^n ($n \geq 2$, an integer) into a *linear combination* of ψ and a constant since $\psi^2 = \psi + 1$.

We now write some *simple relations* between two or three Fibonacci numbers.

Proposition 2.2. *Let* $n \geq 3$ *be a fixed integer and* $f_{n-2} + f_{n-1} = f_n$ *three successive Fibonacci numbers. Then*

(a) $f_{n-1} < (f_n + f_{n-2})/2$.

(b) $2f_{n-2} < f_n$.

(c) $f_n \leq 3f_{n-2}$.

Proof. (a) $f_{n-3} = f_{n-1} - f_{n-2} < f_{n-2}$ implies

$$2f_{n-1} - f_{n-2} < f_{n-1} + f_{n-2}$$

implies

$$2f_{n-1} < f_n + f_{n-2},$$

which implies the desired result.

(b) $f_{n-2} < f_{n-1}$ implies $2f_{n-2} < f_{n-1} + f_{n-2} = f_n$.

(c) $f_{n-3} = f_{n-1} - f_{n-2} < f_{n-2}$ implies

$$f_{n-1} < 2f_{n-2}$$

implies

$$f_{n-1} + f_{n-2} < 3f_{n-2},$$

which implies the desired result. □

Corollary 2.3. *Let $n \geq 3$ be a fixed integer and $f_{n-2} + f_{n-1} = f_n$ three successive Fibonacci numbers. Then*

$$2f_{n-2} \leq f_n \leq 3f_{n-2}.$$

Proof. The proof follows directly from parts (b) and (c) of the proposition above. □

Remark 2.4. It is well-known that the sequence $\{\frac{f_n}{f_{n-1}}\}_2^\infty$ approaches the golden ratio as n goes to infinity. The golden ratio is

$$\frac{1+\sqrt{5}}{2} \approx 1.61803398875\cdots,$$

which is less than 2. But from the corollary above, we have $\frac{f_n}{f_{n-2}} \in [2,3]$ for all $n \geq 3$.

We now write the *limit of the (consecutive) quotients* of the Fibonacci numbers for *a special case* and we end this section with results for the general case.

From the remark above, we have

$$\lim_{n\to\infty} \frac{f_n}{f_{n-2}} = \lim \frac{f_{n-2}+f_{n-1}}{f_{n-2}} = \lim(1 + \frac{f_{n-1}}{f_{n-2}}) = 1 + \phi = \phi^2 \approx 2.61803398875\cdots \in [2,3].$$

We now end the section with a result related to the limits of the (consecutive) quotients of the Fibonacci numbers in general, which is ϕ^m for any integer $m \geq 1$. Note that in the next section we will show that $\phi_1 = \frac{1}{2}\phi^2$ (Example 3.3(b)) and show that $\phi_k \in (1,\phi]$ for all $k \geq 0$ (Proposition 3.5), which is different from ϕ^m for any integer $m \geq 1$, except for the case $\phi_k = \phi$ if and only if $k = 0$.

$$\lim_{n\to\infty} \frac{f_{n+m}}{f_n} = \phi^m,$$

because the ratios between consecutive Fibonacci numbers approaches ϕ. That is,

$$\frac{f_{n+m}}{f_n} = \frac{f_{n+1}}{f_n} \frac{f_{n+2}}{f_{n+1}} \frac{f_{n+3}}{f_{n+2}} \cdots \frac{f_{n+m}}{f_{n+m-1}}$$

which approaches the product of m factors of ϕ as n goes to infinity. Note that by the limit law for product of the convergent sequences, the limit of the product of a finite number of convergent sequences is equal to the product of their limits.

3 The Golden Ratio of Order *k*

In this section, we study some *numerical and algebraic properties* of ϕ_k and ψ_k and their relations to ϕ and ψ, respectively, (Theorems 3.4, 3.6, Remarks 3.2, and 3.9) with some examples for some special values of k (Example 3.3). In particular, it is shown that $\phi_k = \phi$ and $\psi_k = \psi$ if and only if $k = 0$. We show that $\mathbb{Z}[(k+1)\phi_k]$ is a subring of the ring $\mathbb{Z}[\phi_0 = \phi]$ for any nonnegative integer k (Remark 3.9). In a manner similar to the cases of ϕ^n and ψ^n, it is also shown that ϕ_k^n [resp. ψ_k^n] can be decomposed into a linear combination of ϕ_k [resp. ψ_k] and a constant for any integer $n \geq 2$ (Remark 3.8). It is shown that ϕ_k and ψ_k are bijective maps from nonnegative reals to $(1,\phi]$ and $[\psi, 1)$, respectively, (Propositions 3.5 and 3.7). We also define the golden rectangle of order k (or k-golden rectangle for short), as a general form of the classic golden rectangle, with a class of examples for any real $k \geq 0$. We will show that the ratio of height to width of the pages of the Gutenberg Bible is the golden ratio of order $k \neq 0$ (see Example 3.11 and Remark 3.10). Finally, we end the paper by posing a question related to the Penrose tiling and quasicrystals in connection to the golden ratio of order $k > 0$.

We now extend the notion of the golden ratio.

Definition 3.1. Let $a > b > 0$ be two real numbers and $k \geq 0$ a fixed real number. We say that a/b is a golden ratio of order k, denoted ϕ_k, if it satisfies the following identity

$$\frac{a}{b} = \frac{(2k+1)(k+1)a - (k^2+k-1)b}{(k+1)^2 a}$$

From the above identity, we get the quadratic equation

$$(k+1)^2x^2 - (2k+1)(k+1)x + (k^2+k-1) = 0$$

by assuming $x = \frac{a}{b}$.
Thus,

$$\phi_k = \frac{(2k+1)+\sqrt{5}}{2(k+1)} \text{ and } \psi_k = \frac{(2k+1)-\sqrt{5}}{2(k+1)}$$

are *two distinct roots* of the above quadratic equation and they are *algebraic numbers* when $k \geq 0$ is an integer (i.e. a number is algebraic if it is the root of a polynomial equation with integer coefficients). Obviously, $\phi_0 = \phi$ and $\psi_0 = \psi$ (see the previous section for the definition of ϕ and ψ).

As a result of the above definition (Definition 3.1), we can extend the notion of a golden rectangle as follows:

• A rectangle is called a *golden rectangle of order* k (or a k-golden rectangle for short) if the ratio of the longer side to the shorter side is the golden ratio of order k ($k \geq 0$ a fixed real number). For example, a rectangle with sides equal to $(2k+1)+\sqrt{5}$ and $2(k+1)$ is a golden rectangle of order $k \geq 0$.

Note that, from the above quadratic equation, the *sum and the product* of ϕ_k and ψ_k, respectively, are

$$\phi_k + \psi_k = \frac{(2k+1)}{(k+1)}$$

and

$$\phi_k\psi_k = \frac{k^2+k-1}{(k+1)^2}.$$

Clearly, from the above identity, we have

$$\psi_k = \frac{(k^2+k-1)}{(k+1)^2}\phi_k^{-1},$$

which satisfies the special case for $k = 0$.

Remark 3.2. We now write the relationship between ϕ_k and ψ_k with ϕ and ψ, respectively, as follows:

$$\phi_k = \frac{(2k+1)+\sqrt{5}}{2(k+1)} = \frac{1}{k+1}(k+\phi)$$

and

$$\psi_k = \frac{(2k+1)-\sqrt{5}}{2(k+1)} = \frac{1}{k+1}(k+\psi).$$

• Clearly $\phi_k = \frac{a}{b}$ since a/b is larger than 1 and $\psi_k \neq \frac{a}{b}$ since $\psi_k < 1$ for all $k \geq 0$ from the fact that $\psi \approx -0.6180339887 \cdots < 0$ (see Theorem 3.6(c)).
We now write some examples of ϕ_k and ψ_k for some *special values* of k.

Example 3.3. The following are true.

(a) $k = 0$, then $\phi_0 = \phi$ and $\psi_0 = \psi$.

(b) $k = 1$, then $\phi_1 = \frac{1}{2}(1+\phi) = \frac{\phi^2}{2}$.

(c) $k = 1$, then $\psi_1 = \frac{1}{2}(1+\psi) = \frac{\psi^2}{2}$.

(d) $k=\phi$, then $\phi_k = \frac{1}{\phi+1}(\phi+\phi) = \frac{1}{\phi^2}(2\phi) = \frac{2}{\phi} = 2\phi^{-1} = -2\psi$.

(e) $k = \frac{1+\phi}{2}$, then $\phi_k = \frac{1+3\phi}{3+\phi}$.

(f) $k = \frac{1+\psi}{2}$, then $\psi_k = \frac{1+3\psi}{3+\psi}$.

(g) $k=-\psi$, then $\psi_k = 0$.

The following two theorems provide some *numerical properties* of ϕ_k and ψ_k, respectively, for any (fixed) nonnegative real number k.

Theorem 3.4. *The following are true.*

(a) *Let $k \geq 0$ be a real number. Then $\phi_k = \phi$ if and only if $k = 0$.*

(b) *If $k \geq 0$ is a real number, then $\phi_k \neq 1$.*

(c) *If $k \neq 0$ is a positive real number, then $1 < \phi_k < \phi$.*

(d) $\lim_{k\to\infty}\phi_k = 1$.

Proof. (a) The proof of the sufficiency is clear. To prove the necessity, assume that $\phi_k = \phi$. Thus, $(k+1)\phi = k+\phi$ implies $k(\phi-1)=0$, which implies $k=0$ since $\phi \neq 1$.

(b) Suppose $\phi_k = 1$. Then $(k+1) = k+\phi$ implies $\phi = 1$, which is a contradiction. (c) Since $1<\phi$, then $k < k\phi$ implies $(k+\phi) < k\phi+\phi = (k+1)\phi$, which implies $\phi_k < \phi$. Also, $1<\phi$ implies $k+1 < k+\phi$, which implies $1 < \phi_k$.

(d) $\phi_k = \frac{1}{k+1}(k+\phi) = \frac{k+1}{k+1} - \frac{1}{k+1} + \frac{\phi}{k+1} = 1 - \frac{1}{k+1} + \frac{\phi}{k+1}$
Now the result follows as k goes to infinity. □

We now show that ϕ_k is a *bijective map* from nonnegative reals to the half open interval $(1,\phi]$.

Proposition 3.5. *Let $k \geq 0$ be a real number. Then $\phi_k : \mathbb{R}^+ = [0,\infty) \to (1,\phi]$ is a bijection (i.e. one-to-one and onto map).*

Proof. It is injective since $\phi_k = \phi_l$ implies $(l+1)(k+\phi) = (k+1)(l+\phi)$ implies that $(l-k)(\phi-1) = 0$, which implies $l-k=0$ since $\phi \neq 1$ and hence $l=k$.

It is also surjective since for any $\alpha \in (1,\phi]$, $\phi_k = \alpha$ when $k = \frac{\phi-\alpha}{\alpha-1}$. Note that $k \geq 0$ since $(\phi-\alpha) \geq 0$ and $(\alpha-1) > 0$. □

In the following theorem, we will see *some common properties* of ψ_k and ϕ_k as discussed in the above theorem for ϕ_k.

Theorem 3.6. *The following are true.*

(a) *Let $k \geq 0$ be a real number. Then $\psi_k = \psi$ if and only if $k=0$.*

(b) *If $k \geq 0$ is a real number, then $\psi_k \neq 1$.*

(c) *If $k \geq 0$ is a real number, then $\psi_k < 1$.*

(d) *If $k > -\psi$, then $\psi_k > 0$.*

(e) *If $0 \leq k \leq -\psi$, then $\psi_k \leq 0$; and for the special cases, $\psi_k = \psi$ or 0 whenever $k=0$ or $k=-\psi$, respectively.*

(f) $\lim_{k\to\infty}\psi_k = 1$.

Proof. (a) The proof of the sufficiency is clear. To prove the necessity, assume that $\psi_k = \psi$. Thus, $(k+1)\psi = k+\psi$ implies $k(\psi-1)=0$, which implies $k=0$ since $\psi \approx -0.6180339887\cdots \neq 1$.

(b) Suppose $\psi_k = 1$. Then $(k+1) = k+\psi$ implies $\psi = 1$, which is a contradiction since $\psi \approx -0.6180339887\cdots \neq 1$.

(c) Since $\psi < 0$, then $k+\psi < k < k+1$ implies $\psi_k < 1$. Note that $k+1$ is positive for all $k \geq 0$. (d) The proof follows directly from the fact that $k+1$ is positive for all $k \geq 0$ and $k+\psi > 0$ from the hypothesis.

(e) Since $0 \leq k \leq -\psi$, then $k+1 > 0$ and $k+\psi \leq 0$ implies the desired result; and the special cases are clear.

(f) The proof is similar to the proof of Part (d) of Theorem 3.4. □

Now, in a manner similar to the proof of Proposition 3.5, we show that ψ_k is a *bijective map* from nonnegative reals to the half open interval $[\psi, 1)$.

Proposition 3.7. *Let $k \geq 0$ be a real number. Then $\psi_k : \mathbb{R}^+ = [0, \infty) \to [\psi, 1)$ is a bijection (i.e. one-to-one and onto map).*

Proof. ψ_k is injective since $\psi_k = \psi_l$ implies

$$(l+1)(k+\psi) = (k+1)(l+\psi)$$

implies that $(l-k)(\psi-1) = 0$, which implies $l-k=0$ since $\psi \neq 1$; Hence $l = k$.

It is also surjective since for any $\alpha \in [\psi, 1)$, $\psi_k = \alpha$ when $k = \frac{\psi-\alpha}{\alpha-1}$. Note that $k \geq 0$ since $(\psi-\alpha) \leq 0$ and $(\alpha-1) < 0$. □

We now show that ϕ_k^n can be *decomposed into a linear combination* of ϕ_k and a constant for any integer $n \geq 2$ using induction on n.

Remark 3.8. Consider the quadratic equation

$$(k+1)^2x^2 - (2k+1)(k+1)x + (k^2+k-1) = 0.$$

Following Definition 3.1, we have $x^2 = A_k x - B_k$, where $A_k = \frac{(2k+1)}{(k+1)}$ and $B_k = \frac{(k^2+k-1)}{(k+1)^2}$.

From this, for $x = \phi_k$, the constant ϕ_k satisfies the quadratic equation $\phi_k^2 = A_k\phi_k - B_k$. As a result, one can easily decompose any positive integer power of ϕ_k into a linear combination of ϕ_k and a constant.

Thus, we can write x^n for all integers $n \geq 2$ as follows:
$x^3 = x^2x = A_kx^2 - B_kx$, and assume that
$x^n = A_kx^{(n-1)} - B_kx^{(n-2)}$.
Hence, the proof by induction on $n \geq 2$ provides the desired result as follows:

$$x^{(n+1)} = x^nx = (A_kx^{(n-1)} - B_kx^{(n-2)})x = A_kx^n - Bx^{(n-1)}.$$

Note that similar to the argument in the above remark, we can also *decompose* ψ_k^n ($n \geq 2$, an integer) into a *linear combination* of ψ_k and a constant since $\psi_k^2 = A_k\psi_k - B_k$.

We now *construct an algebraic structure* (a ring) based on ϕ_k for any *integer* $k \geq 0$.

Remark 3.9. Let $k \geq 0$ be a fixed integer. Then the set

$$R_k = \mathbb{Z}[(k+1)\phi_k] = \{a + b(k+1)\phi_k \mid a, b \in \mathbb{Z}\}$$

is a ring since

$$(k+1)^2\phi_k^2 = (k+1)^2(A_k\phi_k - B_k) = (2k+1)(k+1)\phi_k - (k^2+k-1),$$

where $A_k = \frac{(2k+1)}{(k+1)}$ and $B_k = \frac{k^2+k-1}{(k+1)^2}$. That is,

$$(a + b(k+1)\phi_k)(c + d(k+1)\phi_k) \in \mathbb{Z}[(k+1)\phi_k].$$

Also, R_k is a subring of $R_0 = \mathbb{Z}[\phi_0 = \phi]$ for each fixed positive integer $k \geq 0$.

• We now write a short note on some erroneous ideas and examples of disputed observations related to the golden ratio. Then in the following remark we will show that one of these measurements which is related to the Gutenberg Bible, corresponds to ϕ_k for some $k \neq 0$ (Example 3.11). Note that the following note is taken from Wikipedia by searching for “golden ratio".

Remark 3.10. The golden ratio has been used to analyze the proportions of natural objects as well as artificial systems such as financial markets, in some cases based on dubious fits to data. The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves and other parts of vegetation.

Some 20th-century artists and architects, including Le Corbusier and Salvador Dalí, have proportioned their works to approximate the golden ratio, believing it to be aesthetically pleasing. These uses often appear in the form of a golden rectangle.
Some specific proportions in the bodies of many animals (including humans) and parts of the shells of mollusks are often claimed to be in the golden ratio. There is a large variation in the real measures of these elements in specific individuals, however, and the proportion in question is often significantly different from the golden ratio. The ratio of successive phalangeal bones of the digits and the metacarpal bone has been said to approximate the golden ratio. The nautilus shell, the construction of which proceeds in a logarithmic spiral, is often cited, usually with the erroneous idea that any logarithmic spiral is related to the golden ratio, but sometimes with the claim that each new chamber is golden-proportioned relative to the previous one. However, measurements of nautilus shells do not support this claim.

Studies by psychologists, starting with Gustav Fechner c. 1876, have been devised to test the idea that the golden ratio plays a role in human perception of beauty. While Fechner found a preference for rectangle ratios centered on the golden ratio, later attempts to carefully test such a hypothesis have been, at best, inconclusive.

In investing, some practitioners of technical analysis use the golden ratio to indicate support of a price level, or resistance to price increases, of a stock or commodity; after significant price changes up or down, new support and resistance levels are supposedly found at or near prices related to the starting price via the golden ratio. The use of the golden ratio in investing is also related to more complicated patterns described by Fibonacci numbers (e.g. Elliott wave principle and Fibonacci retracement). However, other market analysts have published analyses suggesting that these percentages and patterns are not supported by the data.

• Historian John Man states that both the pages and text area of the Gutenberg Bible were “based on the golden section shape". However, according to his own measurements, the ratio of height to width of the pages is 1.45.

Example 3.11. According to the above measurement of the Gutenberg Bible, we can calculate k for $\phi_k = 1.45$, which is the golden ratio of order $k \approx 0.3734 \neq 0$, from the following formula:

$$(k+1)1.45 = k + \phi.$$

Finally, we end the paper by posing a question related to the Penrose tiling and quasicrystals in connection to the golden ratio of order $k > 0$. Between 1973 and 1974, Roger Penrose developed Penrose tiling, a pattern related to the golden ratio both in the ratio of areas of its two rhombic tiles and in their relative frequency within the pattern. This gained in interest after Dan Shechtman’s Nobel-winning 1982 discovery of quasicrystals with icosahedral symmetry, which were soon afterward explained through analogies to the Penrose tiling.

Question: Is there any tiling, in general, or similar to the Penrose tiling whose pattern satisfies the golden ratio of order $k > 0$? Moreover, is there any quasicrystal in nature whose structure satisfies the golden ratio of order $k > 0$?

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