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SEMI-BOUNDED HYPERGROUP JOINS

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Abstract In this paper, hypergroup joins (formed by unions of compact and discrete spaces) involving semi-bounded generalized hypergroups are investigated. Several classic hypergroup spaces associated with joins are expanded to create discrete, semi-bounded generalized hypergroups. The notion of "semi-boundedness", which has previously been defined only for discrete generalized hypergroups, is extended in a natural way to include non-discrete spaces. It is shown, through proof and example, that the join of a compact hypergroup with a semi-bounded discrete generalized hypergroup can result in a locally compact, semi-bounded generalized hypergroup that is neither compact nor discrete.

1 Introduction

The study of hypergroups has led to ever more generalization, as new objects arise from more familiar ones. Hypergroup structures studied in the early 20th century by F. Marty and M.S. Wall involved spaces of conjugacy classes and double cosets of topological groups (see the introduction in [1]). The notion of a signed hypergoup arose from an example, provided by Jewett in the 1970s, where the dual of a relatively simple three element hypergroup failed to be a hypergroup ([3] , Example 9.1C, page 51). Semi-bounded generalized hypergroups were introduced in the later 20th century as part of an investigation of spaces of orthogonal polynomials (see [5] and [4].) It has since been shown that semi-bounded generalized hypergroups can arise naturally as dual spaces of standard hypergroups ([6], Example 6.2, page 79)).

The notion of semi-boundedness has commonly been restricted to discrete generalized hypergroups. However, the concept can be expanded to include non-discrete, locally compact spaces. In this paper it will be shown that the join of a compact group, hypergroup or signed hypergroup with a discrete, semi-bounded generalized hypergroup will result in a semi-bounded locally compact generalized hypergroup that may not be discrete.

A signed hypergroup join is the union of a compact generalized hypergroup, H that admits a Haar measure with full support, and a discrete generalized hypergroup, J, whose intersection is necessarily taken to be the identity of the two spaces. The join, $H \vee J$, is given a generalized hypergroup structure born from the structures of H and J. Hypergroup joins were originally introduced in [3] where it was noted that the join of two hypergroups is again a hypergroup. Hypergroup joins were studied further by Vrem in [10]. In this paper, we expand the notion of joins to generalized hypergroups. In particular, we show that the join of two generalized hypergroups is again a generalized hypergroup, even when the discrete portion is semi-bounded and not bounded.

2 Definition and Structure

The definition of a generalized hypergroup given here matches the discrete generalized hypergroups given in [5]. The setup and notation, however, follows that which is used in [3], [1] and others. An explanation of how the different notations correspond can be found in [7] (Prop 5.2.1, page 315).

PRELIMINARY NOTATION: Let X be a locally compact topological space. Let $C_b(X)$ be the set of continuous and bounded real-valued functions on X. Let $M_b(X)$ represent the space of bounded Radon measures on X, and let $M_c(X)$ represent the set of measures in $M_b(X)$ that have compact support. Let $\Gamma(X)$ represent the set of all locally compact subsets of X. For each $x \in X$, the Dirac measure δ_x is defined by $\delta_x(A) = 1$ if $x \in A \subseteq X$, and 0 otherwise.

Definition 2.1. A generalized hypergroup is a quadruple $(X, *, \check{}, e)$ where X is a locally compact topological space, $*$ is an associative binary convolution function $* : X \times X \to M_c(X)$ (which maps $(x, y) \in X \times X$ to $*(x, y) = \delta_x * \delta_y \in M_c(X)$), $x \mapsto \check{x}$ is an involutive map from X onto itself for which $\check{\check{x}} = x$ for all $x \in X$, and $e \in X$ is an identity element for which $\delta_x * \delta_e = \delta_x = \delta_e * \delta_x$ for all $x \in X$.

For all $x, y \in X$, the involution must satisfy the properties $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$ and $e \in$ supp$(\delta_x * \delta_y) \Leftrightarrow y = \check{x}$.

Additionally, if X is discrete, then $\delta_x * \delta_{\check{x}}(e) > 0$ for all $x \in X$.

If X is not discrete, then the system must satisfy the following continuity requirements:

i) the map $(x, y) \mapsto \delta_x * \delta_y$ must be continuous where $M_b(X)$ is given the topology of pointwise convergence with respect to $C_b(X)$, and

ii) the map $(x, y) \mapsto supp(\delta_x * \delta_y)$ must be continuous where $\Gamma(X)$ is given the Michael topology ([3], 2.5, page 12).

A generalized hypergroup is *commutative* if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in X$, *hermitian* if $\check{x} = x$ for all $x \in X$, and *normal* if $\delta_x * \delta_y(K) = 1$ for all $x, y \in X$.

Definition 2.2. If X is a generalized hypergoup, and $H, K \subseteq X$, then

$$H * K = \bigcup_{(h,k) \in H \times K} support(\delta_h * \delta_k).$$

Definition 2.3. A closed non-void subset H of X is a *generalized subhypergroup* of X if $\check{H} = H$ and $H * H = H$.

A focus of this paper is on the notion of semi-boundness, which has previously been considered only for discrete spaces. The definition given below expands this notion to non-discrete spaces, and matches the more familiar statement for discrete spaces.

Definition 2.4. A generalized hypergroup is *semi-bounded* if, for each $x \in X$,

$$\gamma(x) = \sup_{y \in X} \{\| \delta_x * \delta_y \|\} < \infty,$$

and *bounded* if

$$k = \sup_{x \in X} \{|\gamma(x)|\} < \infty.$$

If each $\delta_x * \delta_y$ is a probability measure, then X is a (standard) *hypergroup*, and X is a group if each $\delta_x * \delta_y$ is another Dirac measure. A generalized hypergroup is a *signed hypergroup* if it is real, normal and bounded. With both groups and hypergroups, the norm-bounding constant $k = 1$, and with signed hypergroups $k \geq 1$.

Definition 2.5. Given $f \in C_b(X)$, $\mu \in M_b(X)$, and $x, y \in X$, the functions f_x , $\delta_x * \mu$, $\check{f}$ and $\check{\mu}$ are defined by

i) $f_x(y) = f(x * y) = \delta_x * \delta_y(f) = \int_X f d(\delta_x * \delta_y)$;

ii) $\delta_x * \mu(f) = \mu(f_x)$;

iii) $\check{f}(x) = f(\check{x})$;

iv) $\check{\mu}(f) = \mu(\check{f})$.

Definition 2.6. A measure m with full support on a generalized hypergroup X is pseudo-invariant if

$\int_X f(x * y)g(y)dm(y) = \int_X f(y)g(\check{x} * y)dm(y)$ for all $f, g \in C_c(X)$ and $x \in X$.

Definition 2.7. A measure m with full support on a generalized hypergroup X is a Haar measure if $\delta_x * m(f) = m(f_x) = m(f) = m * \delta_x(f)$ for all $x \in X$ and $f \in C_c(X)$.

If X is compact, an invariant Haar measure can be normalized so that $\| m \| = 1$. If X is discrete, an invariant Haar measure can be unit-normalized so that $m(e) = 1$.

Lemma 5.1D in [3] (page 24) shows that all Haar measures on hypergroups are pseudo-invariant.

It was shown in [4] (Lemma 2.3, page 371) that every discrete generalized hypergroup X admits a pseudo-invariant measure. Moreover, the pseudo-invariant measure is a Haar measure if and only if X is normal ([4], Theorem 2.5, page 372). Such a measure will invariably be a positive multiple of the function $m(x) = (\delta_x * \delta_{\check{x}}(e))^{-1}$.

For non-discrete, non-compact signed hypergroups, Rösler assumes the existence of a pseudo-invariant measure ([7] page 304), and proves it will be a Haar measure if and only if X is normal ([7], Corollary 3.4, page 305). Rösler also notes the existence of a pseudo-invariant measure implies the involution property $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$ ([8], page 149).

However, in order to ensure that $f_x \in C_c(X)$ whenever $f \in C_c(X)$, it is necessary to either assume $\cup_{x \in X} supp(f_x)$ is a relatively compact subset of X for each $f \in C_c(X)$ (see axiom A1 in [8], page 149), or add an additional axiom concerning the support of the measures $\delta_x * \delta_y$; $c \in \text{supp}(\delta_a * \delta_b) \Leftrightarrow b \in \text{supp}(\delta_{\check{a}} * \delta_c) \;\; \forall a, b, c \in X$. (see [9], page 89).

Example 2.8. For any positive real number $\theta > 0$, the generalized hypergroup $\mathbb{Z}_2(\theta)$ is the two-element set $\{0, a\}$ with identity $e = 0$ and convolution defined by

$$\begin{aligned} \delta_0 * \delta_0 &= \delta_0, \\ \delta_0 * \delta_a &= \delta_a = \delta_0 * \delta_a, \text{ and} \\ \delta_a * \delta_a &= \theta\delta_0 + (1-\theta)\delta_a. \end{aligned}$$

The hermitian space $\mathbb{Z}_2(\theta)$ is a group if $\theta = 1$, a standard hypergroup if $0 < \theta < 1$, and is signed if $\theta > 1$. The space is bounded with $k = \theta + |(1-\theta)|$.

The normalized Haar measure is

$$w = \frac{\theta}{(\theta+1)}\delta_0 + \frac{1}{\theta+1}\delta_a.$$

The unit-normalized Haar measure is

$$m = \delta_0 + \frac{1}{\theta}\delta_a.$$

Example 2.9. An unbounded, semi-bounded, commutative, hermitian, signed, and normal generalized hypergroup structure can be defined on the set $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ by taking $e = 0$ to be the identity, and convolution defined by

$$\begin{aligned} \delta_n * \delta_m &= \delta_{\max\{n,m\}} \text{ for all } n \neq m, \text{ and} \\ \delta_n * \delta_n &= \textstyle\sum_{j=0}^{n-1} \delta_j + (1-n)\delta_n. \end{aligned}$$

Note that, for each $n \in \mathbb{N}$,

$$\gamma(n) = \sup_{m \in \mathbb{N}_0} \{\| \delta_n * \delta_m \|\} = 2n - 1 < \infty.$$

The unit-normalized Haar measure is $w = \sum_{j=0}^{\infty} \delta_j$.

Later in this paper it will be noted that the generalized hypergroup $(\mathbb{N}_0, *, \check{}, e)$ results from an infinite join of signed hypergroups of the form $\mathbb{Z}_2(\theta)$.

3 Generalized Hypergroup Joins

A "join" is formed from the union of a compact and a discrete generalized hypergroup, whose intersection is taken to be the identity element e of each space. The compact portion must be bounded and admit a Haar measure.

Definition 3.1. Let $(H, *)$ be a bounded, compact generalized hypergroup that admits a Haar measure w_H, normalized so that $w_H(H) = 1$. Let $(J, \cdot)$ be any discrete generalized hypergroup. Take $H \cap J = \{e\}$ and endow the union $K = H \cup J$ with the unique topology for which both H and J are closed subspaces of K. Let $\tilde{J} = J \setminus \{e\}$, and define a convolution, # , on K as follows:

i) If $x, y \in H$, then $\delta_x \# \delta_y = \delta_x * \delta_y$.
ii) If $x, y \in J$ and $x \neq \check{y}$, then $\delta_x \# \delta_y = \delta_x \cdot \delta_y$.
iii) If $x \in H$ and $y \in \tilde{J}$, then $\delta_x \# \delta_y = \delta_y$.
iv) If $x \in \tilde{J}$, and $\delta_x \cdot \delta_{\check{x}} = \sum_{z \in J} c_z \delta_z$, then

$$\delta_x \# \delta_{\check{x}} = c_e w_H + \sum_{z \in \tilde{J}} c_z \delta_z.$$

With the above convolution, and with involution as on H and J, respectively, K is called the $join$ of H and J and is written $K = H \vee J$. It is easily shown that H is a generalized subhypergroup of K. For this reason we usually use the symbol $*$ for both the convolution on H and K. A different symbol, usually $\cdot$, is used for the convolution on J, since J is a generalized subhypergroup of K if and only if $H = \{e\}$. Clearly K is commutative/hermitian if and only if both H and J are commutative/hermitian, respectively.

Note that the formula

$$\delta_x \# \delta_y = c_e w_H + \sum_{z \in \tilde{J}} c_z \delta_z$$

holds even when $x, y \in \tilde{J}$ with $x \neq \check{y}$, since in that case $e \notin \text{supp}(\delta_x \cdot \delta_y)$ ensures that $c_e = 0$.

Hereinafter, we let $\tilde{H} = H \setminus \{e\}$ and $\tilde{J} = J \setminus \{e\}$. Also, throughout this paper it is assumed that any suitable H admits a normalized Haar measure w_H.

Theorem 3.2. *Let $(H, *)$ and $(J, \cdot)$ be a suitable compact signed hypergroup and discrete generalized hypergroup. Then the join $K = (H \vee J, *)$ is a generalized hypergroup.*

Proof. It's clear that $\check{\check{x}} = x$, and $\delta_x * \delta_e = \delta_x = \delta_e * \delta_x$ for all $x \in K$.

To see that the involution properties hold, let $x, y \in K$.

If $x, y \in H$, or $x, y \in \tilde{J}$ with $x \neq \check{y}$, then it is clear that $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$.

If $x \in H, y \in \tilde{J}$, then $(\delta_x * \delta_y)\check{} = (\delta_y)\check{} = \delta_{\check{y}} = \delta_{\check{y}} * \delta_{\check{x}}$.

If $x = \check{y} \in \tilde{J}$, then

$$(\delta_x * \delta_{\check{x}})\check{} = (c_e w_H + \sum_{z \in \tilde{J}} c_z \delta_z)\check{} = c_e \check{w}_H + \sum_{z \in \tilde{J}} c_z \check{\delta}_z = c_e w_H + \sum_{z \in \tilde{J}} c_z \check{\delta}_z = \delta_{\check{x}} * \delta_x.$$

It is clear that $e \in \text{supp}(\delta_x * \delta_y) \Leftrightarrow y = \check{x}$, since it is assumed to hold for both $(H, *)$ and $(J, \cdot)$. Also, since K is discrete if and only if H is finite, in that case $\delta_x * \delta_{\check{x}}(e) > 0$ would hold for both H and J, and thus for K as well.

To see that each $\delta_x * \delta_y$ has compact support even when K is not compact (which occurs when J is infinite), let $x, y \in K$. The conclusion is clear if $x, y \in H$, or $x, y \in \tilde{J}$ with $x \neq \check{y}$, or $x \in H$ and $y \in \tilde{J}$. For the remaining case, if $x \in \tilde{J}$, then

$$\delta_x * \delta_{\check{x}} = c_e w_H + \sum_{z \in \tilde{J}} c_z \delta_z$$

has compact support since w_H has compact support (supp$(w_H) = H$) and the sum is a finite sum.

If K is not discrete, then the convolution $*$ on K will be continuous. To see that both the maps $(x, y) \mapsto \delta_x * \delta_y$ and $(x, y) \mapsto supp(\delta_x * \delta_y)$ are continuous, let $(x_\alpha, y_\alpha) \to (x, y) \in K \times K$.

Case a) $x, y \in \tilde{H}$. Since $\tilde{H} \subseteq H \vee J$ is open, we have that $\tilde{H} \times \tilde{H} \subseteq K \times K$ is open which implies there is an α_0 such that $\alpha \geq \alpha_0$ implies $(x_\alpha, y_\alpha) \in \tilde{H} \times \tilde{H} \subset H \times H$ and hence $\delta_{x_\alpha} * \delta_{y_\alpha} \to \delta_x * \delta_y$ (and supp$(\delta_{x_\alpha} * \delta_{y_\alpha}) \to$ supp$(\delta_x * \delta_y)$), since $*$ is cont on H.

Case b) $x, y \in \tilde{J}, x \neq \check{y}$. Then $\tilde{J} \subset H \vee J$ is an open subset and hence $\tilde{J} \times \tilde{J} \subset K \times K$ is open which implies there is an α_0 such that $\alpha \geq \alpha_0$ implies $(x_\alpha, y_\alpha) \in \tilde{J} \times \tilde{J}$ and hence $\delta_{x_\alpha} * \delta_{y_\alpha} \to \delta_x * \delta_y$ (and supp$(\delta_{x_\alpha} * \delta_{y_\alpha}) \to$ supp$(\delta_x * \delta_y)$), since $\tilde{J}$ is discrete.

Case c) $x \in \tilde{H}$ and $y \in \tilde{J}$ implies $\delta_x * \delta_y = \delta_y = \delta_y * \delta_x$. Then $(x, y) \in \tilde{H} \times \tilde{J} \subset K \times K$ is open which implies there is an α_0 s.t. $\alpha \geq \alpha_0$ implies $(x_\alpha, y_\alpha) \in \tilde{H} \times \tilde{J}$ and hence $\delta_{x_\alpha} * \delta_{y_\alpha} = \delta_{y_\alpha}$. Thus $y_\alpha \to y$ implies $\delta_{y_\alpha} \to \delta_y$ (and supp$(\delta_{y_\alpha}) \to$ supp(δ_y)) implies $\delta_{x_\alpha} * \delta_{y_\alpha} \to \delta_x * \delta_y$ (and supp$(\delta_{x_\alpha} * \delta_{y_\alpha}) \to$ supp$(\delta_x * \delta_y)$).

Case d) $x \in \tilde{J}$ (which implies $\check{x} \in \tilde{J}$). Let $(x_\alpha, y_\alpha) \to (x, \check{x})$. Then J discrete implies $(x_\alpha, y_\alpha) = (x, \check{x})$ eventually.

It remains to show that the convolution of point masses is associative. Note that if $\mu \in M_b(K)$ with supp$(\mu) \subseteq H$ and $x \in \tilde{J}$, then $\delta_x * \mu = \mu(H)\delta_x = \mu * \delta_x$. To see this let $f \in C_c(K)$; then

$$\delta_x * \mu(f) = \int_H f(x * t) d\mu(t) = \int_H f(x) d\mu(t) = \mu(H) f(x) = \mu(H)\delta_x(f).$$

Similarly for $\mu * \delta_x$. For the normalized Haar measure w_H, we then have $\delta_x * w_h = \delta_x = w_H * \delta_x$.

Now let $x, y, z \in K$.

Case a) $x, y, z \in H$. This is clear.

Case b) One element (say z) in $\tilde{J}$, the others in H. Then

$$(\delta_x * \delta_y) * \delta_z = \delta_z = \delta_x * (\delta_y * \delta_z).$$

Case c) One element (say z) in H, the others in $\tilde{J}$. Let $\delta_x \cdot \delta_y = \sum_{s \in J} c_s \delta_s$. Then

$$(\delta_x * \delta_y) * \delta_z = (c_e w_H + \sum_{s \in \tilde{J}} c_s \delta_s) * \delta_z = c_e w_H * \delta_z + \sum_{s \in \tilde{J}} c_s (\delta_s * \delta_z) = c_e w_H + \sum_{s \in J} c_s \delta_s = \delta_x * \delta_y = \delta_x * (\delta_y * \delta_z).$$

Case d) $x, y, z \in \tilde{J}$. Let $\delta_x \cdot \delta_y = \sum_{s \in J} c_s \delta_s$, $\delta_{\check{z}} \cdot \delta_z = \sum_{s \in J} d_s \delta_s$, $\delta_y \cdot \delta_z = \sum_{s \in J} p_s \delta_s$, and $\delta_x \cdot \delta_{\check{x}} = \sum_{s \in J} q_s \delta_s$. Then

$$\begin{aligned}
(\delta_x * \delta_y) * \delta_z &= (c_e w_H + \sum_{s \in \tilde{J}} c_s \delta_s) * \delta_z = c_e \delta_z + \sum_{s \in \tilde{J}, s \neq \check{z}} c_s \delta_s \cdot \delta_z + c_{\check{z}} \delta_{\check{z}} * \delta_z \\
&= c_e \delta_z + \sum_{s \in \tilde{J}} c_s \delta_s \cdot \delta_z - c_{\check{z}} \delta_{\check{z}} \cdot \delta_z + c_{\check{z}} \delta_{\check{z}} * \delta_z \\
&= (\delta_x \cdot \delta_y) \cdot \delta_z + c_{\check{z}} (\delta_{\check{z}} * \delta_z - \delta_{\check{z}} \cdot \delta_z) \\
&= \delta_x \cdot (\delta_y \cdot \delta_z) + c_{\check{z}} (d_e w_H - d_e \delta_e).
\end{aligned}$$

Similarly $\delta_x * (\delta_y * \delta_z) = \delta_x \cdot (\delta_y \cdot \delta_z) + p_{\check{x}} (q_e w_H - q_e \delta_e)$. Thus we need to show that $c_{\check{z}} d_e = p_{\check{x}} q_e$. But this holds since

$$\sum_{s \in J} c_s \delta_s \cdot \delta_z = (\delta_x \cdot \delta_y) \cdot \delta_z = \delta_x \cdot (\delta_y \cdot \delta_z) = \sum_{s \in J} p_s \delta_x \cdot \delta_s.$$

Evaluating both sides at e yields $c_{\check{z}} d_e = p_{\check{x}} q_e$ as desired. □

Theorem 3.3. *The join $K = H \vee J$ is normal if and only if both $(H, *)$ and $(J, \cdot)$ are normal.*

Proof. This result is clear for every case other than $\delta_x * \delta_{\check{x}}$ when $x \in \tilde{J}$. For that case, note that

$$\delta_x * \delta_{\check{x}}(K) = c_e w_H(H) + \sum_{z \in \tilde{J}} c_z \delta_z(\tilde{J}) = c_e + \sum_{z \in \tilde{J}} c_z = \delta_x \cdot \delta_{\check{x}}(J).$$

□

Theorem 3.4. *The join $K = (H \vee J, *)$ is semi-bounded if and only if J is semi-bounded, and bounded if and only if J is bounded. If $(H, *)$ and $(J, \cdot)$ are bounded by constants k_H and k_J, respectively, then $k = \max\{k_H, k_J\}$ will be the norm-bounding constant for the join $(K, *)$.*

Proof. Let $x, y \in K$. If $x, y \in H$, then $\| \delta_x * \delta_y \| \leq k_H$, where k_H is the norm-bounding constant for H. Thus K is bounded on H by k_H. If $x \in H$ and $y \in \tilde{J}$, then $\|\delta_x * \delta_y\| = \|\delta_y\| = 1$. If $x, y \in J$, then $\| \delta_x * \delta_y \| = \| \delta_x \cdot \delta_y \|$ since $\delta_x * \delta_y = \delta_x \cdot \delta_y$ if $x \neq y$, and

$$\|\delta_x * \delta_{\check{x}}\| = (|c_e| w_H + \sum_{z \in \tilde{J}} |c_z| \delta_z)(K) = \|\delta_x \cdot \delta_{\check{x}}\|.$$

Hence, for $x \in J$, $\gamma(x) = \sup_{y \in K}\{\| \delta_x * \delta_y \|\} = sup_{y \in J}\{\| \delta_x \cdot \delta_y \|\}$.

Thus the boundedness and/or semi-boundedness of K on J will coincide, and if k_J is the norm-bounding constant for $(J, \cdot)$, then $k = \max\{k_H, k_J\}$ will be the norm-bounding constant for the join $(K, *)$.

□

Theorem 3.5. *Let w_J be the unit-normalized pseudo-invariant measure on J, and let $w_{\tilde{J}}$ be the restriction of w_J to $\tilde{J}$. The measure $m = w_H + w_{\tilde{J}}$ is pseudo-invariant if and only if w_H is pseudo-invariant.*

Proof. Here $w_J(x) = (\delta_{\check{x}} * \delta_x(e))^{-1}$, which is unit-normalized in the manner of discrete signed hypergroups. In particular, $w_J(e) = 1$, even if J happens to be finite.

Suppose that w_H is pseudo-invariant. Let $f, g \in C_c(K)$ and $x \in K$. Then

$$\int_K f(x*y)g(y)dm(y) = \int_H f(x*y)g(y)dw_H(y) + \int_{\tilde{J}} f(x*y)g(y)d\tilde{w}_J(y).$$

Case a) $x \in H$. Then

$$\begin{aligned}
\int_K f(x*y)g(y)dw_H(y) &= \int_H f(y)g(\check{x}*y)dw_H(y) + \int_{\tilde{J}} f(y)g(y)d\tilde{w}_J(y) \\
&= \int_H f(y)g(\check{x}*y)dw_H(y) + \int_{\tilde{J}} f(y)g(\check{x}*y)d\tilde{w}_J(y) \\
&= \int_K f(y)g(\check{x}*y)dm(y).
\end{aligned}$$

Case b) $x \in \tilde{J}$ with $\delta_{\check{x}} \cdot \delta_x = \sum_{z \in J} c_z \delta_z$ and $\delta_x \cdot \delta_{\check{x}} = \sum_{z \in J} d_z \delta_z$. Recall that $w_J(x) = (\delta_{\check{x}} \cdot \delta_x(e))^{-1} = (c_e)^{-1}$ and $w_J(\check{x}) = (\delta_x \cdot \delta_{\check{x}}(e))^{-1} = (d_e)^{-1}$ and $w_J(e) = 1$. Thus

$$\begin{aligned}
&\int_K f(x*y)g(y)dm(y) \\
&\quad= \int_H f(x)g(y)dw_H(y) + \int_{\tilde{J}\setminus\{\check{x}\}} f(x \cdot y)g(y)d\tilde{w}_J(y) + f(x*\check{x})g(\check{x})w_J(\check{x}) \\
&\quad= f(x)w_H(g) + \int_J f(x \cdot y)g(y)dw_J(y) - f(x)g(e) - f(x \cdot \check{x})g(\check{x})w_J(\check{x}) \\
&\qquad+ g(\check{x})w_J(\check{x})(d_e w_H(f) + \sum_{z \in \tilde{J}} f(z)d_z) \\
&\quad= \int_J f(y)g(\check{x} \cdot y)dw_J(y) + f(x)w_H(g) - f(x)g(e) \\
&\qquad- f(x \cdot \check{x})g(\check{x})w_J(\check{x}) + g(\check{x})w_H(f) + g(\check{x})w_J(\check{x}) \sum_{z \in \tilde{J}} f(z)d_z \\
&\quad= g(\check{x})w_H(f) + \int_J f(y)g(\check{x} \cdot y)dw_J(y) + f(x)w_H(g) - f(x)g(e) \\
&\qquad- f(x \cdot \check{x})g(\check{x})w_J(\check{x}) + g(\check{x})w_J(\check{x})f(x \cdot \check{x}) - g(\check{x})w_J(\check{x})f(e)d_e
\end{aligned}$$

$$
\begin{aligned}
&= g(\check{x})w_H(f) + \int_{\tilde{J}\setminus\{x\}} f(y)g(\check{x}\cdot y)dw_J(y) \\
&\quad + f(e)g(\check{x}) + f(x)g(\check{x}\cdot x)w_J(x) + f(x)w_H(g) - f(x)g(e) - g(\check{x})f(e) \\
&= g(\check{x})w_H(f) + \int_{\tilde{J}\setminus\{x\}} f(y)g(\check{x}\cdot y)dw_J(y) \\
&\quad + f(x)w_J(x)(c_e w_H(g) + \sum_{z\in\tilde{J}} g(z)c_z) - f(x)g(e) + f(x)w_J(x)g(e)c_e \\
&= g(\check{x})w_H(f) + \int_{\tilde{J}\setminus\{x\}} f(y)g(\check{x}\cdot y)dw_J(y) + f(x)w_J(x)g(\check{x}*x) \\
&= \int_H f(y)g(\check{x}*y)dm(y) + \int_{\tilde{J}} f(y)g(\check{x}*y)dm(y) = \int_K f(y)g(\check{x}*y)dm(y).
\end{aligned}
$$

Finally, if m is pseudo-invariant, then so is w_H since H is a signed subhypergroup of K.

□

Lemma 3.6. *If $K = H \vee J$ and λ is a Haar measure on H, then $\lambda * \delta_y = \delta_y * \lambda = \lambda$ for all $y \in H$ and $\lambda * \delta_y = \delta_y * \lambda = \lambda(H)\delta_y$ for all $y \in \tilde{J}$.*

Proof. If $y \in H$, then the result is clear. If $y \in \tilde{J}$ and $f \in C_\infty(K)$, then

$$
\begin{aligned}
\int_K fd(\delta_y * \lambda) &= \int_K \int_K f(t)d(\delta_y * \delta_z)(t)d\lambda(z) \\
&= \int_H \int_K f(t)d(\delta_y * \delta_z)(t)d\lambda(z) = \int_H \int_K f(t)d\delta_y(t)d\lambda(z) \\
&= \int_H f(y)d\lambda(z) = f(y)\lambda(H).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\int_K fd(\lambda * \delta_y) &= \int_K \int_K f(t)d(\delta_z * \delta_y)(t)d\lambda(z) \\
&= \int_H \int_K f(t)d(\delta_z * \delta_y)(t)d\lambda(z) = \int_H \int_K f(t)d\delta_y(t)d\lambda(z) \\
&= \int_H f(y)d\lambda(z) = f(y)\lambda(H).
\end{aligned}
$$

□

Theorem 3.7. *Let $K = H \vee J$. If w_H and w_J are normalized (and unit normalized) Haar measures on $(H, *)$ and $(J, \cdot)$, respectively, then the measure $m = w_H + w_{\tilde{J}}$, where $w_{\tilde{J}}$ is w_J restricted to $\tilde{J}$, is a Haar measure on the join $K = (H \vee J, *)$.*

Proof. Here, again, $w_J(x) = (\delta_{\check{x}} * \delta_x(e))^{-1}$ is unit-normalized in the manner of discrete signed hypergroups, Take $m = w_H + w_{\tilde{J}}$ as Haar measure on K. Clearly m is supported on K. We will check that m is left-invariant. Let $f \in C_c^+(K)$ and $x \in K$. We need to show that $\delta_x * m(f) = m(f)$. Note that

$$
\delta_x * m(f) = \delta_x * w_H(f) + \delta_x * w_{\tilde{J}}(f) = \int_H \delta_x * \delta_t(f)dw_H(t) + \sum_{s\in\tilde{J}} \delta_x * \delta_s(f)w_{\tilde{J}}(s).
$$

Case a) If $x \in H$, then $\delta_x * \delta_s = \delta_s$ for all $s \in \tilde{J}$. Thus

$$
\delta_x * w_{\tilde{J}}(f) = \sum_{s\in\tilde{J}} \delta_x * \delta_s(f)w_{\tilde{J}}(s) = \sum_{s\in\tilde{J}} f(s)w_{\tilde{J}}(s).
$$

Using the previous Lemma,

$$\delta_x * m(f) = \delta_x * w_H(f) + \delta_x * w_{\tilde{J}}(f) = w_H(f) + \sum_{s\in\tilde{J}} f(s)w_{\tilde{J}}(s) = w_H(f) + w_{\tilde{J}}(f) = m(f).$$

Case b) If $x \in \tilde{J}$, then using the previous Lemma,

$$\begin{aligned}\delta_x * m(f) &= \delta_x * w_H(f) + \delta_x * w_{\tilde{J}}(f) = f(x) + \delta_x * w_{\tilde{J}}(f)\\ &= f(x)w_J(e) + \sum_{s\in\tilde{J}} \delta_x * \delta_s(f)w_{\tilde{J}}(s) = \sum_{s\in J} \delta_x * \delta_s(f)w_J(s)\\ &= \sum_{s\in J\backslash\{\check{x}\}} \delta_x \cdot \delta_s(f)w_J(s) + \delta_x * \delta_{\check{x}}(f)w_J(\check{x})\\ &= \sum_{s\in J} f(x\cdot s)w_J(s) - \delta_x \cdot \delta_{\check{x}}(f)w_J(\check{x}) + \delta_x * \delta_{\check{x}}(f)w_J(\check{x}).\end{aligned}$$

Now,

$$\delta_x * \delta_{\check{x}} - \delta_x \cdot \delta_{\check{x}} = c_e w_H + \sum_{t\in\tilde{J}} c_t\delta_t - \sum_{t\in J} c_t\delta_t = c_e w_H - c_e\delta_e.$$

So, we have that

$$\delta_x * \delta_{\check{x}}(f)w_J(\check{x}) - \delta_x \cdot \delta_{\check{x}}(f)w_J(\check{x}) = w_J(\check{x})[c_e \int_H f dw_H - c_e f(e)] = \int_H f dw_H - f(e)$$

since $c_e = w_J(\check{x})^{-1}$. Thus,

$$\begin{aligned}\delta_x * m(f) &= \sum_{s\in J} f(s)w_J(s) + \int_H f(t)dw_H(t) - f(e)\\ &= \sum_{s\in\tilde{J}} f(s)w_J(s) + \int_H f(t)dw_H(t) = m(f).\end{aligned}$$

Therefore m is a Haar measure on K.

□

Corollary 3.8. *The join $K = H \vee J$ admits a Haar measure if and only if both $(H, *)$ and $(J, \cdot)$ are normal.*

Proof. By Theorem 2.5 in [4] (page 372), $(J, \cdot)$ admits a Haar measure if and only if $(J, \cdot)$ is normal. By Corollary 3.4 in [7] (page 305), a pseudo-invariant measure on $(H, *)$ is a Haar measure if and only if $(H, *)$ is normal. □

Theorem 3.9. *The join $K = H \vee J$ satisfies the property $c \in \text{supp}(\delta_a * \delta_b) \Leftrightarrow$ $b \in \text{supp}(\delta_{\check{a}} * \delta_c)$ $\forall a, b, c \in X$, if and only if both $(H, *)$ and $(J, \cdot)$ satisfy the property.*

Proof. It is clear that both $(H, *)$ and $(J, \cdot)$ satisfy the property if the join does. Assume $(H, *)$ and $(J, \cdot)$ satisfy the property, and let $a, b, c \in K$.

Case 1) $a, b, c \in H$. Then the property holds since $*_K = *_H$.

Case 2) $a, b, c \in \tilde{J}$. Then

$$c \in \text{supp}(\delta_a * \delta_b) \iff c \in \text{supp}(\delta_a \cdot \delta_b) \iff b \in \text{supp}(\delta_{\check{a}} \cdot \delta_c) \iff b \in \text{supp}(\delta_{\check{a}} * \delta_c).$$

Case 3a) $a \in H$ and $b, c \in \tilde{J}$. Then

$$c \in \text{supp}(\delta_a * \delta_b) = \text{supp}(\delta_b) \iff b \in \text{supp}(\delta_c) = \text{supp}(\delta_{\check{a}} * \delta_c).$$

Case 3b) $a, c \in \tilde{J}$ and $b \in H$. Then

$$c \in \text{supp}(\delta_a * \delta_b) = \text{supp}(\delta_a) \iff H \subseteq \text{supp}(\delta_{\check{a}} * \delta_c) \iff b \in \text{supp}(\delta_{\check{a}} * \delta_c).$$

Case 3c) $a, b \in \tilde{J}$ and $c \in H$. Then

$$c \in \text{supp}(\delta_a * \delta_b) \iff \check{a} = b \iff b \in \text{supp}(\delta_{\check{a}}) = \text{supp}(\delta_{\check{a}} * \delta_c).$$

Case 4) One element (say a) in $\tilde{J}$ and the other two in H. Then it is not possible to have $c \in \text{supp}(\delta_a * \delta_b) = \text{supp}(\delta_a) = \{a\}$. Nor is it possible to have $b \in \text{supp}(\delta_{\check{a}} * \delta_c) = \text{supp}(\delta_{\check{a}}) = \{\check{a}\}$. □

4 Generalized Hypergroup Join Examples

Joins of simple $\mathbb{Z}_2(\theta)$ systems, described in Example 2.8, have previously been developed when the $\mathbb{Z}_2(\theta)$ are groups or hypergroups ($0 < \theta \leq 1$). In the examples given here, the $\mathbb{Z}_2(\theta)$ are allowed to be signed ($\theta > 1$), leading to the rise of semi-bounded systems.

Example 4.1. For $n \in \mathbb{N}$, let $\mathbb{Z}_2(n) = \{0, n\}$ so that each $\mathbb{Z}_2(n)$ is a two-element signed hypergroup with $\delta_n * \delta_n = n\delta_0 + (1-n)\delta_n$, and $\mathbb{Z}_2(n) \cap \mathbb{Z}_2(m) = \{0\}$ whenever $n \neq m$. Further, let $K_1 = \mathbb{Z}_2(1)$ and, for $n > 1$, let $K_n = K_{n-1} \vee \mathbb{Z}_2(n)$.

Note that forming $K_n = K_{n-1} \vee \mathbb{Z}_2(n)$ uses the normalized Haar measure on K_{n-1}, and the discrete unit-normalized Haar measure on $\mathbb{Z}_2(n)$.

Then $K_1 = \{0, 1\}$ is a group and, for $n > 1$, $K_n = \{0, 1, 2, ..., n\}$ is a hermitian signed hypergroup with convolution

$$\begin{gathered}\delta_0 * \delta_n = \delta_n \text{ for all } n \in K_n, \\ \delta_n * \delta_m = \delta_{\max\{n,m\}}, \text{ for all } n \neq m, \text{ and} \\ \delta_n * \delta_n = \textstyle\sum_{j=0}^{n-1} \delta_j + (1-n)\delta_n.\end{gathered}$$

The norm-bounding constant on each K_n is $k_n = 2n - 1$.

The normalized Haar measure on each K_n is

$$w_n = \frac{1}{n}\sum_{j=0}^{n} \delta_j.$$

Example 4.2. The infinite join construction used in this example was originally described by Jewett [3] for hypergroups. Infinite hypergroup joins are studied extensively in [2]. Here, an infinite join of signed hypergroups gives rise to a semi-bounded generalized hypergroup.

Let K be the union of the sets (K_n) described in the previous example.

$$K = \bigcup_n K_n = \{0, 1, 2, 3, ...\}.$$

The involution and convolution are formed, for $n, m \in K$, by noting that $n, m \in K_{\max\{n,m\}}$ and using the involution and convolution as defined on $K_{\max\{n,m\}}$. These are well-defined since, by construction, K_n is a signed subhypergroup of K_m whenever $n \leq m$, and $supp(\delta_n * \delta_m) \subseteq K_{\max\{n,m\}}$.

The resulting space is the discrete, unbounded, semi-bounded, hermitian, commutative, signed and normal generalized hypergroup on $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ that was described in Example 2.9.

Example 4.3. Vrem showed how a hypergroup space very similar to this example can result from a projective limit of joins ([10], Example 4.5, page 494).

Let $(H, *) = \{1/2, 1/3, 1/4, ..., 0\}$ be the one-point compactification of the set $\{1/n : n \in \mathbb{N}, n \geq 2\}$. The space is hermitian, the identity is $e = 0$, and the convolution is given, for $x = 1/n, y = 1/m \in H$, by

$$\delta_x * \delta_y = \delta_{\max\{x,y\}} \text{ if } x \neq y, \text{ and}$$

$$\delta_{1/n} * \delta_{1/n} = -\delta_{1/n} + (3/2)^{n-2} \sum_{p=n+1}^{\infty} (2/3)^{p-2} \delta_{1/p}.$$

The norm bounding constant is $k = 3$. To see this, note that $\|\delta_x * \delta_y\| = 1$ if $x \neq y$, and

$$\|\delta_{1/n} * \delta_{1/n}\| = 1 + (3/2)^{n-2} \sum_{p=n+1}^{\infty} (2/3)^{p-2} = 3.$$

The normalized Haar measure on H is given by $w_H(\{1/n\}) = (2/3)^{n-2}$ and, for $A \subseteq H$,

$$w_H(A) = \sum_{x \in A \setminus \{0\}} w_H(x).$$

Example 4.4. This final example is unbounded, semi-bounded, hermitian, commutative, normal, and has a Haar measure, but it is not discrete or compact.

Let $(H, *) = \{1/2, 1/3, 1/4, ..., 0\}$ be the signed hypergroup from Example 4.3. Let $(J, \cdot) = \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$ be the semi-bounded generalized hypergroup from Example 2.9.

To review, the convolution on $(J, \cdot)$ is

$$\delta_n \cdot \delta_m = \delta_{\max\{n,m\}} \text{ if } n \neq m;$$
$$\delta_n \cdot \delta_n = \textstyle\sum_{j=0}^{n-1} \delta_j + (1-n)\delta_n.$$

Then the join $K = H \vee J = \{0, ..., 1/n, ..., 1/3, 1/2, 1, 2, 3, ...\}$ will be a semi-bounded locally compact generalized hypergroup that is not bounded or discrete or compact.

The identity is $e = 0$, and the convolution is

$$\delta_x * \delta_y = \delta_{\max\{x,y\}} \text{ if } x \neq y;$$

$$\delta_{1/n} * \delta_{1/n} = -\delta_{1/n} + (3/2)^{n-2} \sum_{p>n} (2/3)^{p-2} \delta_{1/p} \quad \text{for} \quad 1/n < 1;$$

$$\delta_n * \delta_n = \sum_{p=2}^{\infty} (2/3)^{p-2} \delta_{1/p} + \sum_{j=1}^{n-1} \delta_j + (1-n)\delta_n \quad \text{for} \quad n \geq 1;$$

For $x, y \in H$, or if $x \neq y$, the norms $\|\delta_x * \delta_y\|$ are bounded by $k = 3$. The norms are not bounded on J and, for $n \in J$, the semi-bounded function is given by

$$\gamma(n) = \sup_{m \in \mathbb{N}_0} \{\| \delta_n * \delta_m \|\} = 2n - 1 < \infty.$$

The Haar measure on K is $w = w_H + w_J$. That is $w(\{1/n\}) = (2/3)^{n-2}$ for $1/n \in H \setminus \{0\}$, $w(\{n\}) = 1$ for $n \in \tilde{J}$ and, for $A \subseteq K$,

$$w(A) = \sum_{x \in A \setminus \{0\}} w(x).$$

Conclusion: In [6] it was shown that discrete semi-bounded generalized hypergroups can arise naturally as the dual space of a standard hypergroup. Here we have shown that semi-bounded generalized hypergroups, both discrete and non-discrete, can arise arise naturally through Joins. In particular, since every compact hermitian group, and every known hypergroup, is normal and has a Haar measure, joining any such group or hypergroup with a semi-bounded generalized hypergroup will result in a semi-bounded generalized hypergroup.

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