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Some characterizations for a (k,μ)-paracontact metric manifold to be an η-Einstein manifold

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Abstract In this paper we introduce the curvature tensors of (k,μ)-paracontact manifold satisfying the conditions $R\cdot W_0^\star=0$, $P\cdot W_0^\star=0$, $\widetilde{Z}\cdot W_0^\star=0$, $\widetilde{C}\cdot W_0^\star=0$ and $W_0^\star\cdot W_0^\star=0$. According these cases, (k,μ)-paracontact manifolds have been characterized such as η-Einstein.

1 Introduction

In the modern geometry, the geometry of paracontact manifolds has become a subject of growing interest for its considerable applications in applied mathematics and physics. Paracontact manifolds are smooth manifolds of dimension $2n+1$ equipped with a $(1,1)-$tensor ϕ, a vector field ξ, and a 1-form η satisfying $\eta(\xi)=1$, $\phi^2=I-\eta\otimes\xi$ and ϕ induces an almost paracomplex structure on each fibre of $D=\ker(\eta)$ [1].
Moreover if the manifold is equipped with a pseudo-Riemannian metric g so that $g(\phi x_1,\phi x_2)=-g(x_1,x_2)+\eta(x_1)\eta(x_2),\ g(x_1,\phi x_2)=d\eta(x_1,x_2)$, for $x_1,x_2\in\chi(M)$ and (M,ϕ,ξ,η,g) is said to be an almost paracontact metric manifold.

In 1985, Kaneyuki and Williams initiated the perspective of paracontact geometry [2]. Zamkovoy achieved a systematic research on paracontact metric manifolds [3]. Recently, B. Cappeletti-Montano, I. Kupeli Erken and C. Murathan introduced a new type of paracontact geometry so-called paracontact metric $(k,\mu)-$space, where k and μ are constant [4].

After then, G.P. Pokhariyal and R. S. Mishra researched curvature tensors and their relativistic significance [5]. R. H. Ojha developed the properties of the $M-$projective tensor in Sasakian and Kaehler manifolds [6]. B. Prasad introduced a pseudo projective curvature tensor on a Riemannian manifold [7]. Additionally, some geometers have done studies on the curvature of manifolds [8, 9, 10, 11, 12, 13].

Motivated by these idea, we make an attempt to the study properties of the some certain curvature tensor in a $(k,\mu)-$paracontact metric manifold. In the present paper we investigate cases $R\cdot W_0^*=0$, $P\cdot W_0^*=0$, $\widetilde{Z}\cdot W_0^*=0$, $\widetilde{C}\cdot W_0^*=0$ and $W_0^*\cdot W_0^*=0$, where R, P, $\widetilde{Z}$, $\widetilde{C}$ and W_0^* denote the curvature tensors of manifold, respectively.

2 Preliminaries

An $(2n+1)$-dimensional manifold M is called to have an paracontact structure if it admits a $(1,1)-$tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions [2]:

$$\phi^2 x_1 = x_1-\eta(x_1)\xi, \tag{2.1}$$

for any vector field $x_1 \in \chi(M)$, where $\chi(M)$ the set of all differential vector fields on M,

$$\eta(\xi) = 1, \eta \circ \phi = 0, \ \phi\xi = 0, \tag{2.2}$$

an almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\phi x_1, \phi x_2) = -g(x_1, x_2) + \eta(x_1)\eta(x_2), \quad g(x_1, \xi) = \eta(x_1) \tag{2.3}$$

for all vector fields $x_1, x_2 \in \chi(M)$ is said almost paracontact metric manifold, where signature of g is $(n+1, n)$. An almost paracontact structure is called a paracontact structure if $g(x_1, \phi Y) = d\eta(x_1, x_2)$ with the associated metric g [3]. We now define a $(1,1)$-tensor field h by $h = \frac{1}{2} L_\xi \phi$, where L denotes the Lie derivative. Then h is symmetric and satisfies the conditions

$$h\phi = -\phi h, \qquad h\xi = 0, \qquad Tr.h = Tr.\phi h = 0. \tag{2.4}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\widetilde{\nabla}_{x_1}\xi = -\phi x_1 + \phi h x_1 \tag{2.5}$$

for any $x_1 \in \chi(M)$[3]. For a paracontact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$, if ξ is a killing vector field or equivalently, $h = 0$, then it is called a K-paracontact manifold.

An almost paracontact manifold is said to be para-Sasakian if and only if the following condition holds [3]

$$(\widetilde{\nabla}_{x_1}\phi)x_2 = -g(x_1, x_2)\xi + \eta(x_2)x_1$$

for all $x_1, x_2 \in \chi(M)$ [14]. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(x_1, x_2)\xi = -(\eta(x_2)x_1 - \eta(x_1)x_2) \tag{2.6}$$

for all $x_1, x_2 \in \chi(M)$, but this is not a sufficient condition for a para-contact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true [15].

A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ is of the from $S(x_1, x_2) = ag(x_1, x_2) + b\eta(x_1)\eta(x_2)$,where a, b are smooth functions on M. If $b = 0$, then the manifold is also called Einstein [14].

A paracontact metric manifold is said to be a $(k, \mu)-$paracontact manifold if the curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(x_1, x_2)\xi = k\left[\eta(x_2)x_1 - \eta(x_1)x_2\right] + \mu\left[\eta(x_2)hx_1 - \eta(x_1)hx_2\right] \tag{2.7}$$

for all $x_1, x_2 \in \chi(M)$, where k and μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(x_1, x_2)\xi = 0$ [23].

In particular, if $\mu = 0$, then the paracontact metric $(k, \mu)-$manifold is called paracontact metric $N(k)$-manifold. Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(x_1, x_2)\xi = k(\eta(x_2)x_1 - \eta(x_1)x_2) \tag{2.8}$$

for all $x_1, x_2 \in \chi(M)$. Though the geometric behavior of paracontact metric $(k, \mu)-$spaces is different according as $k < -1$, or $k > -1$, but there are some common results for $k < -1$ and $k > -1$ [4].

Lemma 2.1. *There does not exist any paracontact $(k, \mu)-$manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$ [4].*

In a paracontact metric $(k, \mu)-$manifold $(M^{2n+1}\phi, \xi, \eta, g)$, $n > 1$, the following relation hold:

$$h^2 = (k+1)\phi^2, \text{ for } k \neq -1, \tag{2.9}$$

$$(\widetilde{\nabla}_{x_1}\phi)x_2 = -g(x_1 - hx_1, x_2)\xi + \eta(x_2)(x_1 - hx_1), \tag{2.10}$$

$$\begin{aligned} S(x_1, x_2) &= [2(1-n)+n\mu]g(x_1, x_2) + [2(n-1)+\mu]g(hx_1, x_2) \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(x_1)\eta(x_2), \end{aligned} \tag{2.11}$$

$$S(x_1, \xi) = 2nk\eta(x_1), \tag{2.12}$$

$$\begin{aligned} Qx_2 &= [2(1-n)+n\mu]x_2 + [2(n-1)+\mu]hx_2 \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(x_2)\xi, \end{aligned} \tag{2.13}$$

$$Q\xi = 2nk\xi, \quad g(QX, x_2) = S(x_1, x_2), \tag{2.14}$$

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi, \tag{2.15}$$

for any vector fields x_1, x_2 on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1}, g), respectively [4].

The concept of quasi-conformal curvature tensor was defined by K. Yano and S. Sawaki [22]. Quasi-conformal, concircular, projective and $W_0^\star$-curvature tensor of a $(2n+1)$-dimensional Riemannian manifold are, respectively, defined as

$$\begin{aligned} \widetilde{C}(x_1, x_2)x_3 &= aR(x_1, x_2)x_3 + b[S(x_2, x_3)x_1 - S(x_1, x_3)x_2 \\ &\quad +g(x_2, x_3)Qx_1 - g(x_1, x_3)Qx_2] \\ &\quad -\frac{\tau}{2n+1}[\frac{a}{2n} + 2b][g(x_2, x_3)x_1 - g(x_1, x_3)x_2], \end{aligned} \tag{2.16}$$

$$\widetilde{Z}(x_1, x_2)x_3 = R(x_1, x_2)x_3 - \frac{\tau}{2n(2n+1)}[g(x_2, x_3)x_1 - g(x_1, x_3)x_2], \tag{2.17}$$

$$P(x_1, x_2)x_3 = R(x_1, x_2)x_3 - \frac{1}{2n}[S(x_2, x_3)x_1 - S(x_1, x_3)x_2], \tag{2.18}$$

$$W_0^*(x_1, x_2)x_3 = R(x_1, x_2)x_3 + \frac{1}{2n}[S(x_2, x_3)x_1 - g(x_1, x_3)Qx_2], \tag{2.19}$$

for all $x_1, x_2, x_3 \in \chi(M)$ [5].

3 Some characterizations for a $(k, \mu)-$Paracontact Manifolds

In this section, we will give the main results for this paper.
Let M be $(2n+1)-$dimensional $(k, \mu)-$paracontact metric manifold and we denote W_0^*-curvature tensor from (2.19), we have for later

$$\begin{aligned} W_0^*(x_1, x_2)\xi &= k(2\eta(x_2)x_1 - \eta(x_1)x_2) + \mu(\eta(x_2)hx_1 - \eta(x_1)hx_2) \\ &\quad -\frac{1}{2n}\eta(x_1)Qx_2. \end{aligned} \tag{3.1}$$

Putting $x_1 = \xi$ in (2.19), we obtain

$$\begin{aligned} W_0^*(\xi, x_2)x_3 &= k(g(x_2, x_3)\xi - \eta(x_3)x_2) + \mu(g(hx_2, x_3)\xi - \eta(x_3)hx_2) \\ &\quad +\frac{1}{2n}(S(x_2, x_3)\xi - \eta(x_3)Qx_2). \end{aligned} \tag{3.2}$$

In (3.2), by choosing $x_3 = \xi$,

$$W_0^*(\xi, x_2)\xi = k(2\eta(x_2)\xi - x_2) - \mu hY - \frac{1}{2n}Qx_2. \tag{3.3}$$

From (2.7), it follows

$$R(\xi, x_2)x_3 = k(g(x_2, x_3)\xi - \eta(x_3)x_2) + \mu(g(hx_2, x_3)\xi - \eta(x_3)hx_2). \tag{3.4}$$

Choosing $x_3 = \xi$ in (3.4), we have

$$R(\xi, x_2)\xi = k(\eta(x_2)\xi - x_2) - \mu h x_2. \tag{3.5}$$

In the same way, choosing $x_1 = \xi$ and by using (3.4) in (2.16), we have

$$\begin{aligned}\widetilde{C}(\xi, x_2)x_3 &= (ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)(g(x_2, x_3)\xi - \eta(x_3)x_2) \\ &\quad + a\mu(g(hx_2, x_3)\xi - \eta(x_3)hY) + b(S(x_2, x_3)\xi - \eta(x_3)Qx_2).\end{aligned} \tag{3.6}$$

In (3.6) choosing $x_3 = \xi$ and using (2.12), we obtain

$$\begin{aligned}\widetilde{C}(\xi, x_2)\xi &= (ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)(\eta(x_2)\xi - x_2) \\ &\quad - a\mu h x_2 + b(2nk\eta(x_2)\xi - Qx_2).\end{aligned} \tag{3.7}$$

In same way, from (3.4) and (2.17), we get

$$\widetilde{Z}(\xi, x_2)x_3 = (k - \frac{r}{2n(2n+1)})(g(x_2, x_3)\xi - \eta(x_3)x_2) + \mu(g(hx_2, x_3)\xi - \eta(x_3)hx_2), \tag{3.8}$$

from which

$$\widetilde{Z}(\xi, x_2)\xi = (k - \frac{r}{2n(2n+1)})(\eta(x_2)\xi - x_2) - \mu h x_2. \tag{3.9}$$

From (2.18) and (3.4), we observe

$$P(\xi, x_2)x_3 = kg(x_2, x_3)\xi + \mu(g(hx_2, x_3)\xi - \eta(x_3)hx_2) - \frac{1}{2n}S(x_2, x_3)\xi. \tag{3.10}$$

Choosing $x_3 = \xi$ in (3.10), we obtain

$$P(\xi, x_2)\xi = -\mu h x_2. \tag{3.11}$$

Theorem 3.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-paracontact space. Then $\widetilde{C} \cdot W_0^* = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $\widetilde{C} \cdot W_0^* = 0$. This implies that

$$\begin{aligned}(\widetilde{C}(x_1, x_2)W_0^*)(x_5, x_4)x_3 &= \widetilde{C}(x_1, x_2)W_0^*(x_5, x_4)x_3 - W_0^*(\widetilde{C}(x_1, x_2)x_5, x_4)x_3 \\ &\quad - W_0^*(x_5, \widetilde{C}(x_1, x_2)x_4)x_3 \\ &\quad - W_0^*(x_5, x_4)\widetilde{C}(x_1, x_2)x_3 = 0,\end{aligned} \tag{3.12}$$

for any $x_1, x_2, x_3, x_4, x_5 \in \chi(M)$. Taking $x_1 = x_3 = \xi$ in (3.12), making use of (3.1), (3.6) and (3.7), for $A = [ak + 2nkb - \frac{r}{(2n+1)}(\frac{a}{2n} + 2b)]$, we have

$$\begin{aligned}(\widetilde{C}(\xi, x_2)W_0^*)(x_5, x_4)\xi &= \widetilde{C}(\xi, x_2)(k(2\eta(x_4)x_5 - \eta(x_5)x_4) + \mu(\eta(x_4)hx_5 \\ &\quad - \eta(x_5)hx_4) - \frac{1}{2n}\eta(x_5)Qx_4) - W_0^*(A(g(x_2, x_5)\xi \\ &\quad - \eta(x_5)x_2) + a\mu(g(hx_2, x_5)\xi - \eta(x_5)hx_2) \\ &\quad + b(S(x_2, x_5)\xi - \eta(x_5)Qx_2), x_4))\xi - W_0^*(x_5, A(g(x_2, x_4)\xi \\ &\quad - \eta(x_4)x_2) + a\mu(g(hx_2, x_4)\xi - \eta(x_4)hx_2) \\ &\quad + b(S(x_2, x_4)\xi - \eta(x_4)Qx_2))\xi - W_0^*(x_5, x_4)(A(2\eta(x_2)\xi \\ &\quad - x_2) - a\mu h x_2 + b(2nk\eta(x_2)\xi - Qx_2) = 0.\end{aligned} \tag{3.13}$$

Taking into account (3.1), (3.3), (3.6) and inner product both sides of (3.13) by $x_3 \in \chi(M)$ and using (2.3), (2.13) and (2.19) choosing $x_4 = x_2 = e_i, \xi$, in (3.13), $1 \leq i \leq n$, for orthonormal basis of $\chi(M)$, we arrive

$$
\begin{aligned}
&(kb-\frac{br}{2n}-A+b[2(1-n)+n\mu])S(x_5,x_3)+(a\mu-b\mu+b[2(n-1)+\mu])S(x_5,hx_3)\\
&+(\frac{Ar}{2n}+a\mu(1+k)[2(n-1)+\mu]+bk[2(n-1)+n(2k-\mu)]+\frac{br}{2n}[2(1-n)+n\mu]\\
&+b(1+k)(1+k)[2(n-1)+\mu]^2+bk[2(n-1)+n(2k-\mu)]\\
&+Ak+a\mu^2(1+k)-kA(2n+1)-kbr)g(x_5,x_3)\\
&+(b\mu[2(n-1)+n(2k-\mu)]+ka\mu+2nA\mu-\mu br)g(x_5,hx_3)\\
&+(-bk[2(n-1)+n(2k-\mu)]+2nk^2b+ka(2n+1)-kbr\\
&-4nkb-a\mu^2(1+k)(2n+1)-2nb\mu(1+k)[2(n-1)+\mu]\\
&-\frac{Ar}{2n}-a\mu(1+k)[2(n-1)+\mu]-\frac{br}{2n}(1+k)[2(n-1)+n(2k-\mu)]\\
&-b(1+k)[2(n-1)+\mu]^2-2bk[2(n-1)+n(2k-\mu)]\\
&-bk[2(1-n)+n\mu])\eta(x_5)\eta(x_3)=0. \qquad (3.14)
\end{aligned}
$$

Replacing hx_3 instead of x_3 in (3.14) and by using (2.9), we get

$$
\begin{aligned}
&(kb-\frac{br}{2n}-A+b[2(1-n)+n\mu])S(x_5,hx_3)+(1+k)(a\mu-b\mu\\
&+b[2(n-1)+\mu])S(x_5,x_3)-2nk(1+k)(a\mu-b\mu\\
&+b[2(n-1)+\mu])\eta(x_5)\eta(x_3)+(\frac{Ar}{2n}+a\mu(1+k)[2(n-1)+\mu]\\
&+bk[2(n-1)+n(2k-\mu)]+\frac{br}{2n}[2(1-n)+n\mu]\\
&+b(1+k)(1+k)[2(n-1)+\mu]^2+bk[2(n-1)+n(2k-\mu)]\\
&+Ak+a\mu^2(1+k)-kA(2n+1)-kbr)g(x_5,hx_3)\\
&+(1+k)+(b\mu[2(n-1)+n(2k-\mu)]+ka\mu\\
&+2nA\mu-\mu br)g(x_5,x_3)-(1+k)(b\mu[2(n-1)\\
&+n(2k-\mu)]+ka\mu+2nA\mu-\mu br)\eta(x_5)\eta(x_3)=0. \qquad (3.15)
\end{aligned}
$$

From (3.14), (3.15) and also using (2.11), for the sake of brevity, we set

$$
\begin{aligned}
p_1 &= (kb-\frac{br}{2n}-A+b[2(1-n)+n\mu])\\
p_2 &= (a\mu-b\mu+b[2(n-1)+\mu])\\
p_3 &= (\frac{Ar}{2n}+a\mu(1+k)[2(n-1)+\mu]+bk[2(n-1)+n(2k-\mu)]\\
&\quad+\frac{br}{2n}[2(1-n)+n\mu]+b(1+k)(1+k)[2(n-1)+\mu]^2\\
&\quad+bk[2(n-1)+n(2k-\mu)]+Ak+a\mu^2(1+k)-kA(2n+1)-kbr)\\
p_4 &= (b\mu[2(n-1)+n(2k-\mu)]+ka\mu+2nA\mu-\mu br)\\
p_5 &= (-bk[2(n-1)+n(2k-\mu)]+2nk^2b+ka(2n+1)-kbr-4nkb\\
&\quad-a\mu^2(1+k)(2n+1)-2nb\mu(1+k)[2(n-1)+\mu]-\frac{Ar}{2n}\\
&\quad-a\mu(1+k)[2(n-1)+\mu]-\frac{br}{2n}(1+k)[2(n-1)+n(2k-\mu)]\\
&\quad-b(1+k)[2(n-1)+\mu]^2-2bk[2(n-1)+n(2k-\mu)]-bk[2(1-n)+n\mu])
\end{aligned}
$$

and

$$\begin{aligned} q_1 &= (p_4p_2(1+k)-p_3p_1)[2(n-1)+\mu]+(p_4p_1-p_3p_2)[2(1-n)+n\mu] \\ q_2 &= (p_1^2-p_2^2(1+k))[2(n-1)+\mu]+(p_4p_1-p_3p_2) \\ q_3 &= (p_4p_1-p_3p_2)[2(n-1)+n(2k-\mu)]-(p_5p_1+2nkp_2^2(1+k)+p_4p_2(1+k))[2(n-1)+\mu], \end{aligned}$$

we conclude

$$q_2S(x_5,x_3)=q_1g(x_5,x_3)+q_3\eta(x_5)\eta(x_3).$$

So, M is an $\eta-$Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta-$Einstein manifold i.e. $q_2S(x_5,x_3)=q_1g(x_5,x_3)+q_3\eta(x_5)\eta(x_3)$, then from (3.15)-(3.12), we have $\widetilde{C}\cdot W_0^*=0$. □

Theorem 3.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $P\cdot W_0^*=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Assume that $P\cdot W_0^*=0$. This yields to

$$\begin{aligned} (P(x_1,x_2)W_0^*)(x_5,x_4)x_3 &= P(x_1,x_2)W_0^*(x_5,x_4)x_3-W_0^*(P(x_1,x_2)x_5,x_4)x_3 \\ &\quad -W_0^*(x_5,P(x_1,x_2)x_4)x_3 \\ &\quad -W_0^*(x_5,x_4)P(x_1,x_2)x_3=0, \end{aligned} \tag{3.16}$$

for any $x_1,x_2,x_3,x_4,x_5\in\chi(M)$. Taking $x_1=x_3=\xi$ in (3.16) and making use of (3.1), (3.10), (3.11), we obtain

$$\begin{aligned} (P(\xi,x_2)W_0^*)(x_5,x_4)\xi &= P(\xi,x_2)(k(2\eta(x_4)x_5-\eta(x_5)x_4)+\mu(\eta(x_4)hx_5-\eta(x_5)hx_4) \\ &\quad -\frac{1}{2n}\eta(x_5)Qx_4)-W_0^*(kg(x_2,x_5)\xi+\mu(g(hx_2,x_5)\xi \\ &\quad -\eta(x_5)hx_2)-\frac{1}{2n}S(x_2,x_5)\xi,x_4)\xi-W_0^*(x_5,kg(x_2,x_4)\xi \\ &\quad +\mu(g(hx_2,x_4)\xi-\eta(x_4)hx_2)-\frac{1}{2n}S(x_2,x_4)\xi)\xi \\ &\quad +W_0^*(x_5,x_4)\mu hx_2=0. \end{aligned} \tag{3.17}$$

Taking into account (3.1), (3.3), (3.10), putting $x_5=\xi$ and inner product both sides of (3.17) by $\xi\in\chi(M)$, we arrive

$$\begin{aligned} & 2n\mu S(x_4,hx_2)-S(x_2,Qx_4)+4n^2k(k-2n)\eta(x_4)\eta(x_2)-4n^2k^2g(x_2,x_4) \\ & -4n^2\mu kg(x_2,hx_4)+(2n\mu-2nk+4n^2k)S(x_2,x_4)=0. \end{aligned} \tag{3.18}$$

Using (2.3), (2.13) and (3.18), we get

$$\begin{aligned} & (2n(2nk+\mu-k)-[2(1-n)+n\mu])S(x_2,x_4)+([2(n-1)+\mu] \\ & +2n\mu)S(x_2,hx_4)-4n^2k^2g(x_2,x_4)-4n^2k\mu g(x_2,hx_4) \\ & +(2nk[2(n-1)+n(2k-\mu)]+4n^2k(k-2n))\eta(x_2)\eta(x_4)=0. \end{aligned} \tag{3.19}$$

Replacing hx_3 of x_3 in (3.19) and making use of (2.9), we have

$$\begin{aligned} & (2n(2nk+\mu-k)-[2(1-n)+n\mu])S(x_2,hx_4)+(1+k)([2(n-1)+\mu] \\ & +2n\mu)S(x_2,x_4)-2nk(1+k)([2(n-1)+\mu]+2n\mu)\eta(x_2)\eta(x_4) \\ & -4n^2k^2g(x_2,hx_4)-4n^2k\mu(1+k)g(x_2,x_4)+4n^2k\mu(1+k)\eta(x_2)\eta(x_4)=0. \end{aligned} \tag{3.20}$$

From (3.19), (3.20) and using (2.11), for the sake of brevity, we put

$$\begin{aligned} p_1 &= (2n(2nk+\mu-k)-[2(1-n)+n\mu]) \\ p_2 &= ([2(n-1)+\mu]+2n\mu) \\ p_3 &= 4n^2k^2 \\ p_4 &= 4n^2k\mu \\ p_5 &= (2nk[2(n-1)+n(2k-\mu)])+4n^2k(k-2n)) \end{aligned}$$

and

$$\begin{aligned} q_1 &= (p_1p_3 - p_4p_2(1+k))[2(n-1)+\mu] + (p_2p_3 - p_1p_4)[2(1-n)+n\mu] \\ q_2 &= (p_1^2 - p_2^2(1+k))[2(n-1)+\mu] + (p_2p_3 - p_1p_4) \\ q_3 &= (p_2p_3 - p_1p_4)[2(n-1)+n(2k-\mu)] + (p_4p_2(1+k) - 2nkp_2^2(1+k) - p_1p_5)[2(n-1)+\mu], \end{aligned}$$

that is,

$$q_2 S(x_2,x_4) = q_1 g(x_2,x_4) + q_3\eta(x_2)\eta(x_4).$$

Thus, M is an η-Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta-$Einstein manifold i.e. $q_2S(x_2,x_4) = q_1g(x_2,x_4) + q_3\eta(x_2)\eta(x_4)$, then from (3.20)-(3.16) we have $P \cdot W_0^* = 0$. □

Theorem 3.3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $R\cdot W_0^* = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $R \cdot W_0^* = 0$. That is,

$$\begin{aligned} (R(x_1,x_2)W_0^*)(x_5,x_4)x_3 &= R(x_1,x_2)W_0^*(x_5,x_4)x_3 - W_0^*(R(x_1,x_2)x_5,x_4)x_3 \\ &\quad -W_0^*(x_5,R(x_1,x_2)x_4)x_3 \\ &\quad -W_0^*(x_5,x_4)R(x_1,x_2)x_3 = 0, \end{aligned} \tag{3.21}$$

for any $x_1,x_2,x_3,x_4,x_5 \in \chi(M)$. Setting $x_1 = x_3 = \xi$ in (3.21) and making use of (3.1), (3.3), (3.4), we obtain

$$\begin{aligned} (R(\xi,x_2)W_0^*)(x_5,x_4)\xi &= R(\xi,x_2)(k(2\eta(x_4)x_5 - \eta(x_5)x_4) \\ &\quad +\mu(\eta(x_4)hx_5 - \eta(x_5)hx_4) - \frac{1}{2n}\eta(x_5)Qx_4) \\ &\quad -W_0^*(k(g(x_2,x_5)\xi - \eta(x_5)x_2) \\ &\quad +\mu(g(hx_2,x_5)\xi - \eta(x_5)hx_2),x_4)\xi \\ &\quad -W_0^*(x_5,k(g(x_2,x_4)\xi - \eta(x_4)x_2) \\ &\quad +\mu(g(hx_2,x_4)\xi - \eta(x_4)hx_2))\xi \\ &\quad -W_0^*(x_5,x_4)(k(\eta(x_2)\xi - x_2) - \mu hx_2) = 0. \end{aligned} \tag{3.22}$$

Inner product both sides of (3.22) by $x_3 \in \chi(M)$ and using of (3.2), (3.3) and (3.5), we get

$$\begin{aligned} &2nkg(W_0^*(x_5,x_4)x_2,x_3) + 2n\mu g(W_0^*(x_5,x_4)hx_2,x_3) \\ &+2nk\mu(\eta(x_4)\eta(x_3)g(x_2,hx_5) - g(hU,x_3)g(x_2,x_4)) \\ &+2n\mu^2(1+k)(\eta(x_4)\eta(x_3)g(x_2,x_5) - \eta(x_5)\eta(x_3)g(x_2,x_4)) \\ &+\mu(\eta(x_5)\eta(x_3)S(x_2,hx_4) - \eta(x_4)\eta(x_5)S(x_2,hx_3)) \\ &+2nk^2(g(x_2,x_5)g(x_4,x_3) - g(x_2,x_4)g(x_5,x_3)) \\ &+2n\mu k(g(x_2,x_5)g(hx_3,x_4) - g(hx_2,x_4)g(x_5,x_3)) \\ &+2n\mu^2(g(hx_2,x_5)g(hx_4,x_3) - g(hx_2,x_4)g(hx_5,x_3)) \\ &+k(g(x_2,x_5)S(x_4,x_3) - g(x_2,x_4)S(x_5,x_3)) \\ &-k(S(x_2,x_4)\eta(x_5)\eta(x_3) + S(x_2,x_3)\eta(x_5)\eta(x_4)) \\ &+2nk^2(g(x_4,x_2)\eta(x_5)\eta(x_3) - g(x_2,x_3)\eta(x_5)\eta(x_4)) \\ &+2nk\mu(\eta(x_5)\eta(x_4)g(hx_2,x_3) + g(hx_2,x_5)g(x_4,x_3)) \\ &-\mu(\eta(x_5)\eta(x_3)S(hx_2,x_4) + g(hx_2,x_4)S(x_5,x_3)) = 0. \end{aligned} \tag{3.23}$$

Making use of (2.9), (2.19) and choosing $x_5 = x_3 = e_i, \xi$, $1 \le i \le n$, for orthonormal basis of $\chi(M)$ in (3.23), we have

$$\begin{aligned} &4nkS(x_4,x_2) + 4n\mu S(x_4,hx_2) + k(2nk - 4n^2k - r)g(x_2,x_4) \\ &+\mu(2nk - 4n^2k - r)g(x_2,hx_4) = 0. \end{aligned} \tag{3.24}$$

Replacing hx_2 instead of x_2 in (3.24) and taking into account (2.9), we get

$$\begin{aligned}&4nkS(x_4,hx_2)+4n\mu(1+k)S(x_4,x_2)-8n^2k\mu(1+k)\eta(x_4)\eta(x_2)\\&+k(2nk-4n^2k-r)g(x_4,hx_2)+\mu(1+k)(2nk-4n^2k-r)g(x_4,x_2)\\&-\mu(1+k)(2nk-4n^2k-r)\eta(x_4)\eta(x_2)=0.\end{aligned}\tag{3.25}$$

From (3.24), (3.25) and by using (2.11), for the sake of brevity, we set

$$\begin{aligned}p_1&=4n(k^2-\mu^2(1+k)),\\p_2&=(\mu^2(1+k)-k^2)(2nk-4n^2k-r),\\p_3&=-\mu^2(1+k)(2nk-4n^2k-r)\end{aligned}$$

and we have

$$p_1S(x_2,x_4)=p_2g(x_2,x_4)+p_3\eta(x_2)\eta(x_4),$$

which verifies our assertion. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta-$Einstein manifold i.e. $p_1S(x_2,x_4)=p_2g(x_2,x_4)+p_3\eta(x_2)\eta(x_4)$, then from (3.25)-(3.21) we have $R\cdot W_0^*=0$. □

Theorem 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-paracontact space. Then $\widetilde{Z}\cdot W_0^*=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Assume that $\widetilde{Z}\cdot W_0^*=0$. This implies that

$$\begin{aligned}(\widetilde{Z}(x_1,x_2)W_0^*)(x_5,x_4,x_3)&=\widetilde{Z}(x_1,x_2)W_0^*(x_5,x_4)x_3-W_0^*(\widetilde{Z}(x_1,x_2)x_5,x_4)x_3\\&\quad-W_0^*(x_5,\widetilde{Z}(x_1,x_2)x_4)x_3\\&\quad-W_0^*(x_5,x_4)\widetilde{Z}(x_1,x_2)x_3=0,\end{aligned}\tag{3.26}$$

for any $x_1,x_2,x_3,x_4,x_5\in\chi(M)$. Setting $x_1=x_3=\xi$ in (3.26) and making use of (3.1), (3.8), (3.9) for $A=k-\frac{r}{2n(2n+1)}$, we obtain

$$\begin{aligned}(\widetilde{Z}(\xi,x_2)W_0^*)(x_5,x_4)\xi&=\widetilde{Z}(\xi,x_2)(k(2\eta(x_4)x_5-\eta(x_5)x_4)\\&\quad+\mu(\eta(x_4)hx_5-\eta(x_5)hx_4)-\frac{1}{2n}\eta(x_5)Qx_4)\\&\quad-W_0^*(A(g(x_2,x_5)\xi-\eta(x_5)x_2)\\&\quad+\mu(g(hx_2,x_5)\xi-\eta(x_5)hx_2,x_4)\xi\\&\quad-W_0^*(x_5,A(g(x_2,x_4)\xi-\eta(x_4)x_2)\\&\quad+\mu(g(hx_2,x_4)\xi-\eta(x_4)hx_2))\xi\\&\quad-W_0^*(x_5,x_4)(A(\eta(x_2)\xi-x_2)-\mu hx_2)=0.\end{aligned}\tag{3.27}$$

By means of (3.1), (3.3), (3.8) and inner product both sides of (3.27) by $x_3\in\chi(M)$, we get

$$\begin{aligned}&2nAg(W_0^*(x_5,x_4)x_2,x_3)+2n\mu g(W_0^*(x_5,x_4)hY,x_3)+2nA\mu(\eta(x_4)\eta(x_3)g(x_2,hU)\\&-\eta(x_5)\eta(x_3)g(x_2,hW))+2n\mu^2(1+k)(\eta(x_4)\eta(x_3)g(x_2,x_5)-\eta(x_5)\eta(x_3)g(x_2,x_4))\\&+2nkA(\eta(x_5)\eta(x_4)g(x_2,x_3)-\eta(x_5)\eta(x_3)g(x_2,x_4))+2nk\mu(\eta(x_5)\eta(x_4)g(hY,x_3)\\&-\eta(x_5)\eta(x_3)g(hY,x_4))+A(g(x_2,x_5)S(x_3,x_4)-g(x_2,x_4)S(x_5,x_3))\\&+\mu(g(hY,x_5)S(x_4,x_3)-g(hY,x_4)S(x_5,x_3))+2n\mu^2(g(hY,x_5)g(hW,x_3)\\&-g(hY,x_4)g(hU,x_3))-A(S(x_2,x_4)\eta(x_5)\eta(x_3)-S(x_2,x_3)\eta(x_4)\eta(x_5))\\&+2n\mu A(g(x_2,x_5)g(hW,x_3)-g(x_2,x_4)g(hU,x_3))+2nk\mu(g(hY,x_5)g(x_4,x_3)\\&-g(hY,x_4)g(x_5,x_3))+2nkA(g(x_2,x_5)g(x_4,x_3)-g(x_2,x_4)g(x_5,x_3))\\&-\mu(\eta(x_5)\eta(x_3)S(x_2,hW)+\eta(x_4)\eta(x_5)S(x_2,hZ))+4nk(Ag(x_2,x_4)\eta(x_5)\eta(x_3)\\&+\mu g(hY,x_4)\eta(x_5)\eta(x_3))-\mu\eta(x_5)\eta(x_4)S(x_2,hZ)=0.\end{aligned}\tag{3.28}$$

Making use of (2.12), (2.19) and choosing $x_5 = x_3 = e_i, \xi$, $1 \leq i \leq n$, for orthonormal basis of $\chi(M)$ in (3.28), we have

$$\begin{aligned} &4nAS(x_4, x_2) + 4n\mu S(x_4, hx_2) + A(2nk - 4n^2k - r)g(x_4, x_2) \\ &+\mu(2nk - 4n^2k - r)g(x_4, hx_2) = 0. \end{aligned} \tag{3.29}$$

Replacing hx_2 instead of x_2 in (3.29) and taking into account (2.9), we arrive

$$\begin{aligned} &4nAS(x_4, hx_2) + 4n\mu(1+k)S(x_4, x_2) - 8n^2k\mu(1+k)\eta(x_4)\eta(x_2) \\ &+A(2nk - 4n^2k - r)g(x_4, hx_2) + \mu(1+k)(2nk - 4n^2k - r)g(x_4, x_2) \\ &-\mu(1+k)(2nk - 4n^2k - r)\eta(x_4)\eta(x_2) = 0. \end{aligned} \tag{3.30}$$

From (3.29), (3.30) and by using (2.11), for the sake of brevity, if we set

$$\begin{aligned} p_1 &= 4n(A^2 - \mu^2(1+k)), \\ p_2 &= (\mu^2(1+k) - A^2)(2nk - 4n^2k - r), \\ p_3 &= -\mu^2(1+k)(4n^2k + 2nk - r) \end{aligned}$$

then (3.30) reduce

$$p_1S(x_4, x_2) = p_2g(x_4, x_2) + p_3\eta(x_4)\eta(x_2).$$

This tell us, M is an η-Einstein manifold. Conversely, let $M^{2n+1}(\phi, \xi, \eta, g)$ be an $\eta-$Einstein manifold i.e. $p_1S(x_4, x_2) = p_2g(x_4, x_2) + p_3\eta(x_4)\eta(x_2)$, then from (3.30)-(3.26) we have $\widetilde{Z} \cdot W_0^* = 0$. □

Theorem 3.5. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a (k, μ)-paracontact space. Then $W_0^* \cdot W_0^* = 0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $W_0^* \cdot W_0^* = 0$. That is,

$$\begin{aligned} (W_0^*(x_1, x_2)W_0^*)(x_5, x_4, x_3) &= W_0^*(x_1, x_2)W_0^*(x_5, x_4)x_3 - W_0^*(W_0^*(x_1, x_2)x_5, x_4)x_3 \\ &\quad -W_0^*(x_5, W_0^*(x_1, x_2)x_4)x_3 \\ &\quad -W_0^*(x_5, x_4)W_0^*(x_1, x_2)x_3 = 0, \end{aligned} \tag{3.31}$$

for any $x_1, x_2, x_3, x_4, x_5 \in \chi(M)$. Setting $x_1 = x_3 = \xi$ in (3.31), in view of (3.1), (3.2), we obtain

$$\begin{aligned} (W_0^*(\xi, x_2)W_0^*)(x_5, x_4)\xi &= W_0^*(\xi, x_2)(k(2\eta(x_4)x_5 - \eta(x_5)x_4) + \mu(\eta(x_4)hx_5 \\ &\quad -\eta(x_5)hx_4) - \frac{1}{2n}\eta(x_5)Qx_4) - W_0^*(k(g(x_2, x_5)\xi \\ &\quad -\eta(x_5)x_2) + \mu(g(hx_2, x_5)\xi - \eta(x_5)hx_2) + \frac{1}{2n}(S(x_2, x_5)\xi \\ &\quad -\eta(x_5)Qx_2), x_4)\xi - W_0^*(x_5, k(g(x_2, x_4)\xi - \eta(x_4)x_2) \\ &\quad +\mu(g(hx_2, x_4)\xi - \eta(x_4)hx_2) + \frac{1}{2n}(S(x_2, x_4)\xi \\ &\quad -\eta(x_4)Qx_2))\xi - W_0^*(x_5, x_4)(k(2\eta(x_2)\xi - x_2) \\ &\quad -\mu hx_2 - \frac{1}{2n}Qx_2) = 0. \end{aligned} \tag{3.32}$$

Using (3.2), (3.3) and inner product both sides of (3.32) by $x_3 \in \chi(M)$, we get

$$\begin{aligned}
&2n^2kg(W_0^*(x_5,x_4)x_2,x_3)+4n^2\mu g(W_0^*(x_5,x_4)hx_2,x_3)+2ng(W_0^*(x_5,x_4)Qx_2,x_3)\\
&+4n^2k\mu(\eta(x_4)\eta(x_3)g(x_2,hx_5)-g(x_5,x_3)g(x_2,hx_4))+4n^2\mu^2(1+k)(\eta(x_4)\eta(x_3)g(x_2,x_5)\\
&-\eta(x_5)\eta(x_3)g(x_2,x_4))+2n\mu(\eta(x_3)\eta(x_4)S(x_2,hx_5)-\eta(x_5)\eta(x_3)S(x_2,hx_4))\\
&+4n^2k^2(\eta(x_5)\eta(x_4)g(x_2,x_3)-g(x_2,x_4)g(x_5,x_3))+2n\mu(g(hx_2,x_5)S(x_3,x_4)\\
&-S(hx_2,x_4)\eta(x_5)\eta(x_3))+4n^2k^2(g(x_2,x_5)g(x_4,x_3)+g(x_2,x_4)\eta(x_5)\eta(x_3))\\
&+4n^2k^2\mu(g(x_2,x_5)g(hx_4,x_3)+g(hx_2,x_5)g(x_4,x_3))+2nk(g(x_2,x_5)S(x_4,x_3)\\
&+S(x_2,x_5)g(x_4,x_3))+2n\mu(S(x_2,x_5)g(hx_4,x_3)-S(x_2,x_4)g(hx_5,x_3))\\
&+4n^2k^2\mu(g(hx_2,x_3)\eta(x_5)\eta(x_4)-g(x_2,x_4)g(hx_5,x_3))+4n^2\mu^2(g(hx_2,x_5)g(hx_4,x_3)\\
&-g(hx_2,x_4)g(hx_5,x_3))-2nk(g(x_2,x_4)S(x_5,x_3)+g(x_5,x_3)S(x_2,x_4))\\
&-2n\mu(g(hx_2,x_4)S(x_5,x_3)+S(x_2,x_4)g(hx_5,x_3))+(S(x_2,x_5)S(x_4,x_3)\\
&-S(x_2,x_4)S(x_5,x_3))-(S(x_2,Qx_4)\eta(x_5)\eta(x_3)+S(Qx_2,x_3)\eta(x_5)\eta(x_4))\\
&+2n(kS(x_2,x_3)\eta(x_5)\eta(x_4)-\mu S(hx_2,x_3)\eta(x_5)\eta(x_4))\\
&+8n^2k^2g(x_4,x_3)\eta(x_5)\eta(x_2)=0. \qquad (3.33)
\end{aligned}$$

Making use of (2.12), (2.19) and choosing $x_4 = x_2 = e_i, \xi$, $1 \leq i \leq n$, for orthonormal basis of $\chi(M)$ in (3.33), we conclude

$$\begin{aligned}
&(2n[2(1-n)+n\mu]-r)S(x_5,x_3)+(4n^2k+2n\mu+2n[2(n-1)+\mu])S(x_5,hZ)\\
&+(4n^2\mu(1+k)[2(n-1)+\mu]+4nk[2(n-1)+n(2k-\mu)]+r[2(1-n)+n\mu]\\
&+2n(1+k)[2(n-1)+\mu]^2+4n^2\mu^2(1+k)-8n^3k^2)g(x_5,x_3)\\
&+(2n\mu[2(n-1)+n(2k-\mu)]-2nr\mu+4n^2k\mu(1-2n))g(hU,x_3)\\
&+(20n^2k^2-8nk[2(n-1)+n(2k-\mu)]-r[2(1-n)+n\mu]\\
&+4n^2k^2(2n+1)-8n^3\mu^2(1+k)-8n^2\mu(1+k)[2(n-1)+\mu]\\
&-2n(1+k)[2(n-1)+\mu]^2)\eta(x_5)\eta(x_3)=0. \qquad (3.34)
\end{aligned}$$

Replacing hx_3 instead of x_3 and putting (2.9) in (3.34), we observe

$$\begin{aligned}
&(2n[2(1-n)+n\mu]-r)S(x_5,hx_3)+(1+k)(4n^2k+2n\mu+2n[2(n-1)+\mu])S(x_5,x_3)\\
&-2nk(1+k)(4n^2k+2n\mu+2n[2(n-1)+\mu])\eta(x_5)\eta(x_3)+(4n^2\mu(1+k)[2(n-1)+\mu]\\
&+4nk[2(n-1)+n(2k-\mu)]+r[2(1-n)+n\mu]+2n(1+k)[2(n-1)\\
&+\mu]^2+4n^2\mu^2(1+k)-8n^3k^2)g(x_5,hx_3)+(1+k)(2n\mu[2(n-1)+n(2k-\mu)]\\
&-2nr\mu+4n^2k\mu(1-2n))g(x_5,x_3)-(1+k)(2n\mu[2(n-1)+n(2k-\mu)]\\
&-2nr\mu+4n^2k\mu(1-2n))\eta(x_5)\eta(x_3)=0. \qquad (3.35)
\end{aligned}$$

From (3.34), (3.35) and by using (2.11), for the sake of brevity, we set

$$\begin{aligned}
p_1 &= (2n[2(1-n)+n\mu]-r)\\
p_2 &= (4n^2k+2n\mu+2n[2(n-1)+\mu])\\
p_3 &= (4n^2\mu(1+k)[2(n-1)+\mu]+4nk[2(n-1)+n(2k-\mu)]\\
&\quad +r[2(1-n)+n\mu]+2n(1+k)[2(n-1)+\mu]^2+4n^2\mu^2(1+k)-8n^3k^2)\\
p_4 &= (2n\mu[2(n-1)+n(2k-\mu)]-2nr\mu+4n^2k\mu(1-2n))\\
p_5 &= (20n^2k^2-8nk[2(n-1)+n(2k-\mu)]-r[2(1-n)+n\mu]+4n^2k^2(2n+1)\\
&\quad -8n^3\mu^2(1+k)-8n^2\mu(1+k)[2(n-1)+\mu]-2n(1+k)[2(n-1)+\mu]^2)
\end{aligned}$$

and

$$\begin{aligned} q_1 &= [p_4p_2(1+k)-p_3p_1][2(n-1)+\mu]+(p_4p_1-p_3p_2)[2(1-n)+n\mu],\\ q_2 &= (p_1^2-p_2^2(1+k))[2(n-1)+\mu]+(p_4p_1-p_3p_2),\\ q_3 &= (p_4p_1-p_3p_2)[2(n-1)+n(2k-\mu)]-(p_5p_1+2nkp_2^2(1+k)+p_4p_2(1+k))[2(n-1)+\mu], \end{aligned}$$

then we have

$$q_2S(x_5,x_3)=q_1g(x_5,x_3)+q_3\eta(x_5)\eta(x_3).$$

So, M is an η-Einstein manifold. Conversely, let $M^{2n+1}(\phi,\xi,\eta,g)$ be an $\eta-$Einstein manifold i.e. $q_2S(x_5,x_3)=q_1g(x_5,x_3)+q_3\eta(x_5)\eta(x_3)$, then from (3.35)-(3.31) we have $W_0^*\cdot W_0^*=0$. □

Example 3.6. We consider the 3-dimensional manifold $M=\{(x_1,x_2,x_3)\in\mathbb{R}^3, x_1\geq 0\}$, where (x_1,x_2,x_3) are standart coordinates of $\mathbb{R}^3$. The vector fields

$$e_1=5x^3\frac{\partial}{\partial x_1}+(x_3^2+5)\frac{\partial}{\partial x_2}+(\sqrt{x_1}-5)\frac{\partial}{\partial x_3},\quad e_2=\frac{\partial}{\partial x_2},\quad e_3=\frac{\partial}{\partial x_3}.$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_2) &= g(e_1,e_3)=g(e_2,e_3)=0,\\ g(e_1,e_1) &= g(e_2,e_2)=1,\quad g(e_3,e_3)=-1 \end{aligned}$$

Let η be the 1-form defined by $\eta(x_1)=g(x_1,e_2)$ for any $x_1\in\chi(M)$. Let ϕ be the (1,1) tensor field defined by

$$\phi(e_2)=0,\qquad \phi(e_3)=-e_1,\qquad \phi(e_3)=-e_1.$$

Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then we get

$$[e_3,e_1]=2ze_2,\quad [e_1,e_2]=0,\quad [e_2,e_3]=0.$$

Then we have

$$\eta(e_2)=g(e_2,e_2)=1,\quad \phi^2x_1=x_1-\eta(x_1)e_2,\quad g(\phi X,\phi Y)=-g(x_1,x_2)+\eta(x_1)\eta(x_4),$$

for any $x_1,x_2\in\chi(M)$. Hence, (ϕ,ξ,η,g) defines a paracontact metric structure on M for $e_2=\xi$.

The Levi-Civita connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_{x_1}x_2,x_3) &= Xg(x_2,x_3)+Yg(x_3,x_1)-Zg(x_1,x_2)\\ &\quad -g(x_1,[x_2,x_3])-g(x_2,[x_1,x_3])+g(x_3,[x_1,x_2]). \end{aligned}$$

Using the above formula we obtain.

$$\begin{aligned} \nabla_{e_1}e_1 &= 0, & \nabla_{e_2}e_1 &= -ze_3, & \nabla_{e_3}e_1 &= ze_2,\\ \nabla_{e_1}e_2 &= -ze_3, & \nabla_{e_2}e_2 &= 0, & \nabla_{e_3}e_2 &= -ze_1,\\ \nabla_{e_1}e_3 &= -ze_2, & \nabla_{e_2}e_3 &= -ze_1, & \nabla_{e_3}e_3 &= 0. \end{aligned}$$

Comparing the above relations with $\nabla_{x_1}e_2=-\phi X+\phi hX$, we get

$$he_1=-(x_3+1)e_1,\quad he_3=-(x_3+1)e_3,\quad he_2=0.$$

Using the formula $R(x_1,x_2)x_3=\nabla_{x_1}\nabla_{x_2}x_3-\nabla_{x_2}\nabla_{x_1}x_3-\nabla_{[x_1,x_2]}x_3$, we calculate the following:

$$\begin{aligned} R(e_1,e_2)e_2 &= x_3(x_3+2)\{\eta(e_2)e_1-\eta(e_1)e_2\}+2z\{\eta(e_2)he_1-\eta(e_1)he_2\}\\ &= -x_3^2e_1 \end{aligned}$$

$$\begin{aligned} R(e_2,e_3)e_2 &= x_3(x_3+2)\{\eta(e_3)e_2-\eta(e_2)e_3\}+2z\{\eta(e_3)he_2-\eta(e_2)he_3\}\\ &= x_3^2e_3 \end{aligned}$$

$$\begin{aligned} R(e_1,e_3)e_2 &= x_3(x_3+2)\{\eta(e_3)e_1-\eta(e_1)e_3\}+2z\{\eta(e_3)he_1-\eta(e_1)he_3\}\\ &= 0. \end{aligned}$$

By the above expressions of the curvature tensor and using (2.9), we conclude that M is a $(k,\mu)-$paracontact metric manifold with $k=x_3(x_3+2)$ and $\mu=2x_3$.

4 Conclusion remarks

This paper aims is to obtain the curvature tensors of (k,μ)-paracontact manifold satisfying the conditions $R \cdot W_0^\star = 0$, $P \cdot W_0^\star = 0$, $\widetilde{Z} \cdot W_0^\star = 0$, $\widetilde{C} \cdot W_0^\star = 0$ and $W_0^\star \cdot W_0^\star = 0$. According these cases, (k, μ)-paracontact manifolds have been characterized such as η-Einstein. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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