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SOLVABILITY OF A BIDIMENSIONAL SYSTEM OF RATIONAL DIFFERENCE EQUATIONS VIA MERSENNE NUMBERS

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Abstract In this paper, we focus on obtaining the closed-form solution for the following bidimensional system of higher-order rational difference equations:

$$u_{n+1}^{(1)} = \frac{1}{3 - 2u_{n-m}^{(2)}}, u_{n+1}^{(2)} = \frac{1}{3 - 2u_{n-m}^{(1)}}, n, m \in \mathbb{N}_0,$$

where the initial values $u_{-j}^{(1)}$ and $u_{-j}^{(2)}$, $j \in \{0, 1, ..., m\}$ are real numbers not equal to $3/2$. We show that the solutions of this system are associated with Mersenne numbers and/or Mersenne-Lucas numbers. It is shown that the global stability of positive solutions of this system holds. Finally, we provide numerical examples to illustrate our results.

1 Introduction

In recent decades, there has been a growing interest in determining the behavior of the solutions of difference equations or systems (see., [1], [2], [21], [22]) as the main problem in the theory of difference equations. One of the ways to treat this problem is to provide solutions of difference equations or systems. So, the search for solutions in the closed form of difference equations and/or systems has attracted the attention of many mathematicians (see., [3]−[4], [15]−[26], [33]−[35], [38]−[39]). Their application is prevalent in modeling the evolutionary patterns of variables like exchange rates, particularly when data is sampled at discrete time intervals. Notably, in econometrics, stochastic terms are often integrated when modeling difference equations (see., [5]−[11], [27]−[32]). Furthermore, at the beginning of the 18th century, the following homogeneous linear difference equations of the 2nd-order were solved by De Moivre [14],

$$u_{n+1} = \alpha u_n + \beta u_{n-1}, n \geq 1,$$

where $\alpha, \beta \in \mathbb{R}$ or $\mathbb{C}$ such that $\beta \neq 0$, in particular, we give information about Mersenne sequence that establishes a significant part of our study, defined as follows

$$M_{n+1} = 3M_n - 2M_{n-1},\ n \geq 1,$$

with initial conditions $M_0 = 0$ and $M_1 = 1$. The following Binet formula of the Mersenne numbers gives, $M_n = (a^n - b^n) / (a - b)$, where $a = 2$ and $b = 1$ (see., [12]), and the closed-form expression for the Mersenne-Lucas numbers are $m_n = a^n + b^n$ (see., [36], [37]). This led to the emergence of a new problem, which is how to turn nonlinear difference equations or systems into linear difference equations or systems. Now, in this paper, we seek to provide a class of system of nonlinear difference equations which can be solved in explicit form, but the solutions are expressed by Mersenne numbers, is the following bidimensional system of nonlinear difference equations,

$$u_{n+1}^{(1)} = \frac{1}{3 - 2u_{n-m}^{(2)}},\ u_{n+1}^{(2)} = \frac{1}{3 - 2u_{n-m}^{(1)}}, n, m \in \mathbb{N}_0, \tag{1.1}$$

and the initial values $u_{-m}^{(1)}, ..., u_0^{(1)}, u_{-m}^{(2)}, ..., u_0^{(2)}$ are real numbers not equal to $3/2$.

2 Main results

To solve system (2.5) , we need to utilize the following lemmas.

Lemma 2.1. *Consider the homogeneous linear difference equation with constant coefficients*

$$\gamma_{n+1} - 3\gamma_n + 2\gamma_{n-1} = 0, n \geq 0, \tag{2.1}$$

with initial conditions $\gamma_0, \gamma_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ \gamma_n = \gamma_0 M_{n+1} - 2\gamma_{-1} M_n,$$

where $(M_n, n \geq 0)$ *is the Mersenne sequence.*

Proof. The difference equation (2.1) is ordinarily solved using the characteristic polynomial,

$$\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0,$$

and the roots of this equation are

$$\lambda_1 = a, \lambda_2 = b.$$

These roots are linked to the roots of the Mersenne number sequence. The closed form general solution of the equation (2.1) is given by

$$\forall n \geq -1,\ \gamma_n = c_1 a^n + c_2 b^n,$$

where γ_0, γ_{-1} are initial values such that

$$\begin{cases} \gamma_0 = c_1 + c_2 \\ \gamma_{-1} = \dfrac{c_1}{a} + \dfrac{c_2}{b} \end{cases},$$

and c_1, c_2 are given by:

$$c_1 = a(\gamma_0 - \gamma_{-1}), c_2 = a\gamma_{-1} - b\gamma_0,$$

after some calculations, we get

$$\begin{aligned} \gamma_n &= a(\gamma_0 - \gamma_{-1}) a^n + (a\gamma_{-1} - b\gamma_0) b^n \\ &= \gamma_0 \left(\frac{a^{n+1} - b^{n+1}}{a - b} \right) - 2\gamma_{-1} \left(\frac{a^n - b^n}{a - b} \right). \end{aligned}$$

The lemma is proved. □

Lemma 2.2. *Consider the homogeneous linear difference equation with constant coefficients*

$$\delta_{n+1} + 3\delta_n + 2\delta_{n-1} = 0, n \geq 0, \tag{2.2}$$

with initial conditions $\delta_0, \delta_{-1} \in \mathbb{R}$. *Then,*

$$\forall n \geq 0,\ \delta_n = (-1)^n (\delta_0 M_{n+1} + 2\delta_{-1} M_n),$$

where $(M_n, n \geq 0)$ *is the Mersenne sequence.*

Proof. The difference equation (2.2) is ordinarily solved using the characteristic polynomial, $\lambda^2+3\lambda+2=0$, and the roots of this equation are

$$\lambda_1=-a,\lambda_2=-b.$$

These roots are linked to the roots of the Mersenne number sequence. The closed form general solution of the equation (2.2) is given by

$$\forall n\geq -1,\ \delta_n=(-1)^n\left(\widetilde{c}_1 a^n+\widetilde{c}_2 b^n\right),$$

where δ_0, δ_{-1} are initial values such that

$$\begin{cases} \delta_0=\widetilde{c}_1+\widetilde{c}_2 \\ \delta_{-1}=-\dfrac{\widetilde{c}_1}{a}-\dfrac{\widetilde{c}_2}{b} \end{cases},$$

and $\widetilde{c}_1,\widetilde{c}_2$ are given by:

$$\widetilde{c}_1=a\left(\delta_0+\delta_{-1}\right),\widetilde{c}_2=-b\delta_0-a\delta_{-1},$$

after some calculations, we get

$$\begin{aligned}\delta_n&=(-1)^n\left(a\left(\delta_0+\delta_{-1}\right)a^n+\left(-b\delta_0-a\delta_{-1}\right)b^{3n}\right)\\&=(-1)^n\left(\delta_0\left(\frac{a^{n+1}-b^{n+1}}{a-b}\right)+2\delta_{-1}\left(\frac{a^n-b^n}{a-b}\right)\right).\end{aligned}$$

The lemma is proved. □

Lemma 2.3. *Consider the following system of difference equations*

$$\begin{cases} v_{n+1}^{(1)}=3v_n^{(2)}-2v_{n-1}^{(1)} \\ v_{n+1}^{(2)}=3v_n^{(1)}-2v_{n-1}^{(2)} \end{cases},n\geq 0, \tag{2.3}$$

with initial conditions $v_{-1}^{(1)},v_0^{(1)},v_{-1}^{(2)},v_0^{(2)}\in\mathbb{R}$. *Then, we have:*

$$\begin{aligned} v_{2n}^{(1)}&=v_0^{(1)}M_{2n+1}-2v_{-1}^{(2)}M_{2n}, & v_{2n}^{(2)}&=v_0^{(2)}M_{2n+1}-2v_{-1}^{(1)}M_{2n},\\ v_{2n+1}^{(1)}&=v_0^{(2)}M_{2(n+1)}-2v_{-1}^{(1)}M_{2n+1}, & v_{2n+1}^{(2)}&=v_0^{(1)}M_{2(n+1)}-2v_{-1}^{(2)}M_{2n+1}.\end{aligned}$$

Proof. From system (2.3), we obtain the following system

$$\begin{cases} v_{n+1}^{(1)}+v_{n+1}^{(2)}=3\left(v_n^{(1)}+v_n^{(2)}\right)-2\left(v_{n-1}^{(1)}+v_{n-1}^{(2)}\right) \\ v_{n+1}^{(1)}-v_{n+1}^{(2)}=-3\left(v_n^{(1)}-v_n^{(2)}\right)-2\left(v_{n-1}^{(1)}-v_{n-1}^{(2)}\right) \end{cases},n\geq 0, \tag{2.4}$$

Using the change of variables $\gamma_n=v_n^{(1)}+v_n^{(2)}$ and $\delta_n=v_n^{(1)}-v_n^{(2)}$, we can write the system(2.4) as

$$\begin{cases} \gamma_{n+1}=3\gamma_n-2\gamma_{n-1} \\ \delta_{n+1}=-3\delta_n-2\delta_{n-1} \end{cases},n\geq 0,$$

by Lemmas 2.1 – 2.2, we have

$$\begin{aligned}&\forall n\geq 0,\ \gamma_n=\gamma_0 M_{n+1}-2\gamma_{-1}M_n,\\&\forall n\geq 0,\ \delta_n=(-1)^n\left(\delta_0 M_{n+1}+2\delta_{-1}M_n\right),\end{aligned}$$

hence, the closed form general solution of the system (2.3) is
$\left(v_n^{(1)},v_n^{(2)}\right)=\left(\left(\gamma_n+\delta_n\right)/2\,,\left(\gamma_n-\delta_n\right)/2\right),n\geq 0$. The lemma is proved. □

2.1 On the system (2.5)

In this subsection, we consider the following 1st-order system of difference equations,

$$u_{n+1}^{(1)} = \frac{1}{3-2u_n^{(2)}}, u_{n+1}^{(2)} = \frac{1}{3-2u_n^{(1)}}, n \in \mathbb{N}_0. \tag{2.5}$$

To find the closed form of the solutions of the system (2.5) we consider the following change variables

$$u_n^{(1)} = \frac{v_{n-1}^{(2)}}{v_n^{(1)}}, u_n^{(2)} = \frac{v_{n-1}^{(1)}}{v_n^{(2)}},$$

then the system (2.5) becomes

$$\begin{cases} v_{n+1}^{(1)} = 3v_n^{(2)} - 2v_{n-1}^{(1)} \\ v_{n+1}^{(2)} = 3v_n^{(1)} - 2v_{n-1}^{(2)} \end{cases}, n \geq 0.$$

By Lemma 2.3, we can easily obtain the closed-form general solution of the equation (2.5). This is summarized in the following theorem:

Theorem 2.4. *Let* $\left\{u_n^{(1)}, u_n^{(2)}, n \geq 0\right\}$ *be a solution of equation* (2.5)*. Then,*

$$u_{2n}^{(1)} = \frac{M_{2n} - 2u_0^{(1)} M_{2n-1}}{M_{2n+1} - 2u_0^{(1)} M_{2n}},$$
$$u_{2n+1}^{(1)} = \frac{M_{2n+1} - 2u_0^{(2)} M_{2n}}{M_{2(n+1)} - 2u_0^{(2)} M_{2n+1}},$$

$$u_{2n}^{(2)} = \frac{M_{2n} - 2u_0^{(2)} M_{2n-1}}{M_{2n+1} - 2u_0^{(2)} M_{2n}},$$
$$u_{2n+1}^{(2)} = \frac{M_{2n+1} - 2u_0^{(1)} M_{2n}}{M_{2(n+1)} - 2u_0^{(1)} M_{2n+1}},$$

where $(M_n, n \geq 0)$ *is the Mersenne sequence.*

Proof. Straightforward and hence omitted. □

2.2 On the system (1.1)

In this paper, we study the System (1.1), which is an extension of System (2.5). Therefore, the System (1.1) can be written as follows

$$u_{(m+1)(n+1)-t}^{(1)} = \frac{1}{3-2v_{(m+1)n-t}^{(2)}}, v_{(m+1)(n+1)-t}^{(2)} = \frac{1}{3-2u_{(m+1)n-t}^{(1)}},$$

for $t \in \{0, 1, ..., m\}$ and $n \in \mathbb{N}$. Now, using the following notation,

$$u_{n,t}^{(1)} = u_{(m+1)n-t}^{(1)}, u_{n,t}^{(2)} = u_{(m+1)n-t}^{(2)}, t \in \{0, 1, ..., m\},$$

we can get $(m+1)$ −systems similar to System (2.5),

$$u_{n+1,t}^{(1)} = \frac{1}{3-2u_{n,t}^{(2)}}, u_{n+1,t}^{(2)} = \frac{1}{3-2u_{n,t}^{(1)}}, n \in \mathbb{N}_0,$$

for $t \in \{0, 1, ..., m\}$. Through the above discussion, we can introduce the following Theorem

Theorem 2.5. *Let* $\left\{u_n^{(1)}, u_n^{(2)}, n \geq -m\right\}$ *be a solution of equation* (1.1). *Then, for* $t \in \{0, 1, ..., m\}$,

$$u_{2(m+1)n-t}^{(1)} = \frac{M_{2n} - 2u_{-t}^{(1)} M_{2n-1}}{M_{2n+1} - 2u_{-t}^{(1)} M_{2n}},$$
$$u_{(m+1)(2n+1)-t}^{(1)} = \frac{M_{2n+1} - 2u_{-t}^{(2)} M_{2n}}{M_{2(n+1)} - 2u_{-t}^{(2)} M_{2n+1}},$$

$$u_{2(m+1)n-t}^{(2)} = \frac{M_{2n} - 2u_{-t}^{(2)} M_{2n-1}}{M_{2n+1} - 2u_{-t}^{(2)} M_{2n}},$$
$$u_{(m+1)(2n+1)-t}^{(2)} = \frac{M_{2n+1} - 2u_{-t-}^{(1)} M_{2n}}{M_{2(n+1)} - 2u_{-t}^{(1)} M_{2n+1}},$$

where $(M_n, n \geq 0)$ *is the Mersenne sequence.*

Proof. The proof of Theorem 2.5 is based on Theorem 2.4 for $(m+1)$ −systems (1.1). □

Corollary 2.6. *Let* $\left\{u_n^{(1)}, u_n^{(2)}, n \geq -m\right\}$ *be a solution of equation* (1.1). *Then, for* $t \in \{0, 1, ..., m\}$,

$$u_{2(m+1)n-t}^{(1)} = \frac{m_{2n+1} - 3 - 4u_{-t}^{(1)} (m_{2n-1} - 2)}{2m_{2n+1} - 4 - 2u_{-t}^{(1)} (m_{2n+1} - 3)},$$
$$u_{(m+1)(2n+1)-t}^{(1)} = \frac{2m_{2n+1} - 4 - 2u_{-t}^{(2)} (m_{2n+1} - 3)}{m_{2n+3} - 3 - 4u_{-t}^{(2)} (m_{2n+1} - 2)},$$

$$u_{2(m+1)n-t}^{(2)} = \frac{m_{2n+1} - 3 - 4u_{-t}^{(2)} (m_{2n-1} - 2)}{2m_{2n+1} - 4 - 2u_{-t}^{(2)} (m_{2n+1} - 3)},$$
$$u_{(m+1)(2n+1)-t}^{(2)} = \frac{2m_{2n+1} - 4 - 2u_{-t-}^{(1)} (m_{2n+1} - 3)}{m_{2n+3} - 3 - 4u_{-t}^{(1)} (m_{2n+1} - 2)},$$

where $(m_n, n \geq 0)$ *is the Mersenne-Lucas sequence.*

Proof. We see that it suffices to remark

$$2M_{2n} = m_{2n+1} - 3 \text{ and } M_{2n+1} = m_{2n+1} - 2, \text{ (see., [37]).}$$

□

Corollary 2.7. *Let* $\left\{u_n^{(1)}, u_n^{(2)}, n \geq -m\right\}$ *be a solution of equation* (1.1). *Then, for* $t \in \{0, 1, ..., m\}$,

$$u_{2(m+1)n-t}^{(1)} = \frac{M_n m_n - 2u_{-t}^{(1)} \left(M_n m_{n-1} - 2^{n-1}\right)}{M_{n+1} m_n - 2^n - 2u_{-t}^{(1)} M_n m_n},$$
$$u_{(m+1)(2n+1)-t}^{(1)} = \frac{M_{n+1} m_n - 2^n - 2u_{-t}^{(2)} M_n m_n}{M_{n+1} m_{n+1} - 2u_{-t}^{(2)} (M_{n+1} m_n - 2^n)},$$

$$u_{2(m+1)n-t}^{(2)} = \frac{M_n m_n - 2u_{-t}^{(2)} \left(M_n m_{n-1} - 2^{n-1}\right)}{M_{n+1} m_n - 2^n - 2u_{-t}^{(2)} M_n m_n},$$
$$u_{(m+1)(2n+1)-t}^{(2)} = \frac{M_{n+1} m_n - 2^n - 2u_{-t-}^{(1)} M_n m_n}{M_{n+1} m_{n+1} - 2u_{-t}^{(1)} (M_{n+1} m_n - 2^n)},$$

where $(M_n, n \geq 0)$ *is the Mersenne sequence and* $(m_n, n \geq 0)$ *is the Mersenne-Lucas sequence.*

Proof. We see that it suffices to remark

$$M_{2n} = M_n m_n \text{ and } M_{2n+1} = M_{n+1} m_n - 2^n \text{ (see., [13]).}$$

□

Remark 2.8. There are many systems whose solutions can be expressed by Mersenne and Mersenne-Lucas numbers, which are

$$u_{n+1}^{(1)} = \frac{1}{a_k - b_k u_{n-m}^{(2)}}, u_{n+1}^{(2)} = \frac{1}{a_k - b_k u_{n-m}^{(1)}}, n, m \in \mathbb{N}_0, k \geq 1,$$

where $(a_k, b_k) = \left(a^k + b^k, (ab)^k\right) \in \{(3,2);(5,4);(9,8);(17,16);...\}, k \geq 1$. Using the results of Theorem 2.5, we get

$$u_{2(m+1)n-t}^{(1)} = \frac{M_{2kn} - 2u_{-t}^{(1)} M_{k(2n-1)}}{M_{k(2n+1)} - 2u_{-t}^{(1)} M_{2kn}},$$
$$u_{(m+1)(2n+1)-t}^{(1)} = \frac{M_{k(2n+1)} - 2u_{-t}^{(2)} M_{2kn}}{M_{2k(n+1)} - 2u_{-t}^{(2)} M_{k(2n+1)}},$$

$$u_{2(m+1)n-t}^{(2)} = \frac{M_{2kn} - 2u_{-t}^{(2)} M_{k(2n-1)}}{M_{k(2n+1)} - 2u_{-t}^{(2)} M_{2kn}},$$
$$u_{(m+1)(2n+1)-t}^{(2)} = \frac{M_{k(2n+1)} - 2u_{-t-}^{(1)} M_{2kn}}{M_{2k(n+1)} - 2u_{-t}^{(1)} M_{k(2n+1)}}, k \geq 1.$$

3 Global stability of positive solutions of (1.1)

In the following, we will study the global stability character of the solutions of system (1.1). Obviously, the positive equilibriums of system (1.1) are

$$E_1 = \left(\overline{u}_1^{(1)}, \overline{u}_2^{(1)}\right) = (1,1), \; E_2 = \left(\overline{u}_1^{(2)}, \overline{u}_2^{(2)}\right) = \frac{1}{2}(1,1).$$

Let the functions $h_1, h_2 : (0, +\infty)^{2(m+1)} \to (0, +\infty)$ defined by

$$h_1\left(\left(\underline{u}_{0:m}^{(1)}\right)', \left(\underline{u}_{0:m}^{(2)}\right)'\right) = \frac{1}{3 - 2u_{n-m}^{(2)}}, h_2\left(\left(\underline{u}_{0:m}^{(1)}\right)', \left(\underline{u}_{0:m}^{(2)}\right)'\right) = \frac{1}{3 - 2u_{n-m}^{(1)}},$$

where $\underline{x}_{0:m} = (x_0, x_1, ..., x_m)'$. Now, it is usually useful to linearized system (1.1) around the equilibrium point E_2 in order to facilitate its study. For this purpose, introducing the vectors $\underline{X}_n' := \left(\left(\underline{U}_n^{(1)}\right)', \left(\underline{U}_n^{(2)}\right)'\right)$ where $\underline{U}_n^{(1)} = \left(u_n^{(1)}, u_{n-1}^{(1)}, ..., u_{n-m}^{(1)}\right)'$ and $\underline{U}_n^{(2)} = \left(u_n^{(2)}, u_{n-1}^{(2)}, ..., u_{n-m}^{(2)}\right)'$. With these notations, we obtain the following representation

$$\underline{X}_{n+1} = F_m \underline{X}_n, \tag{3.1}$$

where

$$F_m = \begin{pmatrix} \underline{O}'_{(m-1)} & 0 & \underline{O}'_{(m-1)} & \frac{1}{2} \\ I_{(m-1)} & \underline{O}_{(m-1)} & O_{(m-1)} & \underline{O}_{(m-1)} \\ \underline{O}'_{(m-1)} & \frac{1}{2} & O_{(m-1)} & 0 \\ O_{(m-1)} & \underline{O}_{(m-1)} & I_{(m-1)} & \underline{O}_{(m-1)} \end{pmatrix},$$

with $O_{(k,l)}$ denotes the matrix of order $k \times l$ whose entries are zeros, for simplicity, we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$ and $I_{(m)}$ is the $m \times m$ identity matrix. We summarize the above discussion in the following theorem

Theorem 3.1. *The positive equilibrium point E_2 is locally asymptotically stable.*

Proof. After some preliminary calculations, the characteristic polynomial of F_m is given by:

$$P_{F_m}(\lambda) = \det\left(F_m - \lambda I_{(2(m+1))}\right) = \Lambda_1(\lambda) - \Lambda_2(\lambda),$$

where $\Lambda_1(\lambda) = \lambda^{2(m+1)}$ and $\Lambda_2(\lambda) = \frac{1}{4}$, then $|\Lambda_2(\lambda)| < |\Lambda_1(\lambda)|, \forall\lambda : |\lambda| = 1$. By Rouche's Theorem, all zeros of $\Lambda_1(\lambda) - \Lambda_2(\lambda) = 0$ lie in the unit disc $|\lambda| < 1$. Thus, the positive equilibrium point E_2 is locally asymptotically stable. □

Corollary 3.2. *For every well defined solution of system* (1.1)*, we have* $\lim u_n^{(1)} = \lim u_n^{(2)} = \frac{1}{2}$.

Proof. From Theorem 2.5, we have

$$\begin{aligned}\lim u^{(1)}_{2(m+1)n-t} &= \lim \frac{M_{2n} - 2u^{(1)}_{-t}M_{2n-1}}{M_{2n+1} - 2u^{(1)}_{-t}M_{2n}} \\ &= \lim \frac{1 - 2u^{(1)}_{-t}\frac{M_{2n-1}}{M_{2n}}}{\frac{M_{2n+1}}{M_{2n}} - 2u^{(1)}_{-t}} \\ &= \frac{1 - u^{(1)}_{-t}}{2 - 2u^{(1)}_{-t}} = \frac{1}{2},\end{aligned}$$

$$\begin{aligned}\lim u^{(1)}_{(m+1)(2n+1)-t} &= \lim \frac{M_{2n+1} - 2u^{(2)}_{-t}M_{2n}}{M_{2(n+1)} - 2u^{(2)}_{-t}M_{2n+1}} \\ &= \lim \frac{1 - 2u^{(2)}_{-t}\frac{M_{2n}}{M_{2n+1}}}{\frac{M_{2(n+1)}}{M_{2n+1}} - 2u^{(2)}_{-t}} \\ &= \frac{1 - u^{(2)}_{-t}}{2 - 2u^{(2)}_{-t}} = \frac{1}{2}.\end{aligned}$$

The rest of the proof, showing that $\lim u_n^{(1)}$ is similar to the proof for $\lim u_n^{(2)}$, completes the proof of Corollary 3.2. □

The following result is an immediate consequence of Theorem 3.1 and Corollary 3.2.

Corollary 3.3. *The unique positive equilibrium point* E_2 *is globally asymptotically stable.*

4 Numerical Examples

In order to clarify and shore theoretical results from the previous section, we present some interesting numerical examples in this section.

Example 4.1. We consider an interesting numerical example for the system of difference equations (1.1) when $m = 1$ with the initial conditions $u^{(1)}_{-1} = 1.3$, $u^{(1)}_0 = 2$, $u^{(2)}_{-1} = 0.2$ and $u^{(2)}_0 = 1.1$. The plot of the solutions is shown in Figure 1.

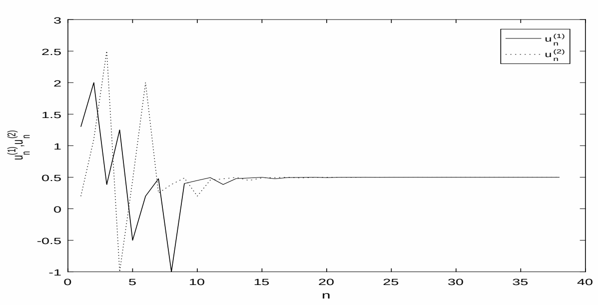

Figure1.The plot of the solutions of system (1.1), when $m = 1$ and we put the initial

conditions $u^{(1)}_{-1} = 1.3, u^{(1)}_0 = 2, u^{(2)}_{-1} = 0.1$ and $u^{(2)}_0 = 1.1$.

Example 4.2. We consider an interesting numerical example for the system of difference equations (1.1) when $m = 2$ with the initial conditions

i	0	1	2
$u_{-i}^{(1)}$	-3	-4	-1
$u_{-i}^{(2)}$	-1	-3	0

Table 1. The initial conditions.

The plot of the solutions is shown in Figure 2.

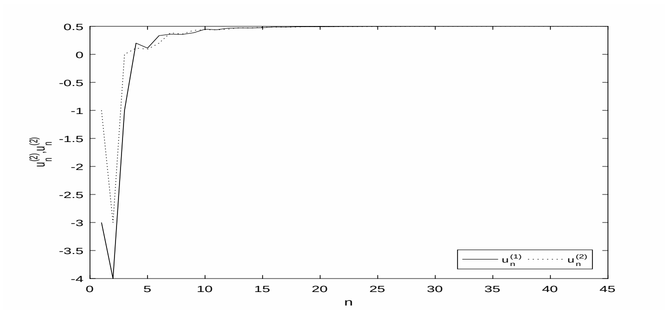

Figure 2. The plot of the solutions of system (1.1); when we put the initial conditions in Table 1.

Example 4.3. We consider an interesting numerical example for the system of difference equations (1.1) when $m = 3$ with the initial conditions

i	0	1	2	3
$u_{-i}^{(1)}$	0.3	0.1	0.2	0.2
$u_{-i}^{(2)}$	0.4	0.3	0.1	0.3

Table 2. The initial conditions.

The plot of the solutions is shown in Figure 3.

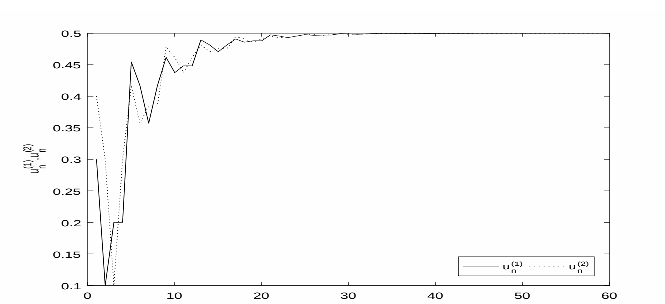

Figure 3. The plot of the solutions of system (1.1); when we put the initial conditions in Table 2.

In these examples, we show that the solutions of the system (1.1) for some cases are globally asymptotically stable.

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