Palestine Journal of Mathematics

Vol 13(2)(2024) , 8–17

A Bochner theorem on a noncommutative hypergroup

Kouakou Germain BROU, Ibrahima TOURE and Kinvi KANGNI

Communicated by Taher Abualrub

MSC 2010 Classifications: 43A62, 43A22, 20N20.

Keywords and phrases: Hypergroups, Gelfand pair, probability measure, positive definite measure, strongly positive definite function.

Abstract Let G be a locally compact hypergroup and let K be a compact subhypergroup of G. (G, K) is a Gelfand pair if $M_c(G//K)$, the algebra of measures with compact support on the double coset $G//K$, is commutative for the convolution. In this paper, assuming that (G, K) is a Gelfand pair, we define and study the inverse Fourier transform on G and then establish a Bochner theorem for the pair (G, K).

1 Introduction

Hypergroups generalize locally compact groups. They appear when the Banach space of all bounded Radon measures on a locally compact space carries a convolution having all properties of a group convolution apart from the fact that the convolution of two point measures is a probability measure with compact support and not necessarily a point measure. The intention was to unify harmonic analysis on duals of compact groups, double cosets spaces $G//H$ (H a compact subgroup of a locally compact group G), and commutative convolution algebras associated with product linearization formulas of special functions. The notion of hypergroup has been sufficiently studied (see for example [2, 5, 8, 9]). Harmonic analysis and probability theory on commutative hypergroups are well developed meanwhile where many results from group theory remain valid (see [1]). When G is a commutative hypergroup, the convolution algebra $M_c(G)$ consisting of measures with compact support on G is commutative. The typical example of a commutative hypergroup is the double coset $G//K$ when G is a locally compact group, K is a compact subgroup of G such that (G, K) is a Gelfand pair. In [5], R. I. Jewett has shown the existence of a positive measure called Plancherel measure on the dual space $\widehat{G}$ of a commutative hypergroup G. In [6], R. Lasser relying on this result, has established a Bochner theorem on a commutative hypergroup G. When the hypergroup G is not commutative, it is possible to involve a compact subhypergroup K of G leading to a commutative subalgebra of $M_c(G)$. In fact, if K is a compact subhypergroup of a hypergroup G, the pair (G, K) is said to be a Gelfand pair if $M_c(G//K)$ the convolution algebra of measures with compact support on $G//K$ is commutative. The notion of Gelfand pairs for hypergroups is well-known (see [3, 10, 11]). When (G, K) is a Gelfand pair; it has been shown in [4] the existence of a Plancherel measure on $\widehat{G}$. The goal of this paper is to extend Lasser's work by obtaining a Bochner theorem over Gelfand pair associated with a noncommutative hypergroup. In the next section, we give notations and setup useful for the remainder of this paper. In section 3, we define the inverse Fourier transform on $M_b(\widehat{G})$, the algebra of bounded measures on $\widehat{G}$ and obtain a relationship between measures in $M_b(G)$ and measures in $M_b(\widehat{G})$. Then, we define the so-called positive (rep. strongly positive) definite measure (resp. function) on $\widehat{G}$ and obtain some of their characterizations. Finally, thanks to these results, we prove that for any strongly positive definite measure, there exists a positive measure in $M_b(G)$ whose Fourier transform coincides with the function on the support of the Plancherel measure.

2 Notations and preliminaries

We use the notations and setup of this section in the rest of the paper without mentioning. Let G be a locally compact space. We denote by:
- $C(G)$ (resp. $M(G)$) the space of continuous complex- valued functions (resp. the space of Radon measures) on G,
- $C_b(G)$ (resp. $M_b(G)$) the space of bounded continuous functions (resp. the space of bounded Radon measures) on G,
- $\mathcal{K}(G)$ (resp. $M_c(G)$) the space of continuous functions(resp. the space of Radon measures) with compact support on G,
- $C_0(G)$ the space of elements in $C(G)$ which are zero at infinity,
- $\mathfrak{C}(G)$ the space of compact subsets of G,
- δ_x the point measure at $x \in G$,
- $spt(f)$ the support of the function f.
- $spt(\mu)$, the support of the measure μ.
Let us notice that the topology on $M(G)$ is the cône topology [5] and the topology on $\mathfrak{C}(G)$ is the topology of Michael [7].

Definition 2.1. [5]. Let G be a locallly compact topological space. G is said to be a *hypergroup* if the following assumptions are satisfied.

(H1) There is a binary operator $*$ named convolution on $M_b(G)$ under which $M_b(G)$ is an associative algebra such that:
i) the mapping $(\mu, \nu) \longmapsto \mu * \nu$ is continuous from $M_b(G) \times M_b(G)$ in $M_b(G)$.
ii) $\forall x, y \in G$, $\delta_x * \delta_y$ is a measure of probability with compact support.
iii) the mapping: $(x, y) \longmapsto$ supp$(\delta_x * \delta_y)$ is continuous from $G \times G$ in $\mathfrak{C}(G)$.

(H2) There is a unique element e (called neutral element) in G such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x, \forall x \in G$.

(H3) There is an involutive homeomorphism: $x \longmapsto \overline{x}$ from G into G, named involution, such that:
i) $(\delta_x * \delta_y)^- = \delta_{\overline{y}} * \delta_{\overline{x}}, \forall x, y \in G$ with $\mu^-(f) = \mu(f^-)$ where $f^-(x) = f(\overline{x}), \forall f \in C(G)$ and $\mu \in M(G)$.
ii) $\forall x, y, z \in G$, $z \in$ supp$(\delta_x * \delta_y)$ if and only if $x \in$ supp$(\delta_z * \delta_{\overline{y}})$.

The hypergroup G is compact if G is a compact topological space.
The hypergroup G is commutative if $\delta_x * \delta_y = \delta_y * \delta_x, \forall x, y \in G$. For $x, y \in G$, $x * y$ is the support of $\delta_x * \delta_y$ and for $f \in C(G)$,

$$f(x * y) = (\delta_x * \delta_y)(f) = \int_G f(z) d(\delta_x * \delta_y)(z).$$

The convolution of two measures μ, ν in $M_b(G)$ is defined by: $\forall f \in C(G)$,

$$(\mu * \nu)(f) = \int_G \int_G (\delta_x * \delta_y)(f) d\mu(x) d\nu(y) = \int_G \int_G f(x * y) d\mu(x) d\nu(y).$$

For μ in $M_b(G)$, $\mu^* = (\overline{\mu})^-$. So $M_b(G)$ is a *-Banach algebra.

Definition 2.2. [5]. Let G be a hypergroup and H subset of G. H is a subhypergroup of G if the following conditions are satisfied.

(i) H is non empty and closed in G,

(ii) $\forall x \in H, \overline{x} \in H$,

(iii) $\forall x, y \in H$, supp$(\delta_x * \delta_y)$ is a subset of H.

Let us now consider a hypergroup G provided with a left Haar measure μ_G and K a compact subhypergroup of G with a normalized Haar measure ω_K. Let us put $M_{\mu_G}(G)$ the space of measures in $M_b(G)$ which are absolutely continuous with respect to μ_G. $M_{\mu_G}(G)$ is a closed self-adjoint ideal in $M_b(G)$. For $x \in G$, the double coset of x with respect to K is $K * \{x\} * K = \{k_1 * x * k_2; k_1, k_2 \in K\}$. We write simply KxK for a double coset and recall that $KxK = \bigcup_{k_1,k_2 \in K} \text{supp}(\delta_{k_1} * \delta_x * \delta_{k_2})$. All double cosets form a partition of G and the quotient topology with respect to the corresponding equivalence relation equips the double cosets space $G//K$ with a locally compact topology ([1], page 53). The natural mapping $p_K : G \longrightarrow G//K$ defined by: $p_K(x) = KxK\ , x \in G$ is an open surjective continuous mapping. A function $f \in C(G)$ is said to be invariant by K or $K-invariant$ if $f(k_1 * x * k_2) = f(x)$ for all $x \in G$ and for all $k_1, k_2 \in K$. We denote by $C^\natural(G)$, (resp. $\mathcal{K}^\natural(G)$) the space of continuous functions (resp. continuous functions with compact support) which are $K-$invariant. For $f \in C^\natural(G)$, one defines the function $\widetilde{f}$ on $G//K$ by $\widetilde{f}(KxK) = f(x)\ \forall x \in G$. $\widetilde{f}$ is well defined and it is continuous on $G//K$. Conversely, for all continuous function φ on $G//K$, the function $f = \varphi \circ p_K \in C^\natural(G)$. One has the obvious consequence that the mapping $f \longmapsto \widetilde{f}$ sets up a topological isomorphism between the topological vector spaces $C^\natural(G)$ and $C(G//K)$ (see [10, 11]). So, for any f in $C^\natural(G)$, $f = \widetilde{f} \circ p_K$. Otherwise, we consider the K- projection $f \longmapsto f^\natural$ (by identifying $f^\natural$ and $\widetilde{f^\natural}$) from $C(G)$ into $C(G//K)$ where for $x \in G, f^\natural(x) = \int_K \int_K f(k_1 * x * k_2) d\omega_K(k_1) d\omega_K(k_2)$. If $f \in \mathcal{K}(G)$, then $f^\natural \in \mathcal{K}(G//K)$. For a measure $\mu \in M(G)$, one defines $\mu^\natural$ by $\mu^\natural(f) = \mu(f^\natural)$ for $f \in \mathcal{K}(G)$. μ is said to be $K-$invariant if $\mu^\natural = \mu$ and we denote by $M^\natural(G)$ the set of all those measures. Considering these properties, one defines a hypergroup operation on $G//K$ by: $\delta_{KxK} * \delta_{KyK}(\widetilde{f}) = \int_K f(x * k * y) d\omega_K(k)$ (see [2, p. 12]). This defines uniquely the convolution $(KxK) * (KyK)$ on $G//K$. The involution is defined by: $\overline{KxK} = K\overline{x}K$ and the neutral element is K. Let us put $m = \int_G \delta_{KxK} d\mu_G(x)$, m is a left Haar measure on $G//K$. We say that (G, K) is a Gelfand pair if the convolution algebra $M_c(G//K)$ is commutative. $M_c(G//K)$ is topologically isomorphic to $M_c^\natural(G)$. Considering the convolution product on $\mathcal{K}(G)$, $\mathcal{K}(G)$ is a convolution algebra and $\mathcal{K}^\natural(G)$ is a subalgebra. Thus (G, K) is a Gelfand pair if and only if $\mathcal{K}^\natural(G)$ is commutative ([3], theorem 3.2.2).

3 Bochner theorem

In this section, we give a Bochner theorem for a noncommutative hypergroup extending Lasser's works on commutative hypergroups. For that we give some characterizations of the inverse Fourier transform as well as positive definite measures on the dual space of the hypergroup.

Let G be a hypergroup provided with a left Haar measure μ_G and K a compact subhypergroup of G such that (G, K) is a Gelfand pair.
Let $\widehat{G}$ be the space of continuous and bounded function ϕ on G satisfying the following conditions:
(i) ϕ is K- invariant,
(ii) $\phi(e) = 1$,
(iii) $\int_K \phi(x * k * y) dw_K(k) = \phi(x)\phi(y)\ \forall x, y \in G$,
(iv) $\phi(\overline{x}) = \overline{\phi(x)}\ \forall x \in G$.
Equipped with the topology of uniform convergence on compact subsets of G, $\widehat{G}$ is a locally compact space (see [4]). $\widehat{G}$ is the *dual space* of the hypergroup G.
We can notice that the function $\mathbf{1} : x \longmapsto 1$ belongs to $\widehat{G}$.

3.1 Inverse Fourier transform

For β belongs to $M_b(G)$, the Fourier transform of β is the mapping

$$\widehat{\beta} : \widehat{G} \longrightarrow \mathbb{C} \text{ defined by : } \widehat{\beta}(\phi) = \int_G \phi(\overline{x}) d\beta(x).$$

The Fourier transform of $f \in \mathcal{K}(G)$ is defined by

$$\widehat{f}(\phi) = \widehat{f\mu_G}(\phi) = \int_G \phi(\overline{x}) f(x) d\mu_G(x)$$

For more details on the Fourier transform, see [4].

Definition 3.1. [1]. Let $\beta \in M_b(\widehat{G})$, we call inverse Fourier transform of β, the mapping

$$\overset{\vee}{\beta} : G \longrightarrow \mathbb{C} \text{ defined by} : \overset{\vee}{\beta}(x) = \int_{\widehat{G}} \phi(x) d\beta(\phi).$$

The inverse Fourier transform of $\varphi \in L^1(\widehat{G}, \pi)$ is defined by

$$\overset{\vee}{\varphi}(x) = (\varphi\pi)^\vee(x) = \int_{\widehat{G}} \phi(x)\varphi(\phi) d\pi(\phi),$$

where π is the Plancherel measure (see [4]) on $\widehat{G}$.

In the following results, we give some proporties of the inverse Fourier transform.

Theorem 3.2. *Let $\beta \in M_b(\widehat{G})$. Then $\overset{\vee}{\beta}$ belongs to $C_b^\natural(G)$ and $(\overset{\vee}{\beta^*}) = \overline{\overset{\vee}{\beta}}$.*

Proof. For $x \in G$, we have

$$\begin{aligned}
\overset{\vee}{\beta}(x) &= \textstyle\int_{\widehat{G}} \phi(x) d\beta(\phi) \\
&= \textstyle\int_{\widehat{G}} (\int_G \phi(y) d\delta_x(y)) d\beta(\phi) \\
&= \textstyle\int_{\widehat{G}} (\int_G \phi(\overline{y}) d\delta_{\overline{x}}(y)) d\beta(\phi) \\
&= \textstyle\int_{\widehat{G}} \widehat{\delta_{\overline{x}}}(\phi)) d\beta(\phi) \\
&= \beta(\widehat{\delta_{\overline{x}}})
\end{aligned}$$

The mapping $x \longmapsto \delta_x$ of G on $M_b(G)$ is continuous (see [5] lemma 2.2B). Since the Fourier transform and β are continuous, then $\overset{\vee}{\beta}$ is continuous. β is also bounded, then $\overset{\vee}{\beta}$ is bounded. Let $k_1, k_2 \in K$ and $x \in G$, we have

$$\begin{aligned}
\overset{\vee}{\beta}(k_1 * x * k_2) &= \textstyle\int_{\widehat{G}} \phi(k_1 * x * k_2) d\beta(\phi) \\
&= \textstyle\int_{\widehat{G}} \phi(x) d\beta(\phi) \\
&= \overset{\vee}{\beta}(x).
\end{aligned}$$

Thus, $\overset{\vee}{\beta}$ is $K-$invariant.
Moreover for $x \in G$, we have

$$\begin{aligned}
(\overset{\vee}{\beta^*})(x) &= \beta^*(\widehat{\delta_{\overline{x}}}) \\
&= \overline{\textstyle\int_{\widehat{G}} \overline{\widehat{\delta_{\overline{x}}}^{\,-}}(\phi) d\beta(\phi)} \\
&= \overline{\textstyle\int_{\widehat{G}} \overline{\int_G \phi^-(\overline{y}) d\delta_{\overline{x}}(y)} d\beta(\phi)} \\
&= \overline{\textstyle\int_{\widehat{G}} \phi(x) d\beta(\phi)} \\
&= \overline{\overset{\vee}{\beta}}(x).
\end{aligned}$$

□

Remark 3.3. The map:

$$\begin{array}{ccc} M_b(\widehat{G}) & \longrightarrow & C_b^{\natural}(G) \\ \beta & \longmapsto & \overset{\vee}{\beta} \end{array}$$

is a continuous linear functional.

Theorem 3.4. *If* $h = f * g$ *with* $f, g \in \mathcal{K}^{\natural}(G)$*, then* $\int_G h(x)d\mu^-(x) = \int_{\widehat{G}} \widehat{h}(\phi)\widehat{\mu}(\phi)d\pi(\phi)$ $\forall \mu \in M_b^{\natural}(G)$.

Proof. Let us put $\widetilde{\pi}$ the Plancherel measure of the commutative double coset hypergroup $G//K$. We know (see [4], Proof. th 3.11) that $\pi(\varphi) = \widetilde{\pi}(\widetilde{\varphi})$ where $\widetilde{\varphi}(\widetilde{\phi}) = \varphi(\phi)$ for $\widetilde{\varphi} \in \mathcal{K}(\widehat{G//K})$, $\varphi \in \mathcal{K}(\widehat{G})$, $\widetilde{\phi} \in \widehat{G//K}$ and $\phi \in \widehat{G}$.

$$\begin{aligned}
\int_G h(x)du(\overline{x}) &= \int_G f * g(x)du(\overline{x}) \\
&= \int_G (\mu * f)(x)g(\overline{x})du_G(x) \\
&= \int_G (\mu * f)(x)\overline{g^*}(x)du_G(x) \\
&= \int_{G//K} (\widetilde{\mu * f})(KxK)\overline{\widetilde{g^*}}(KxK)dm(KxK) \\
&= \int_{G//K} (\widetilde{\mu * f})(KxK)\overline{\widetilde{g}^*}(KxK)dm(KxK) \\
&= \int_{\widehat{G//K}} (\widehat{\widetilde{\mu * f}})(\widetilde{\phi})\widehat{\widetilde{g}}(\widetilde{\phi})d\widetilde{\pi}(\widetilde{\phi}), \text{ (see[5] 12.1.}C) \\
&= \int_{\widehat{G//K}} (\widetilde{\widehat{\mu * f}})(\widetilde{\phi})\widetilde{\widehat{g}}(\widetilde{\phi})d\widetilde{\pi}(\widetilde{\phi}) \\
&= \int_{\widehat{G}} (\widehat{\mu * f})(\phi)\widehat{g}(\phi)d\pi(\phi) \\
&= \int_{\widehat{G}} \widehat{\mu}(\phi)\widehat{f}(\phi)\widehat{g}(\phi)d\pi(\phi) = \int_{\widehat{G}} \widehat{\mu}(\phi)\widehat{h}(\phi)d\pi(\phi).
\end{aligned}$$

□

Theorem 3.5. *(i)* $\left\{\widehat{f};\ f \in \mathcal{K}(G)\right\}$ *is a sup-norm dense subspace of* $C_0(\widehat{G})$.
(ii) Let $\beta \in M_b(\widehat{G})$*. If* $\overset{\vee}{\beta} = 0$*, then* $\beta = 0$.
(iii) Let $\beta \in M_b(\widehat{G})$ *and* $\mu \in M_b^{\natural}(G)$*. Then* $\mu = \overset{\vee}{\beta}\mu_G$ *if and only if* $\beta = \widehat{\mu}\pi$.

Proof. (i) We know that $\left\{\widehat{f};\ f \in \mathcal{K}(G)\right\} \subset C_0(\widehat{G})$. Let suppose that $\overline{\left\{\widehat{f};\ f \in \mathcal{K}(G)\right\}} \neq C_0(\widehat{G})$. By the Hahn-Banach theorem, there exists ψ in $C_0(\widehat{G})'$ such that $\psi \neq 0$ and $\psi(\widehat{f}) = 0\ \forall f \in \mathcal{K}(G)$. Using the Riesz's representation theorem, we can take μ in $M_b(\widehat{G})$ such that $\mu \neq 0$ and $\psi(\varphi) = \int_{\widehat{G}} \varphi(\phi)d\mu(\phi)\ \forall \varphi \in C_0(\widehat{G})$.
Let us consider $f \in \mathcal{K}(G)$ with $sptf \subset K$ such that $\int_G f(x)d\mu_G(x) = 1$.
We have

$$\begin{aligned}
0 &= \int_{\widehat{G}} \widehat{f}(\phi)d\mu(\phi) \\
&= \int_{\widehat{G}} \int_G \phi(\overline{x})f(x)d\mu_G(x)d\mu(\phi) \\
&= \int_{\widehat{G}} \int_K f(x)d\mu_G(x)d\mu(\phi) \text{ since } \phi(x) = 1\ \forall x \in K. \\
&= \mu(\widehat{G}).
\end{aligned}$$

Thus, $\mu = 0$ and we have a contradiction. So $\overline{\left\{\widehat{f};\ f \in \mathcal{K}(G)\right\}} = C_0(\widehat{G})$.
In addition, since $\left\{\widehat{f};\ f \in \mathcal{K}(G)\right\} \subset \{\widehat{\mu};\ \mu \in M_{\mu_G}(G)\} \subset C_0(\widehat{G})$, then we can deduce that $\{\widehat{\mu}; \mu \in M_{\mu_G}(G)\}$ is a sup-norm dense subspace of $C_0(\widehat{G})$.
(ii) Let us suppose that $\beta \neq 0$ and take $\mu \in M_{\mu_G}(G)$ such that $\beta(\widehat{\mu}) \neq 0$.
We have

$$\begin{aligned} 0 \neq \int_{\widehat{G}} \widehat{\mu}(\phi) d\beta(\phi) &= \int_{\widehat{G}} \int_G \phi(\overline{x}) d\mu(x) d\beta(\phi) \\ &= \int_G \int_{\widehat{G}} \phi(x) d\beta(\phi) d\mu^-(x) \\ &= \int_G \overset{\vee}{\beta}(x) d\mu^-(x). \end{aligned}$$

Thus, $\overset{\vee}{\beta} \neq 0$ and we have a contradiction.
(iii) Using (i) and ([1] th.1.4.30), it is sufficient to consider the functions of the form $h = f * g$, with $f, g \in \mathcal{K}^\natural(G)$ to ensure our proof.
Let us note that $\int_G h(x) d(\overset{\vee}{\beta}\mu_G)^-(x) = \int_{\widehat{G}} \widehat{h}(\phi) d\beta(\phi)$. Indeed we have

$$\begin{aligned} \int_G h(x) d(\overset{\vee}{\beta}\mu_G)^-(x) &= \int_G h(\overline{x}) \int_{\widehat{G}} \phi(x) d\beta(\phi) d\mu_G(x) \\ &= \int_{\widehat{G}} \int_G h(x) \phi(\overline{x}) d\mu_G(x) d\beta(\phi) \\ &= \int_{\widehat{G}} \widehat{h}(\phi) d\beta(\phi). \end{aligned}$$

Since $\int_G h(x) d\mu^-(x) = \int_{\widehat{G}} \widehat{h}(\phi) d(\widehat{\mu}\pi)(\phi)$, then we have the result. □

Corollary 3.6. *If $f \in \mathcal{K}^\natural(G)$, then $\widehat{f}^\vee = f$.*

Proof. Let $f \in \mathcal{K}^\natural(G)$, and let us put $\mu = f\mu_G$, $\mu \in M_b^\natural(G)$. We have $\widehat{\mu} = \widehat{f}$ and so $\widehat{\mu}\pi = \widehat{f}\pi$. Using Theorem 3.5, we have $\mu = (\widehat{f}\pi)^\vee \mu_G = \widehat{f}^\vee \mu_G$. □

Since $\widehat{f} = \widehat{f^\natural}$ for $f \in \mathcal{K}(G)$, then $\widehat{f}^\vee = f^\natural\ \forall f \in \mathcal{K}(G)$.

Theorem 3.7. *$(\mathcal{K}(\widehat{G}))^\vee$ is a sup-norm dense subspace of $C_0(G)$.*

Proof. Let us note that $(\mathcal{K}(\widehat{G}))^\vee$ is a subspace of $C_0(G)$. This comes from the continuity of the inverse Fourier transform, Theorem 3.5 and corollary 3.6.
Moreover, assume that $(\mathcal{K}(\widehat{G}))^\vee$ is not sup-norm dense in $C_0(G)$. From the Hanh-Banach theorem and Riesz's representation theorem, we can take $\mu \in M_b(G)$, $\mu \neq 0$ such that $\int_G \overset{\vee}{\varphi}(x) d\mu(x) = 0\ \forall \varphi \in \mathcal{K}(\widehat{G})$.
Thus,

$$\begin{aligned} 0 &= \int_{\widehat{G}} \int_G \phi(x) d\mu(x) \varphi(\phi) d\pi(\phi) \\ &= \int_{\widehat{G}} \widehat{\mu^-}(\phi) \varphi(\phi) d\pi(\phi) \\ &= \widehat{\mu^-}\pi(\varphi)\ \forall \varphi \in \mathcal{K}(\widehat{G}). \end{aligned}$$

This implies that $\widehat{\mu^-}\pi = 0$. By Theorem 3.5 we have $\mu = 0$ and we have a contradiction. □

3.2 Positive definite measures and strongly positive definite functions

Definition 3.8. [6]. A measure $\mu \in M_b(\widehat{G})$ is called positive definite if $\overset{\vee}{\mu} \geq 0$.
Let $\varphi \in C_b(\widehat{G})$. Then φ is called strongly positive definite if for any positive definite measure μ, $\mu(\varphi) = \int_{\widehat{G}} \varphi(\phi) d\mu(\phi) \geq 0$.

The space of positive definite measure is denoted by $PM(\widehat{G})$ and that of strongly positive definite functions is denoted by $SP(\widehat{G})$.

Remark 3.9. If $\varphi \in SP(\widehat{G})$, then $\varphi(\mathbf{1}) \geq 0$.
In fact, $\delta_{\mathbf{1}} \in M_b(\widehat{G})$ and $\overset{\vee}{\delta_{\mathbf{1}}}(x) = \int_{\widehat{G}} \phi(x) d\delta_{\mathbf{1}}(\phi) = \mathbf{1}(x) = 1 \geq 0$. Thus,
$\varphi(\mathbf{1}) = \int_{\widehat{G}} \varphi(\phi) d\delta_{\mathbf{1}}(\phi) \geq 0$.

In the following proposition, we bring out some properties of the strongly positive definite functions and the positive definite measures.

Proposition 3.10. *Let $\varphi \in SP(\widehat{G})$. Then the following statements hold.*
(i) If $\mu \in PM(\widehat{G})$, then $(\varphi\mu)^{\vee} \geq 0$.
(ii) If $\mu \in M_b(\widehat{G})$ such that $\overset{\vee}{\mu}$ is real values, then $\mu(\varphi) = \int_{\widehat{G}} \varphi(\phi) d\mu(\phi) \in \mathbb{R}$.
(iii) $\forall \phi \in \widehat{G}$, $\varphi(\overline{\phi}) = \overline{\varphi(\phi)}$; $\overline{\varphi}$ and $\mathcal{R}e(\varphi)$ belong to $SP(\widehat{G})$.
(iv) If $\psi \in SP(\widehat{G})$, then $\varphi\psi$ and $c_1\varphi + c_2\psi$ belong to $SP(\widehat{G})$ for $c_1, c_2 \geq 0$.

Proof. (i) Let $\mu \in PM(\widehat{G})$. $\forall x \in G, \widehat{\delta_x}\mu \in M_b(\widehat{G})$ and for $y \in G$, we have

$$
\begin{aligned}
(\widehat{\delta_x}\mu)^{\vee}(y) &= \widehat{\delta_x}\mu(\widehat{\delta_{\overline{y}}}) \\
&= \textstyle\int_{\widehat{G}} \widehat{\delta_{\overline{y}}}(\phi)\widehat{\delta_x}(\phi) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \int_G \phi(\overline{z_1}) d\delta_{\overline{y}}(z_1) \int_G \phi(\overline{z_2}) d\delta_x(z_2) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \int_G \phi(z_1) d\delta_y(z_1) \int_G \phi(z_2) d\delta_{\overline{x}}(z_2) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \int_G \int_G \phi(z_1)\phi(z_2) d\delta_y(z_1) d\delta_{\overline{x}}(z_2) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \int_G \int_G \int_K \phi(z_1 * k * z_2) d\omega_K(k) d\delta_y(z_1) d\delta_{\overline{x}}(z_2) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \int_K \int_G \int_G (\int_G \phi(z_1 * z_3 * z_2) d\delta_k(z_3)) d\delta_y(z_1) d\delta_{\overline{x}}(z_2) d\omega_K(k) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \int_K \int_G \int_G \phi(z * z_2) d(\delta_k * \delta_y)(z)) d\delta_{\overline{x}}(z_2) d\omega_K(k) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \int_K \int_G \phi(t) d\left\{(\delta_k * \delta_y) * \delta_{\overline{x}}\right\}(t) d\omega_K(k) d\mu(\phi) \\
&= \textstyle\int_K \int_G \overset{\vee}{\mu}(t) d\left\{(\delta_k * \delta_y) * \delta_{\overline{x}}\right\}(t) d\omega_K(k) \geq 0 \text{ since } \mu \in MP(\widehat{G}).
\end{aligned}
$$

It follows that $\widehat{\delta_x}\mu \in MP(\widehat{G})$. Thus,

$$
\begin{aligned}
(\varphi\mu)^{\vee}(\overline{x}) &= \varphi\mu(\widehat{\delta_x}) \\
&= \textstyle\int_{\widehat{G}} \varphi(\phi)\widehat{\delta_x}(\phi) d\mu(\phi) \\
&= \textstyle\int_{\widehat{G}} \varphi(\phi) d(\widehat{\delta_x}\mu)(\phi) \\
&= \widehat{\delta_x}\mu(\varphi) \geq 0 \text{ since } \varphi \in SP(\widehat{G}).
\end{aligned}
$$

(ii) If $\mu \in M_b(\widehat{G})$, then $(\left\|\overset{\vee}{\mu}\right\|_{\infty} \delta_{\mathbf{1}} + \mu)^{\vee} \geq 0$, where $\left\|\overset{\vee}{\mu}\right\|_{\infty}$ is the sup-norm of $\overset{\vee}{\mu}$.

In fact,$(\left\|\overset{\vee}{\mu}\right\|_\infty \delta_{\mathbf{1}} + \mu)^\vee(x) = \left\|\overset{\vee}{\mu}\right\|_\infty + \overset{\vee}{\mu}(x) \geq 0$. Thus, $(\left\|\overset{\vee}{\mu}\right\|_\infty \delta_{\mathbf{1}} + \mu)(\varphi) \geq 0$.
Since $(\left\|\overset{\vee}{\mu}\right\|_\infty \delta_{\mathbf{1}} + \mu)(\varphi) = \left\|\overset{\vee}{\mu}\right\|_\infty \varphi(\mathbf{1}) + \mu(\varphi)$, then $\left\|\overset{\vee}{\mu}\right\|_\infty \varphi(\mathbf{1}) + \mu(\varphi) \geq 0$.
Thus, $\mu(\varphi) \in \mathbb{R}$.
(iii) Let $\phi \in \widehat{G}$, then $\overline{\phi} \in \widehat{G}$ and

$$\begin{aligned}(\delta_\phi + \delta_{\overline{\phi}})^\vee(x) &= \int_{\widehat{G}} \alpha(x) d\delta_\phi(\alpha) + \int_{\widehat{G}} \alpha(x) d\delta_{\overline{\phi}}(\alpha) \\ &= \phi(x) + \overline{\phi}(x) \\ &= 2\mathrm{Re}(\phi(x)).\end{aligned}$$

Thus, $(\delta_\phi + \delta_{\overline{\phi}})(\varphi) = \varphi(\phi) + \varphi(\overline{\phi}) \in \mathbb{R}$ and we deduce that $\mathrm{Im}(\varphi(\phi)) = -\mathrm{Im}(\varphi(\overline{\phi}))$.
We have also, $((\delta_\phi - \delta_{\overline{\phi}})/i)^\vee = 2\mathrm{Im}(\phi)$. So $\varphi(\phi) - \varphi(\overline{\phi}) \in i\mathbb{R}$ and we have $\mathrm{Re}(\varphi(\phi)) = \mathrm{Re}(\varphi(\overline{\phi}))$.
It comes that $\varphi(\overline{\phi}) = \overline{\varphi(\phi)}$.
Let $\mu \in PM(\widehat{G})$. Then $\mu^* \in PM(\widehat{G})$ and

$$\begin{aligned}\mu(\overline{\varphi}) &= \int_{\widehat{G}} \overline{\varphi(\alpha)} d\mu(\alpha) \\ &= \int_{\widehat{G}} \varphi(\overline{\alpha}) d\mu(\alpha) \\ &= \overline{\int_{\widehat{G}} \varphi(\alpha) d\mu^*(\alpha)} \\ &= \overline{\mu^*(\varphi)} \geq 0,\end{aligned}$$

then $\overline{\varphi} \in SP(\widehat{G})$ and so $Re(\varphi) = \frac{1}{2}(\varphi + \overline{\varphi}) \in SP(\widehat{G})$.
(iv) Let $\varphi, \psi \in SP(\widehat{G})$.
For $\mu \in PM(\widehat{G})$, we have$(\psi\mu)^\vee \geq 0$. So $(\psi\mu)(\varphi) \geq 0$ and it follows that

$$\begin{aligned}\mu(\varphi\psi) &= \int_{\widehat{G}} \varphi(\phi)\psi(\phi) d\mu(\phi) \\ &= \int_{\widehat{G}} \varphi(\phi) d(\psi\mu)(\phi) \\ &= (\psi\mu)(\varphi) \geq 0.\end{aligned}$$

Thus, $\varphi\psi \in SP(\widehat{G})$.
Let $c_1, c_2 \geq 0$. For $\mu \in MP(\widehat{G})$, we have

$$\begin{aligned}\int_{\widehat{G}} (c_1\varphi + c_2\psi)(\phi) d\mu(\phi) &= c_1 \int_{\widehat{G}} \varphi(\phi) d\mu(\phi) + c_2 \int_{\widehat{G}} \psi(\phi) d\mu(\phi) \\ &\geq 0.\end{aligned}$$

Thus, $c_1\varphi + c_2\psi \in SP(\widehat{G})$. □

We have the following result which is the main theorem of this paper.

Theorem 3.11. *Let φ in $SP(\widehat{G})$. Then there exists a unique positive measure $\nu \in M_b^\natural(G)$ such that $\varphi_{|spt(\pi)} = \widehat{\nu}_{|spt(\pi)}$. Conversely, $\widehat{\nu}$ is strongly positive for each positive measure $\nu \in M_b(G)$.*

Proof. If $\nu \in M_b(G)$ such that $\nu \geq 0$, then $\widehat{\nu} \in SP(\widehat{G})$. In fact, for $\mu \in PM(\widehat{G})$, we have

$$\begin{aligned}\int_{\widehat{G}} \widehat{\nu}(\phi) d\mu(\phi) &= \int_{\widehat{G}} \int_G \phi(\overline{x}) d\nu(x) d\mu(\phi) \\ &= \int_G \int_{\widehat{G}} \phi(\overline{x}) d\mu(\phi) d\nu(x) \\ &= \int_G \overset{\vee}{\mu}(\overline{x}) d\nu(x) \geq 0.\end{aligned}$$

Let $\mu \in M_b(\widehat{G})$ such that $\overset{\vee}{\mu}$ is real-valued. Since $\left|\overset{\vee}{\mu}(x)\right| \leq \left\|\overset{\vee}{\mu}\right\|_\infty$, then
$-\left\|\overset{\vee}{\mu}\right\|_\infty \overset{\vee}{\delta_{\mathbf{1}}}(x) \leq \overset{\vee}{\mu}(x) \leq \left\|\overset{\vee}{\mu}\right\|_\infty \overset{\vee}{\delta_{\mathbf{1}}}(x) \forall x \in G$ and so $(\left\|\overset{\vee}{\mu}\right\|_\infty \overset{\vee}{\delta_{\mathbf{1}}} \pm \overset{\vee}{\mu}) \geq 0$. This implies that $\left\|\overset{\vee}{\mu}\right\|_\infty \varphi(1) \pm \int_{\widehat{G}} \varphi(\phi)d\mu(\phi) \geq 0$, so $\left|\int_{\widehat{G}} \varphi(\phi)d\mu(\phi)\right| \leq \left\|\overset{\vee}{\mu}\right\|_\infty \varphi(1)$.
For $\mu \in M_b(\widehat{G})$, let us put $\mu_1 = \frac{1}{2}(\mu+\mu^*)$ and $\mu_2 = \frac{1}{2i}(\mu-\mu^*)$. We have $\overset{\vee}{\mu_1} = \frac{1}{2}(\overset{\vee}{\mu}+\overline{\overset{\vee}{\mu}}) = Re(\overset{\vee}{\mu})$ and $\overset{\vee}{\mu_2} = \frac{1}{2i}(\overset{\vee}{\mu} - \overline{\overset{\vee}{\mu}}) = Im(\overset{\vee}{\mu})$. So μ_1 and μ_2 are in $M_b(\widehat{G})$ such that $\overset{\vee}{\mu_1}$and $\overset{\vee}{\mu_2}$ are real valued. Since $\mu = \mu_1 + i\mu_2$, then

$$\begin{aligned}
\left|\int_{\widehat{G}} \varphi(\phi)d\mu(\phi)\right| &\leq \left|\int_{\widehat{G}} \varphi(\phi)d\mu_1(\phi)\right| + \left|\int_{\widehat{G}} \varphi(\phi)d\mu_2(\phi)\right| \\
&\leq \left\|\overset{\vee}{\mu_1}\right\|_\infty \varphi(1) + \left\|\overset{\vee}{\mu_2}\right\|_\infty \varphi(1) \\
&\leq 2\varphi(1) \left\|\overset{\vee}{\mu}\right\|_\infty .
\end{aligned}$$

The mapping

$$\begin{aligned}
\eta : (M_b(\widehat{G}))^\vee &\longrightarrow \mathbb{C} \\
\overset{\vee}{\mu} &\longmapsto \eta(\overset{\vee}{\mu}) = \int_{\widehat{G}} \varphi(\phi)d\mu(\phi)
\end{aligned}$$

is a sup-norm continuous linear function. The restriction $\eta_{|(\mathcal{K}(\widehat{G}))^\vee}$ of η to $(\mathcal{K}(\widehat{G}))^\vee$ is also linear and continuous. Let η_0 be its extension to $C_0(G)$, there exists $\nu \in M_b(G)$ such that

$$\eta_0(f) = \int_G f(x)d\nu(x)\ \forall f \in C_0(G).$$

So $\forall h \in \mathcal{K}(\widehat{G})$,

$$\begin{aligned}
\int_{\widehat{G}} \varphi(\phi)h(\phi)d\pi(\phi) = \eta_0(\overset{\vee}{h}) &= \int_G \int_{\widehat{G}} \phi(x)h(\phi)d\pi(\phi)d\nu(x) \\
&= \int_{\widehat{G}} \int_G \phi(x)d\nu(x)h(\phi)d\pi(\phi) \\
&= \int_{\widehat{G}} \int_G \phi(\overline{x})d\nu^-(x)h(\phi)d\pi(\phi) \\
&= \int_{\widehat{G}} \widehat{\nu^-}(\phi)h(\phi)d\pi(\phi).
\end{aligned}$$

Since φ and $\widehat{\nu^-}$ are continuous, then

$$\varphi_{|spt(\pi)} = \widehat{\nu^-}_{|spt(\pi)} = \widehat{(\nu^-)^\natural}_{|spt(\pi)}.$$

Furthermore, ν is a positive measure. In fact let $f \in C_0(G)$ such that $f \geq 0$ and $\varepsilon > 0$. Since $(\mathcal{K}(\widehat{G}))^\vee$ is dense in $C_0(G)$, then there exists $h \in \mathcal{K}(\widehat{G})$ such that $\left\|f - \overset{\vee}{h}\right\|_\infty < \varepsilon$. We may assume that $\overset{\vee}{h}$ is real valued. Define $\mu = \varepsilon\delta_{\mathbf{1}} + h\pi$. $\mu \in M_b(\widehat{G})$ and

$$\overset{\vee}{\mu}(x) = \varepsilon + \overset{\vee}{h}(x) > f(x) \geq 0\ \forall x \in G.$$

Thus, $\overset{\vee}{\mu} \geq 0$ and $\eta(\overset{\vee}{\mu}) = \int_{\widehat{G}} \varphi(\phi)d\mu(\phi) \geq 0$.

In addition we have

$$\begin{aligned}\left|\int_G f(x)d\nu(x)-\eta(\overset{\vee}{\mu})\right| &= \left|\int_G f(x)d\nu(x)-\varepsilon\eta(\overset{\vee}{\delta_1})-\int_G \overset{\vee}{h}d\nu(x)\right| \\ &\leq \varepsilon\eta(\overset{\vee}{\delta_1})+\int_G\left|(f-\overset{\vee}{h})(x)\right|d\nu(x) \\ &\leq 2\varepsilon\eta(\overset{\vee}{\delta_1}) \\ &\leq 4\varepsilon\varphi(1).\end{aligned}$$

Thus, $\int_G f(x)d\nu(x)\geq\eta(\overset{\vee}{\mu})-4\varepsilon\varphi(1)$ and taking ε so small that we want, we can conclude that $\int_G f(x)d\nu(x)\geq 0$. So ν and $(\nu^-)^\natural$ are positive. Finally, the uniqueness follows from Theorem 3.5. □

In this paper, we have proved a Bochner theorem for a noncommutative hypergoup G admitting a compact subhypergroup K such that (G,K) is a Gelfand pair. In fact, we have obtained a one to one correspondence between the space of strong positive definite functions on $\widehat{G}$ and the space of K-invariant and positive measures on G. Our results generalize those established on commutative locally compact hypergroups for in this case, $(G,\{e\})$ is a Gelfand pair with e the neutral element of G.

References

[1] W. R. Bloom and H. Heyer, Harmonic analysis of probability measures on hypergroups. de Gruyter Studies in Mathematics. 20. Berlin: de Gruyter. vi, 601 p. 1995.

[2] C. F. Dunkl, The Measure Algebra of a Locally Compact Hypergroup, Trans. Amer. Math. Soc., Volume 179, 1973.

[3] B. K. Germain and K. Kangni, On Gelfand pair over Hypergroups, Far East J. Math, Vol. 132, Number 1, 63-76, 2021.

[4] B. K. Germain, I. Toure, K. Kangni, A Plancherel Theorem On a Noncommutatiove Hypergroup, Int. J. Anal. Appl., 20 (2022), 32.

[5] R. I. Jewett, Spaces with an Abstarct Convolution of Measure, Adv. Math., Vol. 18, 1-101 1975.

[6] R. Lasser, Bochner Theorems for Hypergroups and Their Applications to Orthogonal Polynomial Expansions, Journal Of Approximation Theory 37, 3 1 l-325 (1983)

[7] L. Nachbin, On the finite dimensionality of every irreducible representation of a group, Proc. Amer, Math. Soc. 12, 11-12, 1961.

[8] K. A. Ross, Centers of Hypergroups, Trans. Amer. Math. Soc., Vol. 243, Septembre 1978.

[9] R. Spector, Mesures invariantes sur les Hypergroupes, Trans. Amer. Math. Soc., Vol. 239, 1978.

[10] L. Szekelyhidi, Spherical Spectral Synthetis On Hypergroups, Acta Math. Hungar., https://doi.org/10.1007/ s10474-020-01068-9 (2020)

[11] K. Vatti, Gelfand Pairs over Hypergroups Joins, Acta Math. Hungar., May 2019, Doi: 10.1007/s 10474-019-00946-1.

Author information

Kouakou Germain BROU, Univerité de Man, Côte d'Ivoire.
E-mail: germain.brou@univ-man.edu.ci

Ibrahima TOURE, UFR Mathématiques et Informatique, Université Félix Houphouët-Boigny, Côte d'Ivoire.
E-mail: ibrahima.toure@univ-fhb.edu.ci

Kinvi KANGNI, UFR Mathématiques et Informatique, Université Félix Houphouët-Boigny, Côte d'Ivoire.
E-mail: kinvi.kangni@univ-fhb.edu.ci

Received: 2023-01-11
Accepted: 2023-08-30