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THE NORMING SET OF $T \in \mathcal{L}_s(^n\ell_1^2)$ FOR $n = 3, 4, 5$

Sung Guen Kim

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Abstract Let $n \in \mathbb{N}, n \geq 2$ and $(E, \|\cdot\|)$ a Banach space. An element $(x_1, \dots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \dots, x_n)| = \|T\|$, where $\mathcal{L}(^nE)$ denotes the space of all continuous n-linear forms on E. For $T \in \mathcal{L}(^nE)$, we define

$$\text{Norm}(T) = \Big\{(x_1, \dots, x_n) \in E^n : (x_1, \dots, x_n) \text{ is a norming point of } T\Big\}.$$

$\text{Norm}(T)$ is called the *norming set* of T.

Let $\ell_1^2 = \mathbb{R}^2$ with the ℓ_1-norm. In this paper, we characterize the norming set of $T \in \mathcal{L}(^n\ell_1^2)$. Using this result, we completely describe the norming set of $T \in \mathcal{L}_s(^n\ell_1^2)$ for $n = 3, 4, 5$, where $\mathcal{L}_s(^n\ell_1^2)$ denotes the space of all symmetric n-linear forms on ℓ_1^2.

1 Introduction

In 1961 Bishop and Phelps [2] showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1, \dots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \dots, x_n)| = \|T\|$.

For $T \in \mathcal{L}(^nE)$, we define

$$\text{Norm}(T) = \Big\{(x_1, \dots, x_n) \in E^n : (x_1, \dots, x_n) \text{ is a norming point of } T\Big\}.$$

$\text{Norm}(T)$ is called the *norming set* of T. Notice that $(x_1, \dots, x_n) \in \text{Norm}(T)$ if and only if $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \dots, n)$. Indeed, if $(x_1, \dots, x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \dots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \dots, x_n)| = |T(x_1, \dots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \dots, n)$, then

$$(x_1, \dots, x_n) = \Big(\epsilon_1(\epsilon_1 x_1), \dots, \epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that $\text{Norm}(T) = \emptyset$ or an infinite set.

Examples. (a) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2c_0).$$

We claim that $\text{Norm}(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that $\text{Norm}(T) \neq \emptyset$.

Let $\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big| \le \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i|\,|y_i| \le \sum_{i=1}^{\infty}\frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, $\text{Norm}(T) = \emptyset$.

(b) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = x_1y_1 \in \mathcal{L}_s(^2c_0).$$

Then,

$$\text{Norm}(T) = \Big\{\Big((\pm1, x_2, x_3, \ldots),\ (\pm1, y_2, y_3, \ldots)\Big) \in c_0 \times c_0 : |x_j| \le 1,\ |y_j| \le 1 \text{ for } j \ge 2\Big\}.$$

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1}|P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

$\text{Norm}(P)$ is called the *norming set* of P. Notice that $\text{Norm}(P) = \emptyset$ or a finite set or an infinite set.

Kim [7] classified $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2\ell_\infty^2)$, where $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If $\text{Norm}(T) \neq \emptyset$, $T \in \mathcal{L}(^nE)$ is called a *norm attaining* n-linear form and if $\text{Norm}(P) \neq \emptyset$, $P \in \mathcal{P}(^nE)$ is called a *norm attaining* n-homogeneoue polynomial.(See [3])

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about $\text{Norm}(T)$ for $T \in \mathcal{L}(^nE)$. For $m \in \mathbb{N}$, let $\ell_1^m := \mathbb{R}^m$ with the the ℓ_1-norm and $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = \ell_1^m$ or ℓ_∞^2 and $T \in \mathcal{L}(^nE)$, $\text{Norm}(T) \neq \emptyset$ since S_E is compact. Kim [6, 8–10] classified $\text{Norm}(T)$ for every $T \in \mathcal{L}_s(^2\ell_\infty^2), \mathcal{L}(^2\ell_\infty^2), \mathcal{L}(^2\ell_1^2), \mathcal{L}_s(^2\ell_1^3)$ or $\mathcal{L}_s(^3\ell_1^2)$. Kim [11] classified $\text{Norm}(T)$ for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max\Big\{|y|, |x| + (1-w)|y|\Big\}$.

In this paper, we characterize the norming set of $T \in \mathcal{L}(^n\ell_1^2)$. Using this result, we completely describe the norming set of $T \in \mathcal{L}_s(^n\ell_1^2)$ for $n = 3, 4, 5$. This generalizes the results from [9].

2 The norming set of $T \in \mathcal{L}(^n\ell_1^2)$

Proposition 2.1. ([10]) *Let $n, m \ge 2$. Let $T \in \mathcal{L}(^m\ell_1^n)$ with*

$$T\Big(\big(x_1^{(1)}, \ldots, x_n^{(1)}\big), \ldots, \big(x_1^{(m)}, \ldots, x_n^{(m)}\big)\Big) = \sum_{1\le i_k\le n,\ 1\le k\le m} a_{i_1\cdots i_m}\, x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

for some $a_{i_1\cdots i_m} \in \mathbb{R}$. Then

$$\|T\| = \max\{|a_{i_1\cdots i_m}| : 1 \le i_k \le n,\ 1 \le k \le m\}.$$

By simplicity we denote $T = \left(a_{i_1\cdots i_m}\right)_{1\le i_k\le n, 1\le k\le m}$. We call $a_{i_1\cdots i_m}{}'$s the *coefficients* of T. Notice that if $\|T\| = 1$, then $|a_{i_1\cdots i_m}| \le 1$ for all $1 \le i_k \le n, 1 \le k \le m$.

Theorem 2.2. ([10]) *Let $n, m \ge 2$. Let $T \in \mathcal{L}(^m\ell_1^n)$ be the same as in Theorem A. Suppose that $\left(\left(t_1^{(1)},\ldots,t_n^{(1)}\right),\ldots,\left(t_1^{(m)},\ldots,t_n^{(m)}\right)\right) \in \mathrm{Norm}(T)$. If $|a_{i_1'\cdots i_m'}| < \|T\|$ for $1 \le i_k' \le n,\ 1 \le k \le m$, then $t_{i_1'}^{(1)}\cdots t_{i_m'}^{(m)} = 0$.*

The following characterizes the norming sets of $\mathcal{L}(^n\ell_1^2)$.

Theorem 2.3. *Let $n \in \mathbb{N}$ and $T \in \mathcal{L}(^n\ell_1^2)$ with $\|T\| = 1$. Then,*

$$\mathrm{Norm}(T) = \bigcup_{k=1}^{n} \left(A_k^+ \cup A_k^- \cup B_{k,1} \cup B_{k,2}\right),$$

where $e_1 = (1,0), e_2 = (0,1)$,

$$\begin{aligned}
A_k^+ &= \Big\{\Big(\pm X_1,\ldots,\pm X_{k-1},\pm(t,1-t),\pm X_{k+1},\ldots,\pm X_n\Big) \in \left(S_{\ell_1^2}\right)^n : \\
&\quad T\big(X_1,\ldots,X_{k-1},e_1,X_{k+1},\ldots,\pm X_n\big) \times T\big(X_1,\ldots,X_{k-1},e_2,X_{k+1},\ldots,X_n\big) = 1, \\
&\quad 0 \le t \le 1\Big\}, \\
A_k^- &= \Big\{\Big(\pm X_1,\ldots,\pm X_{k-1},\pm(t,-(1-t)),\pm X_{k+1},\ldots,\pm X_n\Big) \in \left(S_{\ell_1^2}\right)^n : \\
&\quad T\big(X_1,\ldots,X_{k-1},e_1,X_{k+1},\ldots,\pm X_n\big) \times T\big(X_1,\ldots,X_{k-1},e_2,X_{k+1},\ldots,X_n\big) = -1, \\
&\quad 0 \le t \le 1\Big\}, \\
B_{k,1} &= \Big\{\Big(\pm X_1,\ldots,\pm X_{k-1},\pm e_1,\pm X_{k+1},\ldots,\pm X_n\Big) \in \left(S_{\ell_1^2}\right)^n : \\
1 &= \Big|T\big(X_1,\ldots,X_{k-1},e_1,X_{k+1},\ldots,\pm X_n)\Big| > \Big|T\big(X_1,\ldots,X_{k-1},e_2,X_{k+1},\ldots,X_n)\Big|\Big\}, \\
B_{k,2} &= \Big\{\Big(\pm X_1,\ldots,\pm X_{k-1},\pm e_2,\pm X_{k+1},\ldots,\pm X_n\Big) \in \left(S_{\ell_1^2}\right)^n : \\
1 &= \Big|T\big(X_1,\ldots,X_{k-1},e_2,X_{k+1},\ldots,\pm X_n)\Big| > \Big|T\big(X_1,\ldots,X_{k-1},e_1,X_{k+1},\ldots,X_n)\Big|\Big\}.
\end{aligned}$$

Proof. Let $\mathcal{F}_k = A_k^+ \cup A_k^- \cup B_{k,1} \cup B_{k,2}$ for $k = 1,\ldots,n$.

($\subseteq$). Let $\Big(X_1,\ldots,X_n\Big) \in \mathrm{Norm}(T)$. Let $1 \le k \le n$ be fixed. Then $X_k = \lambda_1^{(k)}e_1 + \lambda_2^{(k)}e_2$ for some $\lambda_1^{(k)}, \lambda_2^{(k)} \in \mathbb{R}$ with $|\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$.

Case 1.

$$T\big(X_1,\ldots,X_{k-1},e_1,X_{k+1},\ldots,\pm X_n\big) \times T\big(X_1,\ldots,X_{k-1},e_2,X_{k+1},\ldots,X_n\big) = 1.$$

Since $\|T\| = 1$, we have

$$1 = T\big(X_1,\ldots,X_{k-1},e_1,X_{k+1},\ldots,\pm X_n\big) = T\big(X_1,\ldots,X_{k-1},e_2,X_{k+1},\ldots,X_n\big)$$

or

$$-1 = T\big(X_1,\ldots,X_{k-1},e_1,X_{k+1},\ldots,\pm X_n\big) = T\big(X_1,\ldots,X_{k-1},e_2,X_{k+1},\ldots,X_n\big).$$

Claim 1. $X_k \in \big\{\pm(te_1 + (1-t)e_2) : 0 \le t \le 1\big\}$.

By n-linearity of T, it follows that

$$\begin{aligned}
1 &= T\big(X_1,\ldots,X_n\big) = T\big(X_1,\ldots,X_{k-1},(\lambda_1^{(k)}e_1 + \lambda_2^{(k)}e_2),X_{k+1},\ldots,X_n\big) \\
&= \big|\lambda_1^{(k)}T\big(X_1,\ldots,X_{k-1},W_1,X_{k+1},\ldots,X_n\big) + \lambda_2^{(k)}T\big(X_1,\ldots,X_{k-1},W_2,X_{k+1},\ldots,X_n\big)\big| \\
&= |\lambda_1^{(k)} + \lambda_2^{(k)}| \le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1.
\end{aligned}$$

Thus, $|\lambda_1^{(k)} + \lambda_2^{(k)}| = |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$. Hence, $\text{sign}(\lambda_1^{(k)}) = \text{sign}(\lambda_1^{(k)})$. Thus,

$$X_k \in \{|\lambda_1^{(k)}|e_1 + |\lambda_2^{(k)}|e_2, -(|\lambda_1^{(k)}|e_1 + |\lambda_2^{(k)}|e_2)\} \subseteq \{\pm (te_1 + (1-t)e_2) : 0 \le t \le 1\}.$$

Therefore, $X \in A_k^+ \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

Case 2.

$$T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, \pm X_n) \times T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n) = -1.$$

Since $\|T\| = 1$, we have

$$1 = T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, \pm X_n) = -T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n)$$

or

$$-1 = T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, \pm X_n) = -T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n).$$

Claim 2. $X_k \in \{\pm (te_1 - (1-t)e_2) : 0 \le t \le 1\}$.

It follows that

$$\begin{aligned} 1 &= T(X_1, \ldots, X_n) = T(X_1, \ldots, X_{k-1}, (\lambda_1^{(k)}e_1 + \lambda_2^{(k)}e_2), X_{k+1}, \ldots, X_n) \\ &= \big|\lambda_1 T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, X_n) + \lambda_2 T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n)\big| \\ &= |\lambda_1^{(k)} - \lambda_2^{(k)}| \le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1. \end{aligned}$$

Thus, $|\lambda_1^{(k)} - \lambda_2^{(k)}| = |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1$. Hence, $\text{sign}(\lambda_1^{(k)}) = -\text{sign}(\lambda_2^{(k)})$. Thus,

$$X_k \in \{|\lambda_1^{(k)}|e_1 - |\lambda_2^{(k)}|e_2, -(|\lambda_1^{(k)}|e_1 - |\lambda_2^{(k)}|e_2)\} \subseteq \{\pm (te_1 - (1-t)e_2) : 0 \le t \le 1\}.$$

Therefore, $X \in A_k^- \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

Case 3.

$$1 = \big|T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, \pm X_n)\big| > \big|T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n)\big|.$$

Claim 3. $\lambda_2^{(k)} = 0$.

Assume that $\lambda_2^{(k)} \neq 0$. It follows that

$$\begin{aligned} 1 &= \big|T(X_1, \ldots, X_n)\big| = \big|T(X_1, \ldots, X_{k-1}, (\lambda_1^{(k)}e_1 + \lambda_2^{(k)}e_2), X_{k+1}, \ldots, X_n)\big| \\ &\le |\lambda_1^{(k)}|\,\big|T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, X_n)\big| + |\lambda_2^{(k)}|\,\big|T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n)\big| \\ &< |\lambda_1^{(k)}|\big|T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, X_n)\big| + |\lambda_2^{(k)}| \\ &\le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| = 1, \end{aligned}$$

which is a contradiction. Thus, $\lambda_2^{(k)} = 0$ and so $X_k = e_1$. Therefore, $X \in B_{k,1} \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

Case 4.

$$1 = \big|T(X_1, \ldots, X_{k-1}, W_2, X_{k+1}, \ldots, \pm X_n)\big| > \big|T(X_1, \ldots, X_{k-1}, W_1, X_{k+1}, \ldots, X_n)\big|.$$

Claim 4. $\lambda_1^{(k)} = 0$.

Assume that $\lambda_1^{(k)} \neq 0$. By $(*)$, it follows that

$$\begin{aligned} 1 &= \big|T(X_1, \ldots, X_n)\big| = \big|T(X_1, \ldots, X_{k-1}, (\lambda_1^{(k)}e_1 + \lambda_2^{(k)}e_2), X_{k+1}, \ldots, X_n)\big| \\ &\le |\lambda_1^{(k)}|\,\big|T(X_1, \ldots, X_{k-1}, e_1, X_{k+1}, \ldots, X_n)\big| + |\lambda_2^{(k)}|\,\big|T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n)\big| \\ &< |\lambda_1^{(k)}| + |\lambda_2^{(k)}|\big|T(X_1, \ldots, X_{k-1}, e_2, X_{k+1}, \ldots, X_n)\big| \\ &\le |\lambda_1^{(k)}| + |\lambda_2^{(k)}| \le 1, \end{aligned}$$

which is a contradiction. Thus, $\lambda_1 = 0$ and so $X_k = e_2$. Therefore, $X \in B_{k,2} \subseteq \mathcal{F}_k \subseteq \bigcup_{j=1}^n \mathcal{F}_j$.

($\supseteq$). We will show that $\mathcal{F}_k \subseteq \text{Norm}(T)$ for every $1 \le k \le n$.
Let $1 \le k \le n$ be fixed and $Y = (Y_1, \ldots, Y_n) \in \mathcal{F}_k$.
Suppose that $Y \in A_k^+$.
Then $Y_k = \pm(t_k W_1 + (1-t_k)W_2)$ for some $0 \le t_k \le 1$ and

$$T\big(Y_1, \ldots, Y_{k-1}, e_1, Y_{k+1}, \ldots, \pm Y_n\big) \cdot T\big(Y_1, \ldots, Y_{k-1}, e_2, Y_{k+1}, \ldots, Y_n\big) = 1.$$

It follows that

$$\begin{aligned} 1 &= T\big(Y_1, \ldots, Y_n\big) = T\big(Y_1, \ldots, Y_{k-1}, (t_k e_1 + (1-t_k)e_2), Y_{k+1}, \ldots, Y_n\big) \\ &= \big|t_k T\big(Y_1, \ldots, Y_{k-1}, e_1, Y_{k+1}, \ldots, Y_n\big) + (1-t_k) T\big(Y_1, \ldots, Y_{k-1}, e_2, Y_{k+1}, \ldots, Y_n\big)\big| \\ &= |t_k + (1-t_k)| = 1. \end{aligned}$$

Thus, $Y \in \text{Norm}(T)$.
Suppose that $Y \in A_k^-$.
Then $Y_k = \pm(t_k e_1 - (1-t_k)e_2)$ for some $0 \le t_k \le 1$ and

$$T\big(Y_1, \ldots, Y_{k-1}, e_1, Y_{k+1}, \ldots, \pm Y_n\big) \cdot T\big(Y_1, \ldots, Y_{k-1}, e_2, Y_{k+1}, \ldots, Y_n\big) = -1.$$

It follows that

$$\begin{aligned} 1 &= T\big(Y_1, \ldots, Y_n\big) = T\big(Y_1, \ldots, Y_{k-1}, (t_k e_1 - (1-t_k)e_2), Y_{k+1}, \ldots, Y_n\big) \\ &= \big|t_k T\big(Y_1, \ldots, Y_{k-1}, e_1, Y_{k+1}, \ldots, Y_n\big) - (1-t_k) T\big(Y_1, \ldots, Y_{k-1}, e_2, Y_{k+1}, \ldots, Y_n\big)\big| \\ &= |t_k + (1-t_k)| = 1. \end{aligned}$$

Thus, $Y \in \text{Norm}(T)$.
Suppose that $Y \in B_{k,1}$.
Then $Y_k = \pm e_1$ and $\big|T\big(Y_1, \ldots, Y_n\big)\big| = \big|T\big(Y_1, \ldots, Y_{k-1}, e_1, Y_{k+1}, \ldots, \pm Y_n\big)\big| = 1$. Thus, $Y \in \text{Norm}(T)$.
Suppose that $Y \in B_{k,2}$.
Then $Y_k = \pm e_2$ and $\big|T\big(Y_1, \ldots, Y_n\big)\big| = \big|T\big(Y_1, \ldots, Y_{k-1}, e_2, Y_{k+1}, \ldots, \pm Y_n\big)\big| = 1$. Thus, $Y \in \text{Norm}(T)$. We complete the proof. □

3 Classification of the norming set of $T \in \mathcal{L}_s({}^n\ell_1^2)$ for $n = 3, 4, 5$

In this section we completely describe the norming set of $T \in \mathcal{L}_s({}^n\ell_1^2)$ for $n = 3, 4, 5$.
Let $\mathcal{W} \subseteq (S_{\ell_1^2})^n$. We denote

$$\begin{aligned} \text{Sym}(\mathcal{W}) = \Big\{ &\big((x_{\sigma(1)}, y_{\sigma(1)}), \ldots, (x_{\sigma(n)}, y_{\sigma(n)})\big) : X = \big((x_1, y_1), \ldots, (x_n, y_n)\big) \in \mathcal{W}, \\ &\sigma \text{ is a permutation on } \{1, \ldots, n\} \Big\}. \end{aligned}$$

Lemma 3.1. *Let $T \in \mathcal{L}_s({}^5\ell_1^2)$ be of the form*

$$\begin{aligned} &T\big((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)\big) = a x_1x_2x_3x_4x_5 + b y_1y_2y_3y_4y_5 \\ &+ c\big(x_2x_3x_4x_5y_1 + x_1x_3x_4x_5y_2 + x_1x_2x_4x_5y_3 + x_1x_2x_3x_5y_4 + x_1x_2x_3x_4y_5\big) \\ &+ d\big(x_3x_4x_5y_1y_2 + x_2x_4x_5y_1y_3 + x_2x_3x_5y_1y_4 + x_2x_3x_4y_1y_5 + x_1x_4x_5y_2y_3 \\ &+ x_1x_3x_5y_2y_4 + x_1x_3x_4y_2y_5 + x_1x_2x_5y_3y_4 + x_1x_2x_4y_3y_5 + x_1x_2x_3y_4y_5\big) \\ &+ e\big(x_1x_2y_3y_4y_5 + x_1x_3y_2y_4y_5 + x_1x_4y_2y_3y_5 + x_1x_5y_2y_3y_4 + x_2x_3y_1y_4y_5 \\ &+ x_2x_4y_1y_3y_5 + x_2x_5y_1y_3y_4 + x_3x_4y_1y_2y_5 + x_3x_5y_1y_2y_4 + x_4x_5y_1y_2y_3\big) \\ &+ f\big(y_2y_3y_4y_5x_1 + y_1y_3y_4y_5x_2 + y_1y_2y_4y_5x_3 + y_1y_2y_3y_5x_4 + y_1y_2y_3y_4x_5\big) \end{aligned}$$

for some $a, b, c, d, e, f \in \mathbb{R}$.

Then there is $T^{'} = (a^{'}, b^{'}, c^{'}, d^{'}, e^{'}, f^{'}) \in \mathcal{L}_s({}^5\ell_1^2)$ such that $\|T^{'}\| = \|T\|$ and $a^{'} \geq |b^{'}|$ and $c^{'} \geq 0$.

Proof. If $a < 0$, we let $T_1 = -T$. It is immediate that $a^{'} := -a > 0$ and $\|T_1\| = \|T\|$. Thus, we may assume that $a \geq 0$.

If $a < |b|$, we let

$$\begin{aligned} & T_2\big((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)\big) \\ = \; & T\big((y_1, x_1), (y_2, x_2), (y_3, x_3), (y_4, x_4), (y_5, x_5)\big). \end{aligned}$$

It is obvious that $\|T_2\| = \|T\|$. Let $a^{'} := b$ and $b^{'} := a$ Thus, we may assume that $a \geq |b|$. If $c < 0$, we let

$$\begin{aligned} & T_3\big((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)\big) \\ = \; & T\big((x_1, -y_1), (x_2, -y_2), (x_3, -y_3), (x_4, -y_4), (x_5, -y_5)\big). \end{aligned}$$

It is immediate that $c^{'} := -c > 0$ and $\|T_3\| = \|T\|$. This completes the proof. □

We are in a position to classify Norm(T) for every $T \in \mathcal{L}_s({}^5\ell_1^2)$.

Theorem 3.2. *Let $T \in \mathcal{L}_s({}^5\ell_1^2)$ be of the form*

$$\begin{aligned} & T\big((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)\big) = a x_1x_2x_3x_4x_5 + b y_1y_2y_3y_4y_5 \\ + \; & c\big(x_2x_3x_4x_5y_1 + x_1x_3x_4x_5y_2 + x_1x_2x_4x_5y_3 + x_1x_2x_3x_5y_4 + x_1x_2x_3x_4y_5\big) \\ + \; & d\big(x_3x_4x_5y_1y_2 + x_2x_4x_5y_1y_3 + x_2x_3x_5y_1y_4 + x_2x_3x_4y_1y_5 + x_1x_4x_5y_2y_3 \\ + \; & x_1x_3x_5y_2y_4 + x_1x_3x_4y_2y_5 + x_1x_2x_5y_3y_4 + x_1x_2x_4y_3y_5 + x_1x_2x_3y_4y_5\big) \\ + \; & e\big(x_1x_2y_3y_4y_5 + x_1x_3y_2y_4y_5 + x_1x_4y_2y_3y_5 + x_1x_5y_2y_3y_4 + x_2x_3y_1y_4y_5 \\ + \; & x_2x_4y_1y_3y_5 + x_2x_5y_1y_3y_4 + x_3x_4y_1y_2y_5 + x_3x_5y_1y_2y_4 + x_4x_5y_1y_2y_3\big) \\ + \; & f\big(y_2y_3y_4y_5x_1 + y_1y_3y_4y_5x_2 + y_1y_2y_4y_5x_3 + y_1y_2y_3y_5x_4 + y_1y_2y_3y_4x_5\big) \\ & \textit{for some } a \geq |b|, c \geq 0, \|T\| = 1. \end{aligned}$$

Then the following assertions hold: let $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Case 1. $a = |b| = c = |d| = |e| = |f| = 1$

Subcase 1. $a = c = -b = d = e = f = 1$

$$\begin{aligned} & \text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm(ue_1 + (1-u)e_2), \\ & \pm(ve_1 + (1-v)e_2), \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t, s, u, v \leq 1\Big\}\Big). \end{aligned}$$

Subcase 2. $a = c = -b = -d = e = f = 1$

$$\begin{aligned} & \text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_1, \pm e_2\big), \\ & \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big), \\ & \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t, s \leq 1\Big\}\Big). \end{aligned}$$

Subcase 3. $a = c = -b = d = -e = f = 1$

$$\begin{aligned} & \text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm(ue_1 - (1-u)e_2), \pm e_2, \pm e_2\big), \\ & \big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm e_1, \pm e_1, \pm e_1\big) : 0 \leq t, s, u \leq 1\Big\}\Big). \end{aligned}$$

Subcase 4. $a = c = -b = d = e = -f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm(ue_1 + (1-u)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t, s \le 1\Big\}\Big).$$

Subcase 5. $a = c = -b = -d = -e = f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big) : 0 \le t, s \le 1\Big\}\Big).$$

Subcase 6. $a = c = -b = -d = e = -f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big) : 0 \le t, s \le 1\Big\}\Big).$$

Subcase 7. $a = c = -b = d = -e = -f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t, s \le 1\Big\}\Big).$$

Subcase 8. $a = c = -b = -d = -e = -f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm(ue_1 + (1-u)e_2), \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big) : 0 \le t, s, u \le 1\Big\}\Big).$$

Subcase 9. $a = c = b = d = e = f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm(ue_1 + (1-u)e_2), \pm(ve_1 + (1-v)e_2), \pm(we_1 + (1-w)e_2)\big) : 0 \le t, s, u, v, w \le 1\Big\}\Big).$$

Subcase 10. $a = c = b = -d = e = f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big) : 0 \le t, s \le 1\Big\}\Big).$$

Subcase 11. $a = c = b = d = -e = f = 1$

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_2, \pm e_2\big), \big(\pm(te + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t, s \le 1\Big\}\Big).$$

Subcase 12. $a = c = b = d = e = -f = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm(ue_1 + (1-u)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t, s, u \le 1\Big\}\Big).$$

Subcase 13. $a = c = b = -d = -e = f = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big), \big(\pm(te + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

Subcase 14. $a = c = b = -d = e = -f = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm(ue_1 - (1-u)e_2), \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big) : 0 \le t, s, u \le 1\Big\}\Big).$$

Subcase 15. $a = c = b = d = -e = -f = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t, s \le 1\Big\}\Big).$$

Subcase 16. $a = c = b = -d = -e = -f = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm e_1, \pm e_2, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big) : 0 \le t, s \le 1\Big\}\Big).$$

Case 2. $|a_{i_1 \cdots i_5}| < 1$ *for some* $i_k \in \{1, 2\}$ $(k = 1, \ldots, 5)$,
where $a_{11111} = a, a_{22222} = b, a_{11112} = c, a_{11122} = d, a_{11222} = e, a_{12222} = f$ *and* $a_{i_1 \cdots i_5} = a_{\sigma(i_1) \cdots \sigma(i_5)}$ *for every permutation* σ *on* $\{1, \ldots, 5\}$.

Let $M = \{(i_1, \ldots, i_5) : |a_{i_1 \cdots i_5}| < 1\}$ *and define* $S = \big(b_{i_1 \cdots i_5}\big) \in \mathcal{L}({}^5\ell_1^2)$ *be such that* $b_{i_1 \cdots i_5} = a_{i_1 \cdots i_5}$ *if* $(i_1, \ldots, i_5) \notin M$ *and* $b_{i_1 \cdots i_5} = 1$ *if* $(i_1, \ldots, i_5) \in M$. *(Notice that S is included in Case 1.) Then,*

$$\text{Norm}(T) = \bigcap_{(i_1, \ldots, i_5) \in M} \text{Sym}\Big(\Big\{\big(t_1^{(1)} e_1 + t_2^{(1)} e_2, \ldots, t_1^{(5)} e_1 + t_2^{(5)} e_2\big) \in \text{Norm}(S) : t_{i_1}^{(1)} \cdots t_{i_5}^{(5)} = 0\Big\}\Big),$$

where $e_1 = (1, 0), e_2 = (0, 1)$.

Proof. Notice that

$$(*)\qquad T\big((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4),(x_5,y_5)\big) =$$
$$x_1\Big\{x_2\big[x_3(x_4[ax_5+cy_5]+y_4[cx_5+dy_5])+y_3(x_4[cx_5+dy_5]+y_4[dx_5+ey_5])\big]$$
$$+\quad y_2\big[x_3(x_4[cx_5+dy_5]+y_4[dx_5+ey_5])+y_3(x_4[cx_5+ey_5]+y_4[ex_5+fy_5])\big]\Big\}$$
$$+\quad y_1\Big\{x_2\big[x_3(x_4[cx_5+dy_5]+y_4[dx_5+ey_5])+y_3(x_4[dx_5+ey_5]+y_4[ex_5+fy_5])\big]$$
$$+\quad y_2\big[x_3(x_4[dx_5+ey_5]+y_4[ex_5+fy_5])+y_3(x_4[ex_5+fy_5]+y_4[fx_5+by_5]\big]\Big\}.$$

By $(*)$, it follows that

$$(**)\qquad \text{Norm}(T)\supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_1\big),$$
$$\big(\pm(te_1+d(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_2\big),$$
$$\big(\pm(te_1+de(1-t)e_2),\pm e_1,\pm e_1,\pm e_2,\pm e_2\big),$$
$$\big(\pm te_1+ef(1-t)e_2),\pm e_1,\pm e_2,\pm e_2,\pm e_2\big),$$
$$\big(\pm(te_1+bf(1-t)e_2),\pm e_2,\pm e_2,\pm e_2,\pm e_2\big) : 0\le t\le 1\Big\}\Big).$$

Case 1. $a=|b|=c=|d|=|e|=|f|=1$

Subcase 1. $a=c=-b=d=e=f=1$

By $(**)$,

$$\text{Norm}(T)\supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_1\big),$$
$$\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_2\big),$$
$$\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_2,\pm e_2\big),$$
$$\big(\pm te_1+(1-t)e_2),\pm e_1,\pm e_2,\pm e_2,\pm e_2\big),$$
$$\big(\pm(te_1-(1-t)e_2),\pm e_2,\pm e_2,\pm e_2,\pm e_2\big) : 0\le t\le 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 1 follows.

Subcase 2. $a=c=-b=-d=e=f=1$

By $(**)$,

$$\text{Norm}(T)\supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_1\big),$$
$$\big(\pm(te_1-(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_2\big),$$
$$\big(\pm(te_1-(1-t)e_2),\pm e_1,\pm e_1,\pm e_2,\pm e_2\big),$$
$$\big(\pm te_1+(1-t)e_2),\pm e_1,\pm e_2,\pm e_2,\pm e_2\big),$$
$$\big(\pm(te_1-(1-t)e_2),\pm e_2,\pm e_2,\pm e_2,\pm e_2\big) : 0\le t\le 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 2 follows.

Subcase 3. $a=c=-b=d=-e=f=1$

By $(**)$,

$$\begin{aligned}\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1-(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1-(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 3 follows.

Subcase 4. $a = c = -b = d = e = -f = 1$

By $(**)$,

$$\begin{aligned}\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1-(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1+(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

By Theorem C, the assertion of Subcase 4 follows.

Subcase 5. $a = c = -b = -d = -e = f = 1$

By $(**)$,

$$\begin{aligned}\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1-(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1-(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 5 follows.

Subcase 6. $a = c = -b = -d = e = -f = 1$

By $(**)$,

$$\begin{aligned}\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1-(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1+(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 6 follows.

Subcase 7. $a = c = -b = d = -e = -f = 1$

By $(**)$,

$$\begin{aligned}&\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\&\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\&\big(\pm te_1-(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 7 follows.

Subcase 8. $a = c = -b = -d = -e = -f = 1$

By $(**)$,

$$\begin{aligned}&\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\&\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\&\big(\pm te_1+(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 8 follows.

Subcase 9. $a = c = -b = d = e = f = 1$

By $(**)$,

$$\begin{aligned}&\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\&\big(\pm te_1+(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 9 follows.

Subcase 10. $a = c = b = -d = e = f = 1$

By $(**)$,

$$\begin{aligned}&\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\&\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\&\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\&\big(\pm te_1+(1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\&\big(\pm(te_1+(1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 10 follows.

Subcase 11. $a = c = b = d = -e = f = 1$

By $(**)$,

$$\begin{aligned}&\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1 + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 11 follows.

Subcase 12. $a = c = b = d = e = -f = 1$

By $(**)$,

$$\begin{aligned}&\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 12 follows.

Subcase 13. $a = c = b = -d = -e = f = 1$

By $(**)$,

$$\begin{aligned}&\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1 + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 13 follows.

Subcase 14. $a = c = b = -d = e = -f = 1$

By $(**)$,

$$\begin{aligned}&\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_1\big),\\ &\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1, \pm e_2\big),\\ &\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_2, \pm e_2\big),\\ &\big(\pm te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2, \pm e_2\big),\\ &\big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 14 follows.

Subcase 15. $a = c = b = d = -e = -f = 1$

By $(**)$,

$$\begin{aligned}&\text{Norm}(T)\supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_1\big),\\&\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_2\big),\\&\big(\pm(te_1-(1-t)e_2),\pm e_1,\pm e_1,\pm e_2,\pm e_2\big),\\&\big(\pm te_1+(1-t)e_2),\pm e_1,\pm e_2,\pm e_2,\pm e_2\big),\\&\big(\pm(te_1-(1-t)e_2),\pm e_2,\pm e_2,\pm e_2,\pm e_2\big):0\leq t\leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 15 follows.

Subcase 16. $a=c=b=-d=-e=-f=1$

By $(**)$,

$$\begin{aligned}&\text{Norm}(T)\supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_1\big),\\&\big(\pm(te_1-(1-t)e_2),\pm e_1,\pm e_1,\pm e_1,\pm e_2\big),\\&\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_2,\pm e_2\big),\\&\big(\pm te_1+(1-t)e_2),\pm e_1,\pm e_2,\pm e_2,\pm e_2\big),\\&\big(\pm(te_1-(1-t)e_2),\pm e_2,\pm e_2,\pm e_2,\pm e_2\big):0\leq t\leq 1\Big\}\Big).\end{aligned}$$

By Theorem 2.3, the assertion of Subcase 16 follows.

The proof of Case 2 follows from Theorem 2.2 and Case 1.

Therefore, we complete the proof. □

Proposition 3.3. *Let $m,n\geq 2$. Let $T\in\mathcal{L}(^m\ell_1^n)$. Then,*

$$\begin{aligned}\text{Norm}(T)&=\Big\{\big(X_1,\dots,X_m\big):\big(X_1,\dots,X_m,e_1\big)\in\text{Norm}(\overline{T})\Big\}\\&=\Big\{\big(X_1,\dots,X_m\big):\big(X_1,\dots,X_m,e_2\big)\in\text{Norm}(\overline{T})\Big\},\end{aligned}$$

where $e_1=(1,0\dots,0),e_2=(0,1,0\dots,0)$ and $\overline{T}\in\mathcal{L}(^{m+1}\ell_1^n)$ is defined by

$$\begin{aligned}&\overline{T}\big((x_1^{(1)},\dots,x_n^{(1)}),\dots,(x_1^{(m+1)},\dots,x_n^{(m+1)})\big)=\\&T\big((x_1^{(1)},\dots,x_n^{(1)}),\dots,(x_1^{(m)},\dots,x_n^{(m)})\big)\cdot(x_1^{(m+1)}+\cdots+x_n^{(m+1)}).\end{aligned}$$

Proof. It follows from the fact that $\|T\|=\|\overline{T}\|$. □

Let

$$\begin{aligned}&T\big((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)\big)=ax_1x_2x_3x_4+by_1y_2y_3y_4\\+\ &c\big(x_2x_3x_4y_1+x_1x_3x_4y_2+x_1x_2x_4y_3+x_1x_2x_3y_4\big)\\+\ &d\big(x_3x_4y_1y_2+x_2x_4y_1y_3+x_2x_3y_1y_4+x_1x_4y_2y_3+x_1x_3y_2y_4+x_1x_2y_3y_4\big)\\+\ &e\big(x_1y_2y_3y_4+x_2y_1y_3y_4+x_3y_1y_2y_4+x_4y_1y_2y_3\big)\in\mathcal{L}_s(^4l_1^2)\\&\text{with } a,b,c,d,e\in\mathbb{R}.\end{aligned}$$

By an analogous argument as in the proof of Lemma 3.1, we may assume that $a\geq|b|,e\geq 0$.

We classify the norming set of $T\in\mathcal{L}_s(^4\ell_1^2)$.

Theorem 3.4. *Let*

$$\begin{aligned} & T\big((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)\big) = ax_1x_2x_3x_4 + by_1y_2y_3y_4 \\ + \quad & c\big(x_2x_3x_4y_1 + x_1x_3x_4y_2 + x_1x_2x_4y_3 + x_1x_2x_3y_4\big) \\ + \quad & d\big(x_3x_4y_1y_2 + x_2x_4y_1y_3 + x_2x_3y_1y_4 + x_1x_4y_2y_3 + x_1x_3y_2y_4 + x_1x_2y_3y_4\big) \\ + \quad & e\big(x_1y_2y_3y_4 + x_2y_1y_3y_4 + x_3y_1y_2y_4 + x_4y_1y_2y_3\big) \in \mathcal{L}_s({}^4\ell_1^2) \\ & \textit{with } \|T\| = 1, a \geq |b|, e \geq 0, c, d \in \mathbb{R}. \end{aligned}$$

Then the following assertions hold:

Case 1. $a = |b| = |c| = |d| = e = 1$

Subcase 1. $a = b = c = d = e = 1$

$$\text{Norm}(T) = \Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm(ue_1 + (1-u)e_2), \pm(ve_1 + (1-v)e_2)\big) : 0 \leq t, s, u, v \leq 1\Big\}.$$

Subcase 2. $a = b = -c = d = e = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_1, \big), \big(\pm(t, 1-t), \pm(s, 1-s), \pm e_2, \pm e_2\big) : 0 \leq t, s \leq 1\Big\}\Big).$$

Subcase 3. $a = b = c = -d = e = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t, s \leq 1\Big\}\Big).$$

Subcase 4. $a = b = -c = -d = e = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).$$

Subcase 5. $a = -b = c = d = e = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm(se_1 + (1-s)e_2), \pm(ue_1 + (1-u)e_2), \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t, s, u \leq 1\Big\}\Big).$$

Subcase 6. $a = -b = -c = d = e = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t, s \leq 1\Big\}\Big).$$

Subcase 7. $a = -b = c = -d = e = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm(se_1 - (1-s)e_2), \pm(ue_1 - (1-u)e_2), \pm e_1\big) : 0 \leq t, s, u \leq 1\Big\}\Big).$$

Subcase 8. $a=-b=-c=-d=e=1$

$$\mathrm{Norm}(T)=\mathrm{Sym}\Big(\Big\{\big(\pm(te_1-(1-t)e_2),\pm(se_1-(1-s)e_2),\pm e_2,\pm e_2\big),$$
$$\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_2\big),\big(\pm(te_1-(1-t)e_2),\pm e_1,\pm e_1,\pm e_1\big):0\leq t,s\leq 1\Big\}\Big).$$

Case 2. $|a_{i_1\cdots i_4}|<1$ *for some* $i_k\in\{1,2\}$ $(k=1,\ldots,4)$,
where $a_{1111}=a, a_{2222}=b, a_{1112}=c, a_{1122}=d, a_{1222}=e$ *and* $a_{i_1\cdots i_4}=a_{\sigma(i_1)\cdots\sigma(i_4)}$ *for every permutation* σ *on* $\{1,\ldots,4\}$.

Let $M=\{(i_1,\ldots,i_4):|a_{i_1\cdots i_4}|<1\}$ *and define* $S=\big(b_{i_1\cdots i_4}\big)\in\mathcal{L}(^4\ell_1^2)$ *be such that* $b_{i_1\cdots i_4}=a_{i_1\cdots i_4}$ *if* $(i_1,\ldots,i_4)\notin M$ *and* $b_{i_1\cdots i_4}=1$ *if* $(i_1,\ldots,i_4)\in M$. (*Notice that* S *is included in Case 1.*) *Then,*

$$\mathrm{Norm}(T)=\bigcap_{(i_1,\ldots,i_4)\in M}\mathrm{Sym}\Big(\Big\{\big(t_1^{(1)}e_1+t_2^{(1)}e_2,\ldots,t_1^{(4)}e_1+t_2^{(4)}e_2\big)\in\mathrm{Norm}(S):$$
$$t_{i_1}^{(1)}\cdots t_{i_4}^{(4)}=0\Big\}\Big).$$

Proof. We will slightly modify the proof of Theorem 3.2.
Notice that

$$(*)\qquad T\big((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)\big)=$$
$$x_1\Big\{x_2\big(x_3[ax_4+cy_4]+y_3[cx_4+dy_4]\big)+y_2\big(x_3[cx_4+dy_4]+y_3[dx_4+ey_4]\big)\Big\}$$
$$+\quad y_1\Big\{x_2\big(x_3[cx_4+dy_4]+y_3[dx_4+ey_4]\big)+y_2\big(x_3[dx_4+ey_4]+y_3[ex_4+by_4]\big)\Big\}.$$

By $(*)$, it follows that

$$(**)\qquad \mathrm{Norm}(T)\supseteq\mathrm{Sym}\Big(\Big\{\big(\pm(te_1+c(1-t)e_2),\pm e_1,\pm e_1,\pm e_1\big),$$
$$\big(\pm(te_1+cd(1-t)e_2),\pm e_1,\pm e_1,\pm e_2\big),\big(\pm(te_1+de(1-t)e_2),\pm e_1,\pm e_2,\pm e_2\big),$$
$$\big(\pm(te_1+b(1-t)e_2),\pm e_2,\pm e_2,\pm e_2\big):0\leq t\leq 1\Big\}\Big).$$

Case 1. $a=|b|=|c|=|d|=e=1$

Subcase 1. $a=b=c=d=e=1$

By $(**)$,

$$\mathrm{Norm}(T)\supseteq\mathrm{Sym}\Big(\Big\{\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_1\big),$$
$$\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_1,\pm e_2\big),\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_2,\pm e_2\big),$$
$$\big(\pm te_1+(1-t)e_2),\pm e_2,\pm e_2,\pm e_2\big):0\leq t\leq 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 1 follows.

Subcase 2. $a=b=-c=d=e=1$

By $(**)$,

$$\mathrm{Norm}(T)\supseteq\mathrm{Sym}\Big(\Big\{\big(\pm(te_1-(1-t)e_2),\pm e_1,\pm e_1,\pm e_1\big),$$
$$\big(\pm(te_1-(1-t)e_2),\pm e_1,\pm e_1,\pm e_2\big),\big(\pm(te_1+(1-t)e_2),\pm e_1,\pm e_2,\pm e_2\big),$$
$$\big(\pm te_1+(1-t)e_2),\pm e_2,\pm e_2,\pm e_2\big):0\leq t\leq 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 2 follows.

Subcase 3. $a = b = c = -d = e = 1$

By $(**)$,

$$\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big),$$
$$\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big),$$
$$\big(\pm te_1 + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 3 follows.

Subcase 4. $a = b = -c = -d = e = 1$

By $(**)$,

$$\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big),$$
$$\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big),$$
$$\big(\pm te_1 + (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 4 follows.

Subcase 5. $a = -b = c = d = e = 1$

By $(**)$,

$$\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big),$$
$$\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big),$$
$$\big(\pm te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 5 follows.

Subcase 6. $a = -b = -c = d = e = 1$

By $(**)$,

$$\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big),$$
$$\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big),$$
$$\big(\pm te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 6 follows.

Subcase 7. $a = -b = c = -d = e = 1$

By $(**)$,

$$\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big),$$
$$\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big),$$
$$\big(\pm te_1 - (1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \leq t \leq 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 7 follows.

Subcase 8. $a = -b = -c = -d = e = 1$

By $(**)$,

$$\text{Norm}(T) \supseteq \text{Sym}\Big(\Big\{\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1, \pm e_1\big),$$
$$\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1, \pm e_2\big), \big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_2, \pm e_2\big),$$
$$\big(\pm te_1 - +(1-t)e_2), \pm e_2, \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 8 follows.
The proof of Case 2 follows from Theorem 2.2 and Case 1.
This completes the proof. □

Let

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = ax_1x_2x_3 + by_1y_2y_3 + c\big(x_2x_3y_1 + x_1x_3y_2 + x_1x_2y_3\big) +$$
$$d\big(x_3y_1y_2 + x_2y_1y_3 + x_1y_2y_3\big) \in \mathcal{L}_s(^3\ell_1^2) \text{ with } a,b,c,d \in \mathbb{R}.$$

By an analogous argument as in the proof of Lemma 3.1, we may assume that $a \ge b \ge 0$.

Kim [9] classified the norming set of $T \in \mathcal{L}_s(^3\ell_1^2)$. The following classifies the norming set of $T \in \mathcal{L}_s(^3\ell_1^2)$ in a different way.

Theorem 3.5. *Let*

$$T\big((x_1,y_1),(x_2,y_2),(x_3,y_3)\big) = ax_1x_2x_3 + by_1y_2y_3 + c\big(x_2x_3y_1 + x_1x_3y_2 + x_1x_2y_3\big) +$$
$$d\big(x_3y_1y_2 + x_2y_1y_3 + x_1y_2y_3\big) \in \mathcal{L}_s(^3\ell_1^2) \textit{ with } \|T\| = 1, a \ge b \ge 0, c,d \in \mathbb{R}.$$

Then the following assertions hold:

Case 1. $a = |b| = |c| = |d| = e = 1$

Subcase 1. $a = b = c = d = 1$

$$\text{Norm}(T) = \Big\{\big(\pm(t,1-t), \pm(s,1-s), \pm(u,1-u)\big) : 0 \le t,s,u \le 1\Big\}.$$

Subcase 2. $a = b = -c = d = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1-(1-t)e_2), \pm(se_1-(1-s)e_2), \pm e_1\big),$$
$$\big(\pm(te_1+(1-t)e_2), \pm e_2, \pm e_2\big) : 0 \le t,s \le 1\Big\}\Big).$$

Subcase 3. $a = b = c = -d = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1-(1-t)e_2), \pm(se_1-(1-s)e_2), \pm e_2\big),$$
$$\big(\pm(te_1+(1-t)e_2), \pm e_1, \pm e_1\big) : 0 \le t,s \le 1\Big\}\Big).$$

Subcase 4. $a = b = -c = -d = 1$

$$\text{Norm}(T) = \text{Sym}\Big(\Big\{\big(\pm(te_1-(1-t)e_2), \pm e_1, \pm e_1\big),$$
$$\big(\pm(te_1-(1-t)e_2), \pm e_2, \pm e_2\big), \pm(te_1+(1-t)e_2), \pm e_1, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

Case 2. $|a_{i_1i_2i_3}| < 1$ *for some* $i_k \in \{1,2\}$ $(k = 1,2,3)$,
where $a_{111} = a, a_{222} = b, a_{112} = c, a_{122} = d$ *and* $a_{i_1i_2i_3} = a_{\sigma(i_1)\sigma(i_2)\sigma(i_3)}$ *for every permutation* σ *on* $\{1,2,3\}$.

Let $M = \{(i_1, i_2, i_3) : |a_{i_1 i_2 i_3}| < 1\}$ *and define* $S = \big(b_{i_1 i_2 i_3}\big) \in \mathcal{L}({}^3\ell_1^2)$ *be such that* $b_{i_1 i_2 i_3} = a_{i_1 i_2 i_3}$ *if* $(i_1, i_2, i_3) \notin M$ *and* $b_{i_1 i_2 i_3} = 1$ *if* $(i_1, i_2, i_3) \in M$. *(Notice that* S *is included in Case 1.) Then,*

$$\mathrm{Norm}(T) = \bigcap_{(i_1,\ldots,i_3)\in M} \mathrm{Sym}\Big(\Big\{\big(t_1^{(1)}e_1 + t_2^{(1)}e_2, \ldots, t_1^{(3)}e_1 + t_2^{(3)}e_2\big) \in \mathrm{Norm}(S) : t_{i_1}^{(1)} \cdots t_{i_3}^{(3)} = 0\Big\}\Big).$$

Proof. We will slightly modify the proof of Theorem 3.2.

Notice that

$$(*) \qquad T\big((x_1, y_1), (x_2, y_2), (x_3, y_3)\big) = x_1\Big\{x_2[ax_4 + cy_4] + y_2[cx_3 + dy_3]\Big\} + y_1\Big\{x_2[cx_3 + dy_3] + y_2[dx_3 + by_3]\Big\}.$$

By $(*)$, it follows that

$$(**) \qquad \mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + c(1-t)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 + cd(1-t)e_2), \pm e_1, \pm e_2\big), \big(\pm(te_1 + de(1-t)e_2), \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

Case 1. $a = b = |c| = |d| = 1$

Subcase 1. $a = b = c = d = 1$

By $(**)$,

$$\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 1 follows.

Subcase 2. $a = b = -c = d = 1$

By $(**)$,

$$\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2\big), \big(\pm(te_1 + (1-t)e_2), \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

By Theorem C, the assertion of Subcase 2 follows.

Subcase 3. $a = b = c = -d = 1$

By $(**)$,

$$\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 3 follows.

Subcase 4. $a = b = -c = -d = 1$

By $(**)$,

$$\mathrm{Norm}(T) \supseteq \mathrm{Sym}\Big(\Big\{\big(\pm(te_1 - (1-t)e_2), \pm e_1, \pm e_1\big), \big(\pm(te_1 + (1-t)e_2), \pm e_1, \pm e_2\big), \big(\pm(te_1 - (1-t)e_2), \pm e_2, \pm e_2\big) : 0 \le t \le 1\Big\}\Big).$$

By Theorem 2.3, the assertion of Subcase 4 follows.

The proof of Case 2 follows from Theorem 2.2 and Case 1.

This completes the proof. □

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Author information

Sung Guen Kim, Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea.
E-mail: sgk317@knu.ac.kr

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