

ON THE CATEGORY OF STRUCTURE SPECIES

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Abstract The purpose of the present paper is to make a mathematical study of the differences and relations among possible structures inherent in an object, as well as of the whole structure constituted by them (i.e., the structure of structures), against the background of the structuralism by Claude Lévi-Strauss and others. Our discussion focuses on Blanchard's categorical reformulation of the notion of structure species introduced originally by Bourbaki. Under the equivalence between structure species and constructive functors (in the sense of Blanchard), we discuss the issue of how much information concerning a given category is contained in the knowledge about structure species on that category. The main result of the present paper asserts that a category can be reconstructed, up to a certain slight indeterminacy, from the category of structure species on it. This result is partially motivated by various reconstruction theorems that have been shown in the context of anabelian geometry.

1 Introduction: Lévi-Strauss' structuralism

1.1 Structure of structures

Structuralism is a general theory of culture and methodology that focuses on relationships rather than individual objects, or alternatively, where objects are defined by the set of relationships of which they are part and not by the qualities possessed by them taken in isolation. According to this mode of knowledge, phenomena of human life are not intelligible except through their interrelations. These relations constitute a *structure*, and behind local variations in the surface phenomena there are constant laws of abstract structure (cf. [4], [21], [2]).

Structuralism in Europe developed in the early 20th century from insights in the field of linguistics of, mainly, Ferdinand de Saussure and the subsequent Prague, Moscow, and Copenhagen schools of linguistics. After World War II, an array of scholars in the humanities borrowed Saussure's concepts for use in their respective fields. By the early 1960s, structuralism as a movement was coming into its own and some believed that it offered a single unified approach to human life that would embrace all disciplines. Claude Lévi-Strauss, a French anthropologist, was the first such scholar, sparking a widespread interest in structuralism.

Lévi-Strauss defined "structure" as a whole consisting of elements and relations between elements, which retain their invariant properties through a series of transformations. To be precise, a model with structural value satisfies several requirements described as follows (cf. [13], Part 5, Chap. XV):

- (i) The structure exhibits the characteristics of a system. It is made up of several elements, none of which can undergo a change without effecting changes in all the other elements.
- (ii) For any given model there should be a possibility of ordering a series of transformations resulting in a group of models of the same type.
- (iii) The above properties make it possible to predict how the model will react if one or more of its elements are submitted to certain modifications.

- (iv) The model should be constituted so as to make immediately intelligible all the observed facts.

The structures listed, at least, as illustrations are kinships, political ideologies, mythologies, ritual, art, code of etiquette, and even cooking (cf. [21]). The discernment of these structures and their comparative analysis, which takes into account their distribution both historically and geographically, is indeed the subject matter of structural anthropology.

In “*The Elementary Structures of Kinship*” (cf. [14]), Claude Lévi-Strauss examined kinship systems from a structural point of view and demonstrated how apparently different social organizations were different permutations of a few basic kinship structures. The kinship system provides a way to order individuals according to certain rules; social organization is another way of ordering individuals and groups; social stratifications, whether economic or political, provide us with a third type; and all these orders can themselves be ordered by showing the kinds of relationship which exist among them, i.e., how they interact with one another on both the synchronic and the diachronic levels.

One may construct models valid not only for one type of order (kinship, social organization, economic relations, etc.) but where numerous models for all types of order are themselves ordered inside a total model. In [21], Chap. II, N. Rotenstreich explained that

We do not start out with scattered concepts: we start with structures and move to further structures. We start with order and move from another order or to an order of orders. There is a structure to the relation between these orders or structures. This “structure of structures”¹ is not just a static relation of coexistence, i.e. language beside kinship, etc., or even not one of subordination whereby a narrow structure such as rites is comprised in or is secondary to a wider structure such as society.

1.2 Main result of this paper

In the present paper, we make a mathematical study of the differences and relations among possible structures inherent in an object, as well as the whole structure constituted by them (i.e., the structure of structures), against the background of the structuralism by Lévi-Strauss and others². Our discussion focuses on some objects treated in structural theory, starting with Bourbaki. As is well known, Bourbaki, the collective pseudonym of a group of predominantly French mathematicians, undertook, in the mid-20th century, the task of making a unified development of central parts of modern mathematics in largely formalized language.³ This resulted in a long series of books “*Éléments de Mathématique*” that became very influential. In a manifesto written by Bourbaki in 1950, some main principles of their structuralist view of mathematics were presented.

In one of these books (cf. [3]), Bourbaki developed their theory of structures in a set-theoretic manner; the building blocks are called *structure species*. Roughly speaking, a structure species is a set, or a collection of sets, endowed with relations and operations not only among their members but also among collections of elements of these sets, relations among them, etc. A basic example is the structure species of ordered sets, where, from a set S , we obtain (by a suitable echelon construction scheme) the power set $P(S \times S)$ of the product $S \times S$ and a binary relation $s \in P(S \times S)$ (called the typical characterization) equipped with a relation describing the axiom of an ordered set.

In [1], Blanchard introduced the concept of structure species on a category very close to the concept of structure species in the sense of Bourbaki and proved that it is equivalent to the concept of structure species in the sense of Sonner (cf. [22]). The purpose of the present paper is to consider “structure of structures” formulated in terms of Blanchard’s structure species and investigate how this concept can capture the essence of things. That is, we discuss the issue

¹Lévi-Strauss developed a similar idea with the term “order of orders” (cf. [13]).

²Our work was written with the aim of building new bridges between mathematics and humanities studies, taking on the work by Lévi-Strauss. It is partly inspired by some interesting articles by P. J. Larcombe, which incorporate a multicultural point of view (cf., e.g., [10], [11]).

³In 1943, André Weil, one of the collaborators in Bourbaki, met Claude Lévi-Strauss in New York, which led to a small collaboration. By using a mathematical model based on group theory, Weil described marriage rules for four classes of people within Australian aboriginal society. This contribution appeared in an appendix of Lévi-Strauss’s book (cf. [14]).

of how much information concerning a given category is contained in the knowledge about the *structure of structure species* on that category.

The following is the main result of the present paper, which asserts that a category can be reconstructed, up to a certain slight indeterminacy, from the category of structure species on it (the proof will be given in § 3):

Theorem 1.1. *Let us fix a universe \mathcal{U} . Let X and X' be connected \mathcal{U} -small categories. Denote by Sp_X and $Sp_{X'}$ the categories of structure species on X and X' respectively. Also, denote by X^{op} the opposite category of X . Then, the following two conditions are equivalent to each other:*

- (a) $X \cong X'$ or $X^{\text{op}} \cong X'$.
- (b) $Sp_X \cong Sp_{X'}$.

Here, given two categories \mathcal{C} , \mathcal{D} , we write $\mathcal{C} \cong \mathcal{D}$ if \mathcal{C} is equivalent to \mathcal{D} .

1.3 Remark on Theorem 1.1

Finally, we remark that Theorem 1.1 can be regarded as a variant of various reconstruction theorems that have been shown in the context of anabelian geometry (cf. Remark 3.11). In 1983, A. Grothendieck wrote a letter to G. Faltings (cf. [7]), outlining what is today known as the anabelian conjectures. (Many of the claims based on these conjectures have now been proved by mathematicians.) These conjectures concern the possibility of reconstructing certain arithmetic varieties (e.g., hyperbolic curve over a number or a p -adic local field) from their étale fundamental groups.

Our study is partially motivated by the anabelian philosophy of Grothendieck; this is because a structure species, or equivalently a constructive functor (cf. Definition 2.4, (i)), on a category X may be regarded, in some sense, as categorical realization of coverings over X (cf. [22], § 3). Moreover, if X is a (profinite) groupoid, then one can interpret Theorem 1.1 as a reconstruction assertion for X by means of the fundamental group associated with the Galois category of constructive functors over X with finite fibers (cf. Remark 2.10). Similar category-theoretic reconstructions can be found in [16], [17], and [23].

2 Structures species and constructive functors

In this section, we recall structure species on a category defined by Blanchard and the equivalence between structure species and constructive functors. After that, we examine constructive functors on a groupoid.

2.1 Preliminaries on categories

Let \mathcal{C} be a category. We denote by $\text{Ob}(\mathcal{C})$ (resp., $\overline{\text{Ob}}(\mathcal{C})$; resp., $\text{Mor}(\mathcal{C})$) the set of objects (resp., the set of isomorphism classes of objects; resp., the set of morphisms) in \mathcal{C} . For two objects $a, b \in \text{Ob}(\mathcal{C})$, we denote by $\text{Mor}_{\mathcal{C}}(a, b)$ or $\text{Mor}(a, b)$ the set of morphisms $a \rightarrow b$ in \mathcal{C} . Also, we write

$$\text{Mor}\{a, b\} := \text{Mor}(a, b) \cup \text{Mor}(b, a)$$

(hence $\text{Mor}\{a, b\} = \text{Mor}\{b, a\}$ and $\text{Mor}\{a, a\} = \text{Mor}(a, a)$). Also, write

$$\text{Mor}(a, b)^{\neq} \left(\text{resp., } \text{Mor}\{a, b\}^{\neq} \right)$$

for the subset of $\text{Mor}(a, b)$ (resp., $\text{Mor}\{a, b\}$) consisting of non-invertible morphisms. An object a in \mathcal{C} is said to be **minimal** if it is not an initial object and any monomorphism $b \hookrightarrow a$ in \mathcal{C} , where b is not an initial object, is necessarily an isomorphism.

Next, we denote by \mathcal{C}^{op} the opposite category of \mathcal{C} . Also, given each functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we shall write $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ for the functor between the respective opposite categories naturally

induced by F . We denote by \mathcal{C}^{\cong} the category whose objects are the elements of $\text{Ob}(\mathcal{C})$ and whose morphisms are the isomorphisms in $\text{Mor}(\mathcal{C})$. Given two categories \mathcal{C}, \mathcal{D} , we shall write $\mathcal{C} \cong \mathcal{D}$ (resp., $\mathcal{C} \stackrel{\text{isom}}{\cong} \mathcal{D}$) if \mathcal{C} is equivalent (resp., isomorphic) to \mathcal{D} .

Recall that a category \mathcal{C} is said to be a **groupoid** if the morphisms in \mathcal{C} are all invertible. By a **groupoid in \mathcal{C}** , we mean a subcategory \mathcal{D} of \mathcal{C} forming a groupoid such that, if a, a' are objects in \mathcal{D} and $u : a \xrightarrow{\sim} a'$ is an isomorphism in \mathcal{C} , then u lies in $\text{Mor}(\mathcal{D})$. A groupoid \mathcal{D} in \mathcal{C} is called **maximal** if every object a in \mathcal{C} isomorphic to some object in \mathcal{D} belongs to $\text{Ob}(\mathcal{D})$.

A category \mathcal{C} is said to be a **preorder** if $\text{Mor}(a, b)$ has at most one element for all objects $a, b \in \text{Ob}(\mathcal{C})$. We write $a < b$ if $\text{Mor}(a, b)$ is nonempty; the binary relation “ $<$ ” in $\text{Ob}(\mathcal{C})$ is reflexive and transitive. If in addition “ $<$ ” is symmetric, \mathcal{C} is said to be an **order**. Note that “ $<$ ” is symmetric precisely when the only isomorphisms in the preorder \mathcal{C} are the identity morphisms. If $(T, <)$ is a partial order, then we write T^\perp for the order defined in such a way that $\text{Ob}(T^\perp) = T$ and $\text{Mor}(a, b) \neq \emptyset$ precisely when $a < b$.

Throughout the present paper, we shall fix a universe \mathcal{U} . Denote by Set the category consisting of \mathcal{U} -small sets and maps between them. Denote by Cat the category consisting of \mathcal{U} -small categories and functors between them. Moreover, denote by Ord the full subcategory of Cat consisting of \mathcal{U} -small orders. To each \mathcal{U} -small set T , we associate the category $\text{Dis}(T)$, whose objects are the elements in T and whose only morphisms are identity morphisms. The assignment $T \mapsto \text{Dis}(T)$ defines a functor $\text{Dis} : \text{Set} \rightarrow \text{Ord}$.

2.2 Structure species

Let us recall the categorical reformulation of Bourbaki’s structure species discussed in [1]. Fix a \mathcal{U} -small category X .

Definition 2.1 (cf. [1], § 2, Definition 2.1). Denote by $J : X^{\cong} \hookrightarrow X$ the natural inclusion.

(i) Let us consider a pair

$$\Sigma := (E, S)$$

consisting of two functors $E : X \rightarrow \text{Ord}, S : X^{\cong} \rightarrow \text{Set}$. We say that Σ is a **(covariant) structure species** on X if the composite $\text{Dis} \circ S$ is a subfunctor of $E \circ J$, meaning that, for every $a \in \text{Ob}(X^{\cong})$, the category $(\text{Dis} \circ S)(a)$ is a subcategory of $(E \circ J)(a)$ and the inclusion $(\text{Dis} \circ S)(a) \hookrightarrow (E \circ J)(a)$ is functorial with respect to a .

(ii) Let $\Sigma := (E, S)$ and $\Sigma' := (E', S')$ be structure species on X . A **morphism of structure species** from Σ to Σ' is defined as a natural transformation $\phi : S \rightarrow S'$ such that, for every morphism $u : a \rightarrow b$ in X and for every $U \in \text{Ob}(S(a))$ and $V \in \text{Ob}(S(b))$, we have $E'(u)(\phi_a(U)) < \phi_b(V)$ whenever $E(u)(U) < V$.

Remark 2.2. Let $\Sigma := (E, S)$ be a structure species on X . As mentioned in the Remark following [1], § 2, Definition 2.1, E and S respectively correspond to the *echelon* and *structure scheme* of Σ in the traditional terminology of Bourbaki (cf. [3], Chap. IV, § 1).

Example 2.3 (Continuous maps on topological spaces, [1], § 5). For each \mathcal{U} -small set T , we shall write $P(T)$ for the power set (i.e., the set of subsets) of T . Let P^+ (resp., P^-) be the functor $\text{Set} \rightarrow \text{Set}$ (resp., $\text{Set}^{\text{op}} \rightarrow \text{Set}$) defined as follows:

- For each object T in Set , we set $P^+(T) := P(T)$ (resp., $P^-(T) := P(T)$).
- For each morphism $f : T \rightarrow T'$ in Set , we set $P^+(f)$ (resp., $P^-(f)$) to be the map $P^+(T) \rightarrow P^+(T')$ given by $U \mapsto f(U)$ for every $U \in P^+(T)$ (resp., the map $P^-(T') \rightarrow P^-(T)$ given by $U' \mapsto f^{-1}(U')$ for every $U' \in P^-(T')$).

The maps $P^+(f)$ and $P^-(f)$ defined for any morphism $f : T \rightarrow T'$ in Set are non-decreasing when both $P(T) (= P^+(T) = P^-(T))$ and $P(T') (= P^+(T') = P^-(T'))$ are equipped with the order structures determined by “ \subseteq ”. Hence, the assignments $T \mapsto (P^+(T), \subseteq)^\perp$ and $T \mapsto (P^-(T), \subseteq)^\perp$, respectively, induce functors $\mathcal{P}^+ : \text{Set} \rightarrow \text{Ord}$ and $\mathcal{P}^- : \text{Set}^{\text{op}} \rightarrow \text{Ord}$.

Now, let Z be a \mathcal{U} -small subcategory of Set^{op} . We write $E := \mathcal{P}^+ \circ (P^-|_Z) : Z \rightarrow Set \rightarrow Ord$. For each $T \in \text{Ob}(Z)$, denote the set of topologies on T by $\mathcal{Top}(T)$. If $f : T \xrightarrow{\sim} T'$ is a bijection of \mathcal{U} -small sets, then $P^+(P^-(f))$ induces a bijection $\mathcal{Top}(f) : \mathcal{Top}(T') \xrightarrow{\sim} \mathcal{Top}(T)$. The assignments $T \mapsto \mathcal{Top}(T)$ and $f \mapsto \mathcal{Top}(f)$ together yield a functor $\mathcal{Top} : Z^{\cong} \rightarrow Set$. The resulting pair

$$\Sigma_{\text{top}} := (E, \mathcal{Top})$$

forms a structure species on Z .

We shall denote by

$$\mathcal{Sp}_X$$

the category consisting of structure species on X and morphisms between them. If there exists an equivalence of categories $X \xrightarrow{\sim} X'$ (where X' is another \mathcal{U} -small category), then we can construct an equivalence of categories $\mathcal{Sp}_X \xrightarrow{\sim} \mathcal{Sp}_{X'}$ in an evident manner.

2.3 Constructive functors

Next, let us recall an equivalent realization of a structure species, i.e., a constructive functor in the sense of [1]. In [22], § 2, Definition 3, this notion was introduced under the name “structure species”.

Definition 2.4 (cf. [1], § 3, Definitions 3.1 and 3.2). (i) Let $F : Y \rightarrow X$ be a functor between \mathcal{U} -small categories. We say that F is a **constructive functor** on X (or simply, F is **constructive**) if it satisfies the following conditions:

- F is faithful.
- For every $a \in \text{Ob}(Y)$ and for every isomorphism u in X with domain $F(a)$, there exists uniquely an isomorphism u_Y in Y with domain a satisfying the equality $F(u_Y) = u$.

We often refer to the second condition as the *lifting property* on F .

- (ii) Let $F : Y \rightarrow X$ and $F' : Y' \rightarrow X$ be constructive functors on X . A **morphism of constructive functors** from F to F' is defined as a functor $\Phi : Y \rightarrow Y'$ satisfying the equality $F' \circ \Phi = F$.

We shall denote by

$$\mathcal{Con}_X$$

the category consisting of constructive functors on X and morphisms between them. One may verify that \mathcal{Con}_X admits finite coproducts and fiber products. If $f : X' \rightarrow X$ is a functor between \mathcal{U} -small categories, then the assignment $Y \mapsto f^*(Y) := Y \times_X X'$ defines a functor

$$f^* : \mathcal{Con}_X \rightarrow \mathcal{Con}_{X'}. \quad (2.1)$$

Here, we shall construct a constructive functor associated to a structure species. Let $\Sigma := (E, S)$ be a structure species on X . Denote by Y_Σ the category defined as follows:

- The objects of Y_Σ are pairs (a, T) such that $a \in \text{Ob}(X)$ and $T \in S(a)$.
- The morphisms from (a, T) to (a', T') are morphisms $u : a \rightarrow a'$ in X such that $E(u)(T) < T'$, where “ $<$ ” is the relation defined on the order $E(a')$. The composite law for morphisms in Y_Σ is defined in a natural manner.

Moreover, we set

$$F_\Sigma : Y_\Sigma \rightarrow X$$

to be the functor given by $(a, T) \mapsto a$ and $u \mapsto u$. This functor forms a constructive functor on X (cf. [1], § 2, Propositions 2.1 and 2.2). To a morphism of structure species $\phi : \Sigma \rightarrow \Sigma'$, we can associate a morphism of constructive functors $F_\phi : F_\Sigma \rightarrow F_{\Sigma'}$ (but we will omit the details of this construction). According to [1], § 4, Theorem 2, the assignments $\Sigma \mapsto F_\Sigma$ and $\phi \mapsto F_\phi$ together define an equivalence of categories

$$Sp_X \xrightarrow{\sim} Con_X. \tag{2.2}$$

2.4 Property of constructive functors

In this section, we observe one property of a constructive functor (cf. Proposition 2.7). Before doing so, let us recall the connectedness of a category.

Definition 2.5. A category \mathcal{C} is said to be **connected** if it is nonempty and for any pair of objects $a, b \in \text{Ob}(\mathcal{C})$, there exists a finite sequence (c_1, \dots, c_n) of objects in \mathcal{C} such that $c_1 = a, c_n = b$, and $\text{Mor}\{c_j, c_{j+1}\} \neq \emptyset$ for any $j = 1, \dots, n - 1$.

Example 2.6. A category \mathcal{C} is connected if it has either an initial object or a terminal object. In particular, a category consisting of \mathcal{U} -small sets containing either the empty set or a singleton is connected.

Next, we prove the following assertion, which will be used in the subsequent discussion.

Proposition 2.7. *Let $F : Y \rightarrow X$ and $F' : Y' \rightarrow X$ be constructive functors on X such that $\text{Ob}(Y) \neq \emptyset$ and Y' is a connected groupoid. Also, let $\Phi : F \rightarrow F'$ be a morphism of constructive functors. Then, the maps $\text{Ob}(\Phi) : \text{Ob}(Y) \rightarrow \text{Ob}(Y')$ and $\text{Mor}(\Phi) : \text{Mor}(Y) \rightarrow \text{Mor}(Y')$ induced by Φ are surjective.*

Proof. By the assumption $\text{Ob}(Y) \neq \emptyset$, there exists an object a_Y in Y . Write $a_{Y'} := \Phi(a_Y)$. Now, let us take an arbitrary object $b_{Y'}$ in Y' . Since Y' is a connected groupoid, we can find an isomorphism $u_{Y'} : a_{Y'} \xrightarrow{\sim} b_{Y'}$ in Y' . If we write $u := F'(u_{Y'})$, then the lifting property on F implies that there exists an isomorphism $u_Y : a_Y \xrightarrow{\sim} b_Y$ in Y (for some $b_Y \in \text{Ob}(Y)$) with $F(u_Y) = u$. Since both $u_{Y'}$ and $\Phi(u_Y)$ are isomorphisms with domain $a_{Y'}$, lifting u , the lifting property on F again implies that $u_{Y'} = \Phi(u_Y)$ and hence $b_{Y'} = \Phi(b_Y)$. This shows the surjectivity of $\text{Ob}(\Phi)$. The surjectivity of $\text{Mor}(\Phi)$ can be proved by using a similar argument, so we will finish the proof here. \square

2.5 Constructive functor on a groupoid

This subsection deals with a specific constructive functor on a groupoid. From Proposition 2.9 described later, this constructive functor can be thought of as the universal covering of a topological space or a graph (cf. Example 2.8, Remark 2.10).

Let G be a \mathcal{U} -small groupoid and e an object in G . Suppose that G is connected, which implies $\text{Mor}(a, a') \neq \emptyset$ for any $a, a' \in \text{Ob}(G)$. We shall set $Y_{G,e}$ to be the category defined as follows:

- The objects in $Y_{G,e}$ are pairs (a, u) consisting of $a \in \text{Ob}(G)$ and $u \in \text{Mor}(e, a)$.
- The morphisms in $Y_{G,e}$ from (a, u) to (a', u') are morphisms $v : a \rightarrow a'$ satisfying $u' = v \circ u$. The composition law for morphisms in $Y_{G,e}$ is defined in an evident manner.

It is clear that $Y_{G,e}$ is a connected groupoid and, for any $(a, u), (a', u') \in \text{Ob}(Y_{G,e})$, the set $\text{Mor}((a, u), (a', u'))$ has exactly one element. The assignments $(a, u) \mapsto a$ and $v \mapsto v$ together define a functor

$$F_{G,e} : Y_{G,e} \rightarrow G,$$

which forms a constructive functor on G .

Example 2.8. Let us consider the case where $\text{Ob}(G)$ has exactly one element, which we denote by \otimes . Denote by $\text{Aut}(\otimes)$ the automorphism group of \otimes . Then, the category $Y_{G,\otimes}$ may be regarded as the Cayley graph $\text{Cay}(\text{Aut}(\otimes))$ of the group $\text{Aut}(\otimes)$ by taking the connection set as $\text{Aut}(\otimes)$ itself, where the vertices and arcs in $\text{Cay}(\text{Aut}(\otimes))$ are associated with the objects and morphisms, respectively, in $Y_{G,\otimes}$.

We shall prove the following proposition concerning the constructive functor $F_{G,e}$.

Proposition 2.9. *Let $F : Y \rightarrow G$ be a constructive functor on G .*

(i) *Denote by $F^{-1}(e)$ the preimage of the subset $\{e\} (\subseteq \text{Ob}(G))$ via the map $\text{Ob}(Y) \rightarrow \text{Ob}(G)$ induced by F . Then, the map of sets*

$$\text{ev}_{F,e} : \text{Mor}(F_{G,e}, F) \xrightarrow{\sim} F^{-1}(e)$$

obtained by assigning $\Phi \mapsto \Phi((e, \text{id}_e))$ is bijective.

(ii) *Suppose that Y is a connected groupoid. Then, any morphism of constructive functors $\Phi : F \rightarrow F_{G,e}$ is an isomorphism. In particular, any endomorphism of $F_{G,e}$ is an isomorphism.*

(iii) *Suppose that $\text{Ob}(Y) \neq \emptyset$. Then, any monomorphism of constructive functors $\Phi : F \hookrightarrow F_{G,e}$ is an isomorphism.*

Proof. First, we prove the surjectivity of $\text{ev}_{F,e}$ in assertion (i). Let e_Y be an element of $F^{-1}(e)$. In what follows, we shall construct a morphism of constructive functors $F_{G,e} \rightarrow F$ that is mapped to e_Y via $\text{ev}_{F,e}$. Let us take an arbitrary object (a, u) of $Y_{G,e}$. Since F is constructive, there exists a unique pair (a_Y, u_Y) consisting of an object a_Y in Y and a morphism $u_Y : e_Y \rightarrow a_Y$ with $F(u_Y) = u$. Next, let $v : (a, u) \rightarrow (a', u')$ be a morphism in $Y_{G,e}$. Denote by v_Y the unique lifting of v with domain a_Y . Since F is constructive, the morphism $u' (= v \circ u)$ lifts uniquely to a morphism in Y with domain e_Y . This implies $a'_Y = v_Y \circ a_Y$. Moreover, the lifting property on F again implies that $(\text{id}_a)_Y = \text{id}_{a_Y}$ and $(v' \circ v)_Y = v'_Y \circ v_Y$ for any morphism $v' : (a', u') \rightarrow (a'', u'')$. Thus, the assignments $(a, u) \mapsto a_Y$ and $v \mapsto v_Y$ define a morphism $\Phi_{e_Y} : F_{G,e} \rightarrow F$, and this morphism satisfies $\text{ev}_{F,e}(\Phi_{e_Y}) = e_Y$ by construction. This implies the surjectivity of $\text{ev}_{F,e}$.

Next, we prove the injectivity of $\text{ev}_{F,e}$. Let Φ be a morphism $F_{G,e} \rightarrow F$ and write $e_Y := \text{ev}_{F,e}(\Phi)$. The problem is reduced to proving the equality $\Phi = \Phi_{e_Y}$. To this end, let us take an arbitrary element (a, u) of $Y_{G,e}$. Since u defines a morphism $(e, \text{id}_e) \rightarrow (a, u)$ in $Y_{G,e}$, the image $\Phi(u)$ defines a morphism $e_Y (= \Phi((e, \text{id}_e))) \rightarrow \Phi((a, u))$. This morphism is a unique lifting of $u \in \text{Mor}(G)$ with domain e_Y , so we have $\Phi((a, u)) = \Phi_{e_Y}((a, u))$. Thus, we obtain the equality $\Phi = \Phi_{e_Y}$, as desired. This completes the proof of the bijectivity of $\text{ev}_{F,e}$.

We prove assertion (ii). From Proposition 2.7 and the fact that $Y_{G,e}$ is a connected groupoid, it suffices to prove the injectivities of $\text{Ob}(\Phi)$ and $\text{Mor}(\Phi)$. Suppose that there exists two objects c_Y, c'_Y in Y with $\Phi(c_Y) = \Phi(c'_Y) =: (c, v)$. Since Y is a connected groupoid, there exists an isomorphism $v_Y : c_Y \xrightarrow{\sim} c'_Y$ in Y . The set $\text{Mor}((c, v), (c, v))$ coincides with $\{\text{id}_v\}$, so the equality $\Phi(v_Y) = \text{id}_v$ holds. The equality $F = F_{G,e} \circ \Phi$ implies $F(v_Y) = \text{id}_Y$. But, by the lifting property on F , v_Y must be equal to id_{c_Y} , which implies $c_Y = c'_Y$. This completes the proof of the injectivity of $\text{Ob}(\Phi)$. A similar argument can be applied to prove the injectivity of $\text{Mor}(\Phi)$ (so we omit the details).

Finally, we prove assertion (iii). Let $\Phi : F \hookrightarrow F_{G,e}$ be a monomorphism. Since $Y \neq \emptyset$, there exists a connected groupoid G_0 in Y . The restriction $F|_{G_0} : G_0 \rightarrow G$ of F to G_0 forms a constructive functor on G . Then, it follows from assertion (ii) that the morphism $\Phi|_{G_0} : F|_{G_0} \rightarrow F_{G,e}$ obtained by restricting Φ is an isomorphism. Here, suppose that there exists an object $a \in \text{Ob}(Y) \setminus \text{Ob}(G_0)$. Denote by G_1 the groupoid in Y containing a (hence $\text{Ob}(G_0) \cap \text{Ob}(G_1) = \emptyset$). For the same reason as above, the restriction $\Phi|_{G_1} : F|_{G_1} \rightarrow F_{G,e}$ of Φ to G_1 forms an isomorphism of constructive functors. Thus, we obtain two distinct morphisms $(\Phi|_{G_0})^{-1}$, $(\Phi|_{G_1})^{-1}$ belonging to $\text{Mor}(F_{G,e}, F)$ that coincide with each other after composing with Φ . This contradicts the assumption that Φ is a monomorphism. This implies $\text{Ob}(G_0) = \text{Ob}(Y)$. Next, if there exists a morphism $u \in \text{Mor}(Y) \setminus \text{Mor}(G_0)$, then the two morphisms id_Y and $(\Phi|_{G_0})^{-1} \circ \Phi$ are distinct but identical to each other when composed with Φ . This is a contradiction, so the equality $\text{Mor}(Y) = \text{Mor}(G_0)$ must hold. This implies that $Y = G_0$ and $\Phi = \Phi|_{G_0}$. Thus, Φ turns out to be an isomorphism. \square

Let e' be an object in G and $w : e \rightarrow e'$ a morphism in G . Then, we obtain a morphism of constructive functors

$$\Phi_w := \text{ev}_{F_{G,e'}}^{-1}((e, w^{-1})) : F_{G,e} \rightarrow F_{G,e'}. \tag{2.3}$$

To be precise, this morphism is obtained by assigning $(a, u) \mapsto (a, u \circ w^{-1})$ (for each $(a, u) \in \text{Ob}(Y_{G,e})$) and $v \mapsto v$ (for each $v \in \text{Mor}(Y_{G,e})$). From Proposition 2.9, (ii), Φ_w turns out to be an isomorphism. The existence of Φ_w implies that the isomorphism class of the constructive functor $F_{G,e}$ does not depend on the choice of e . By considering the case of $e = e'$ and applying again Proposition 2.9, (ii), we see that the assignment $w \mapsto \Phi_w$ defines an isomorphism of groups

$$\phi_e := \text{ev}_{F_{G,e,e}}^{-1} : (F_{G,e}^{-1}(e) =) \text{Aut}(e) \rightarrow \text{Aut}_{\text{Con}_G}(F_{G,e}), \tag{2.4}$$

where $\text{Aut}_{\text{Con}_G}(F_{G,e})$ denotes the automorphism group of $F_{G,e}$ in Con_G .

Remark 2.10. One may verify that the full subcategory $\text{Con}_G^{\text{fin}}$ of Con_G consisting of constructive functors with finite fibers forms a Galois category (cf. [6], Exposé V, Théorème 4.1 and Définition 5.1) by setting $\text{ev}_{F,e}$ as the fiber functor. The above discussion shows that the resulting fundamental group $\pi_1(\text{Con}_G^{\text{fin}}; \text{ev}_{F,e})$ of $\text{Con}_G^{\text{fin}}$ with base point $\text{ev}_{F,e}$ is isomorphic to the profinite completion of the group $\text{Aut}(e)$. Hence, if G is a profinite groupoid, then its equivalence class can be reconstructed from the fundamental group $\pi_1(\text{Con}_G^{\text{fin}}; \text{ev}_{F,e})$. This fact is reminiscent of “Grothendieck conjecture”-type theorems in anabelian geometry (cf. Remark 3.11).

Remark 2.11. The cultural model, as well as the paradigm for knowledge and practice, in the modern Western world envisioned by the structuralism of Lévi-Strauss and others is characterized by an organizational structure of an *arborescence* system that looks for the origin of “things” and for the culmination or conclusion of those “things”. In this model, a small idea, like a seed, takes root and grows into a tree with a sturdy trunk supporting numerous branches, all linked to and traceable back to the original idea.⁴ The results of Proposition 2.9 together with the viewpoint of the previous Remark suggest that, in accordance with the picture of Lévi-Strauss’ structuralism, $F_{G,e}$ plays the role of a seed or a trunk in the system of structure species on G .

3 Proof of the main theorem

This section is devoted to proving Theorem 1.1. The nontrivial portion of that theorem is the implication (b) \Rightarrow (a), which asserts that the equivalence class of a category X may be characterized uniquely, up to a certain indeterminacy, from the categorical structure of $\mathcal{S}p_X$, or equivalently, of Con_X (cf. (2.3)).

In the following discussion, we will often speak of various things concerning Con_X as being “reconstructed (or characterized) category-theoretically”. By this, we mean that they are preserved by an arbitrary equivalence of categories $\text{Con}_X \xrightarrow{\sim} \text{Con}_{X'}$ (where X' is another \mathcal{U} -small category). For instance, the set of monomorphisms in Con_X may be *characterized category-theoretically* as the morphisms $\Phi : F \rightarrow F'$ such that, for any F'' , the map of sets $\text{Map}(F'', F) \rightarrow \text{Map}(F'', F')$ obtained by composing with Φ is injective. To simplify our notation, however, we will omit explicit mention of this equivalence $\text{Con}_X \xrightarrow{\sim} \text{Con}_{X'}$, of X' , and of the various “primed” objects and morphisms corresponding to the original objects and morphisms, respectively, in Con_X .

Our tactic for completing the proof of the implication (b) \Rightarrow (a) (i.e., recognizing the structure of X) is, as in [16], [17], and [23], to reconstruct step-by-step various partial information about X from the categorical structure of Con_X .

3.1 Reconstruction of the objects and their automorphisms

The first step of the proof is to reconstruct the set of isomorphism classes of objects in Con_X and their automorphism groups. Let us fix a \mathcal{U} -small category X . Also, let us fix a skeleton \bar{X} of X

⁴Due to criticism of how an object can be explained only with such a picture, a perspective that focuses (not only on the origin and conclusion but) on the process of its creation and the possibility of its change arose later. It led to the establishment of the position known today as *post-structuralism*. In this context, G. Deleuze and F. Guattari used the term “*rhizome*” to describe a process of existence and growth that does not come from a single central point of origin (cf. [5]).

(i.e., a full subcategory \overline{X} of X such that the inclusion $\overline{X} \hookrightarrow X$ is an equivalence of categories and no two distinct objects in \overline{X} are isomorphic).

If G and G' are distinct connected maximal groupoids in X , then we see that $\text{Ob}(G) \cap \text{Ob}(G') = \emptyset$. Moreover, for an object e of X , there exists a unique connected maximal groupoid G_e in X containing e . Thus, the assignment $e \mapsto G_e$ defines a bijective correspondence between $\text{Ob}(\overline{X})$ and the set of connected maximal groupoids in X . Also, we have a decomposition

$$\text{Ob}(X) = \coprod_G \text{Ob}(G),$$

where the disjoint union in the right-hand side runs over the set of connected maximal groupoids in X .

Now, let e be an object in \overline{X} . Denote by G the connected maximal groupoid in X containing e (hence $\text{Ob}(\overline{X}) \cap \text{Ob}(G) = \{e\}$) and by ι_G the natural inclusion $G \hookrightarrow X$. The assignment $F \mapsto \iota_{G*}(F) := \iota_G \circ F$ defines a fully faithful functor

$$\iota_{G*} : \text{Con}_G \rightarrow \text{Con}_X.$$

The following proposition can be proved immediately from the definitions of a constructive functor and ι_{G*} (so we will omit their proofs.)

Proposition 3.1. (i) *The functor ι_{G*} is left adjoint to the functor $\iota_G^* : \text{Con}_X \rightarrow \text{Con}_G$ (cf. (2.3)). This means that for $F \in \text{Ob}(\text{Con}_X)$ and $F' \in \text{Ob}(\text{Con}_G)$, there exists a functorial bijection*

$$\text{Mor}(\iota_{G*}(F'), F) \xrightarrow{\sim} \text{Mor}(F', \iota_G^*(F)). \quad (3.1)$$

(ii) *Let F be a constructive functor on X . Suppose that there exist a constructive functor F' on G and a morphism $F \rightarrow \iota_{G*}(F')$. Then, there exists a constructive functor F'' on G with $\iota_{G*}(F'') \cong F$.*

The following assertion is a direct consequence of assertion (ii) above.

Corollary 3.2. *Let F be a constructive functor on G . Then, F is minimal in Con_G if and only if $\iota_{G*}(F)$ is minimal in Con_X .*

We shall set

$$F_{G,e}^+ := \iota_{G*}(F_{G,e}) : Y_{G,e} \rightarrow X.$$

When there is no fear of confusion, we will write $F_G^+ := F_{G,e}^+$ for simplicity. (This abbreviation of the notation can be justified because of the existence of the isomorphism (2.5) and the fully faithfulness of ι_{G*} . In fact, these facts show that the isomorphism class of $F_{G,e}^+$ depends only on G ; i.e., it does not depend on the choice of the skeleton \overline{X} .)

Denote by

$$\dagger \text{Ob}(\overline{X}) (\subseteq \overline{\text{Ob}}(\text{Con}_X))$$

the set of isomorphism classes of objects in Con_X of the form F_G^+ for some connected groupoid G in X . If G and G' are connected maximal groupoids in X , then $F_G^+ \cong F_{G'}^+$ precisely when $G = G'$. This implies that the assignment from each element $e' \in \text{Ob}(\overline{X})$ to the constructive functor $F_{G'}^+$, where G' denotes the connected maximal groupoid in X containing e' , defines a bijection of sets

$$\xi_{\overline{X}} : \text{Ob}(\overline{X}) \xrightarrow{\sim} \dagger \text{Ob}(\overline{X}).$$

Proposition 3.3. *Let F be a constructive functor on X . Then, F is minimal in Con_X if and only if F is isomorphic to F_G^+ for some connected maximal groupoid G in X . In particular, the subset $\dagger \text{Ob}(\overline{X})$ of $\overline{\text{Ob}}(\text{Con}_X)$ can be reconstructed category-theoretically from the data Con_X (i.e., of a category).*

Proof. The “if” part of the required equivalence follows immediately from Proposition 2.9, (iii), and Corollary 3.2.

In what follows, we prove the “only if” part. Let $F : Y \rightarrow X$ be a minimal object in Con_X . Since this object is nonempty, there exists an object a in Y . Denote by G the connected maximal groupoid in X containing $F(a)$. By the bijection $\text{ev}_{F,F(a)}$ in Proposition 2.9, (i), together with the adjunction relation (3.1), we can find a morphism of constructive functors $\Phi : F_G^+ \rightarrow F$. It follows from Proposition 2.7 that the maps $\text{Ob}(\Phi) : \text{Ob}(Y_{G,e}) \rightarrow \text{Ob}(Y)$, $\text{Mor}(\Phi) : \text{Mor}(Y_{G,e}) \rightarrow \text{Ob}(Y)$ induced by Φ are surjective.

Next, suppose that we are given two objects $(a, u), (a, u')$ in $Y_{G,e}$ with $\Phi((a, u)) = \Phi((a, u'))$. To prove the equality $(a, u) = (a, u')$, we may assume, after possibly composing Φ with a suitable automorphism of F_G^+ , that $a = e$. Denote by v the unique endomorphism of e in G satisfying $\{v\} = \text{Mor}((e, u), (e, u'))$. Then, from the bijectivities of $\text{ev}_{F,e}$ and $\text{ev}_{F_{G,e}}$, it can immediately be seen that $v = \text{id}_e$, i.e., $(e, u) = (e, u')$. This implies the injectivity of $\text{Ob}(\Phi)$. Moreover, since $\text{Mor}((a, u), (b, v))$ is a singleton for any objects $(a, u), (b, v)$ in $Y_{G,e}$, the injectivity of $\text{Ob}(\Phi)$ implies that of $\text{Mor}(\Phi)$. Consequently, Φ turns out to be an isomorphism. This completes the proof of the “only if” part. \square

Let e and G be as above. Then, for each $w \in \text{Aut}(e)$, the automorphism Φ_w (cf. (2.5)) defines, via ι_{G*} , an automorphism Φ_w^+ of F_G^+ . By the bijectivity of ϕ_e (cf. (2.5)) and the fully faithfulness of ι_{G*} , the assignment $w \mapsto \Phi_w^+$ defines an isomorphism of groups

$$\xi_e^{\cong} : \text{Aut}(e) \xrightarrow{\sim} \dagger \text{Aut}(e), \quad (3.2)$$

where we set $\dagger \text{Aut}(e) := \text{Aut}_{\text{Con}_X}(F_G^+)$. The following assertion can be verified immediately.

Proposition 3.4. *The subset $\dagger \text{Aut}(e)$ of $\text{Mor}(\text{Con}_X)$ together with its group structure can be reconstructed category-theoretically from the data (Con_X, F_G^+) (i.e., of a category and a minimal object in this category).*

3.2 Reconstruction of the non-invertible morphisms

Suppose that we are given two distinct objects e, e' in \overline{X} . Denote by G and G' the connected maximal groupoids in X containing e and e' , respectively. Then, each non-invertible morphism $v \in \text{Mor}(e, e')$ (i.e., $v \in \text{Mor}(e, e')^{\neq}$) associates the category

$$Y_{G,G',v}, \text{ or simply, } Y_v,$$

defined as follows:

- $\text{Ob}(Y_{G,G',v}) = \text{Ob}(Y_{G,e}) \sqcup \text{Ob}(Y_{G',e'})$;
- $\text{Mor}(Y_{G,G',v}) = \text{Mor}(Y_{G,e}) \sqcup \text{Mor}(Y_{G',e'}) \sqcup \coprod_{\substack{(a,u) \in \text{Ob}(Y_{G,e}), \\ (a',u') \in \text{Ob}(Y_{G',e'})}} \text{Mor}((a, u), (a', u'))$, where $\text{Mor}((a, u), (a', u')) := \{u' \circ v \circ u^{-1}\}$ for each $(a, u) \in \text{Ob}(Y_{G,e}), (a', u') \in \text{Ob}(Y_{G',e'})$;
- The composition law for morphisms in $Y_{G,G',v}$ is defined in a natural manner.

Denote by

$$F_{G,G',v}^+ : Y_{G,G',v} \rightarrow X$$

the functor given by $(a, u) \mapsto a$ (for each $(a, u) \in \text{Ob}(Y_{G,G',v})$) and $w \mapsto w$ for each $w \in \text{Mor}(Y_{G,G',v})$. Then, $F_{G,G',v}^+$ forms a constructive functor on X , and the natural inclusions $F_{G,e}^+ \hookrightarrow F_{G,G',v}^+$ and $F_{G',e'}^+ \hookrightarrow F_{G,G',v}^+$ together yield a monomorphism

$$\Upsilon_{G,G',v} : F_{G,e}^+ \sqcup F_{G',e'}^+ \hookrightarrow F_{G,G',v}^+,$$

which is not invertible because of the existence of $v \in \text{Mor}(Y_{G,G',v})$. When there is no fear of confusion, we will write $F_v^+ := F_{G,G',v}^+$ and $\Upsilon_v := \Upsilon_{G,G',v}$.

Next, denote by

$$\mathcal{C}_{e,e'}$$

the category defined as follows:

- The objects in $\mathcal{C}_{e,e'}$ are pairs (F, Υ) consisting of a constructive functor $F : Y \rightarrow X$ on X and a non-invertible monomorphism $\Upsilon : F_{G,e}^+ \sqcup F_{G',e'}^+ \hookrightarrow F$ in Con_X .
- The morphisms from (F, Υ) to (F', Υ') are morphisms of constructive functors $\Psi : F \rightarrow F'$ satisfying $\Upsilon' = \Psi \circ \Upsilon$.

For each non-invertible morphism $v : e \rightarrow e'$ in \overline{X} , the pair $(F_{G,G',v}^+, \Upsilon_{G,G',v})$ introduced above specifies an object of $\mathcal{C}_{e,e'}$. The assignment $(F, \Upsilon) \mapsto F$ defines a functor

$$\mathcal{C}_{e,e'} \rightarrow \text{Con}_X. \quad (3.3)$$

By taking account of Proposition 3.3, we see that the category $\mathcal{C}_{e,e'}$ together with the functor (3.2) can be reconstructed category-theoretically from the data $(\text{Con}_X, F_{G,e}^+, F_{G',e'}^+)$ (i.e., of a category and two distinct minimal objects in this category). We identify $\mathcal{C}_{e,e'}$ with $\mathcal{C}_{e',e}$ via the equivalence of categories $\mathcal{C}_{e,e'} \xrightarrow{\sim} \mathcal{C}_{e',e}$ obtained by switching the factors $F_{G,e}^+ \sqcup F_{G',e'}^+ \xrightarrow{\sim} F_{G',e'}^+ \sqcup F_{G,e}^+$. Under this identification, $F_{G',G,v'}^+$ and $\Upsilon_{G',G,v'}$ (where $v' \in \text{Mor}(e', e)^\neq$) may be regarded as elements of $\text{Ob}(\mathcal{C}_{e,e'})$ and $\text{Mor}(\mathcal{C}_{e,e'})$, respectively.

Moreover, we denote by

$$\dagger \text{Mor}\{e, e'\}^\neq (\subseteq \text{Mor}(\text{Con}_X))$$

the set of morphisms in Con_X of the form Υ_v for some $v \in \text{Mor}\{e, e'\}^\neq$. Since $v \neq v'$ implies $(F_v^+, \Upsilon_v) \not\cong (F_{v'}^+, \Upsilon_{v'})$, the assignment $v \mapsto \Upsilon_v$ defines a bijection of sets

$$\xi_{e,e'}^\neq : \text{Mor}\{e, e'\}^\neq \xrightarrow{\sim} \dagger \text{Mor}\{e, e'\}^\neq. \quad (3.4)$$

Proposition 3.5. *Let (F, Υ) be an object in $\mathcal{C}_{e,e'}$. Then, the following two conditions are equivalent to each other:*

- (a) (F, Υ) is minimal in $\mathcal{C}_{e,e'}$ and there is no triple of morphisms

$$(\Upsilon_0 : F_{G,e}^+ \rightarrow F_0, \Upsilon'_0 : F_{G',e'}^+ \rightarrow F_0, \Phi : F_0 \sqcup F_0 \rightarrow F)$$

in Con_X such that Φ is an isomorphism and satisfies the equality $\Upsilon = \Phi \circ (\Upsilon_0 \sqcup \Upsilon'_0)$.

- (b) (F, Υ) is isomorphic to (F_v^+, Υ_v) for some $v \in \text{Mor}\{e, e'\}^\neq$.

In particular, the subset $\dagger \text{Mor}\{e, e'\}^\neq$ of $\text{Mor}(\text{Con}_X)$ can be reconstructed category-theoretically from the data $(\text{Con}_X, F_{G,e}^+, F_{G',e'}^+)$ (i.e., of a category and two distinct minimal objects in this category).

Proof. Since the implication (b) \Rightarrow (a) can be immediately verified from the definition of (F_v^+, Υ_v) , we only consider the inverse direction. Let $(F : Y \rightarrow X, \Upsilon)$ be an object in $\mathcal{C}_{e,e'}$ satisfying the conditions in (a). Denote by $\overline{Y}_{G,e}$ and $\overline{Y}_{G',e'}$ the subcategories in Y defined as the images via Υ of $Y_{G,e}$ and $Y_{G',e'}$, respectively. By the lifting property on F , $\overline{Y}_{G,e}$ and $\overline{Y}_{G',e'}$, respectively, specify connected maximal groupoids in Y . Now, suppose that there exists an object a_0 in $\text{Ob}(Y) \setminus (\text{Ob}(\overline{Y}_{G,e}) \sqcup \text{Ob}(\overline{Y}_{G',e'}))$. If G_0 denotes the connected maximal groupoid in Y containing a_0 , then the minimality of (F, Υ) implies that Y must be equal to the disjoint union $\overline{Y}_{G,e} \sqcup \overline{Y}_{G',e'} \sqcup G_0$. But this is a contradiction because of the second condition in (a). Hence, we have $\text{Ob}(Y) = \text{Ob}(\overline{Y}_{G,e}) \sqcup \text{Ob}(\overline{Y}_{G',e'})$. Since Υ is a monomorphism, the functors $Y_{G,e} \rightarrow \overline{Y}_{G,e}$, $Y_{G',e'} \rightarrow \overline{Y}_{G',e'}$ obtained by restricting Υ are an isomorphism (cf. the proof of Proposition 3.3); by using these isomorphisms, we consider $Y_{G,e} \sqcup Y_{G',e'}$ to be a subcategory of Y . From the second condition in (a), there exists objects $a \in \text{Ob}(\overline{Y}_{G,e})$, $a' \in \text{Ob}(\overline{Y}_{G',e'})$ such that $\text{Mor}\{a, a'\} \neq \emptyset$. Hence, the subcategory $Y_{G,e} \sqcup Y_{G',e'}$ of Y extends to a subcategory of the form $Y_{G,G',v}$ for some $v \in \text{Mor}\{e, e'\}$. It follows from the minimality of (F, Υ) that the inclusion $Y_{G,G',v} \hookrightarrow Y$ must be an isomorphism. This completes the proof of the implication (a) \Rightarrow (b). \square

3.3 Reconstruction of the composition law

Let e, e', G , and G' be as above. We shall set

$$\begin{aligned} \text{Com}(e, e')^{\cong} &:= \left\{ (u, u', u'') \in \text{Aut}(e) \times \text{Mor}(e, e')^{\neq} \times \text{Mor}(e, e')^{\neq} \mid u'' = u' \circ u \right\}, \\ \text{Com}(e, e')_{\text{op}}^{\cong} &:= \left\{ (u, u', u'') \in \text{Aut}(e) \times \text{Mor}(e', e)^{\neq} \times \text{Mor}(e', e)^{\neq} \mid u'' = u \circ u' \right\}. \end{aligned}$$

We will prove the following assertion.

Proposition 3.6. *Let us choose $u \in \text{Aut}(e)$ and $v_1, v_2 \in \text{Mor}\{e, e'\}^{\neq}$. Then, the following two conditions are equivalent to each other:*

- (a) *Either $v_2 = v_1 \circ u$ or $v_2 = u \circ v_1$ holds.*
- (b) *Either $\Upsilon_{v_2} = \Upsilon_{v_1} \circ (\text{id}_{F_G^+} \sqcup \Phi_{u^{-1}})$ or $\Upsilon_{v_2} = \Upsilon_{v_1} \circ (\text{id}_{F_G^+} \sqcup \Phi_u)$ holds.*

In particular the image of $\text{Com}(e, e')^{\cong} \sqcup \text{Com}(e, e')_{\text{op}}^{\cong}$ via the bijection

$$\begin{aligned} \xi_{e, e, e'}^{\cong} &:= \xi_e^{\cong} \times \xi_{e, e'}^{\neq} \times \xi_{e, e'}^{\neq} : \text{Aut}(e) \times \text{Mor}\{e, e'\}^{\neq} \times \text{Mor}\{e, e'\}^{\neq} \\ &\xrightarrow{\cong} \dagger \text{Aut}(e) \times \dagger \text{Mor}\{e, e'\}^{\neq} \times \dagger \text{Mor}\{e, e'\}^{\neq} \end{aligned}$$

can be reconstructed category-theoretically from the data $(\text{Con}_X, F_{G, e}^+, F_{G', e'}^+)$, i.e., of a category and two distinct minimal objects in this category (cf. Proposition 3.4 for the category-theoretic reconstruction of $\dagger \text{Aut}(-)$).

Proof. The assertion follows from the various definitions involved. □

Let G, G', G'' be connected maximal groupoids in X . Denote by e, e' , and e'' the objects of \bar{X} belonging to G, G' , and G'' , respectively. Moreover, set

$$\begin{aligned} \text{Com}(e, e', e'')^{\neq} &:= \left\{ (u, u', u'') \in \text{Mor}(e, e')^{\neq} \times \text{Mor}(e', e'')^{\neq} \times \text{Mor}(e, e'')^{\neq} \mid u'' = u' \circ u \right\}, \\ \text{Com}(e, e', e'')_{\text{op}}^{\neq} &:= \left\{ (u, u', u'') \in \text{Mor}(e', e)^{\neq} \times \text{Mor}(e'', e')^{\neq} \times \text{Mor}(e'', e)^{\neq} \mid u'' = u \circ u' \right\}. \end{aligned}$$

We will prove the following assertion.

Proposition 3.7. *Suppose that we are given morphisms $v \in \text{Mor}\{e, e'\}^{\neq}$, $v' \in \text{Mor}\{e', e''\}^{\neq}$, and $v'' \in \text{Mor}\{e, e''\}^{\neq}$. Then, the following two conditions are equivalent to each other*

- (a) *Either $v \circ v'$ or $v' \circ v$ can be defined and one of the equalities $v'' = v \circ v'$, $v'' = v' \circ v$ holds.*
- (b) *The colimit F_{limit}^+ of the diagram*

$$\begin{array}{ccc} F_{G'}^+ & \xrightarrow{\text{inclusion}} & F_{G', G'', v''}^+ \\ \text{inclusion} \downarrow & & \\ & & F_{G, G', v}^+ \end{array}$$

exists in Con_X , and there exists a monomorphism $F_{G, G'', v''}^+ \hookrightarrow F_{\text{limit}}^+$ in Con_X via which the morphism $\Upsilon_{v''} : F_{G, e}^+ \sqcup F_{G'', e''}^+ \hookrightarrow F_{v''}^+$ is compatible with the morphism $F_{G, e}^+ \sqcup F_{G'', e''}^+ \hookrightarrow F_{\text{limit}}^+$ arising naturally from Υ_v and $\Upsilon_{v'}$.

In particular, the image of $\text{Com}(e, e', e'')^{\neq} \sqcup \text{Com}(e, e', e'')_{\text{op}}^{\neq}$ via the bijection

$$\begin{aligned} \xi_{e, e', e''}^{\neq} &:= \xi_{e, e'}^{\neq} \times \xi_{e', e''}^{\neq} \times \xi_{e, e''}^{\neq} : \text{Mor}\{e, e'\}^{\neq} \times \text{Mor}\{e', e''\}^{\neq} \times \text{Mor}\{e, e''\}^{\neq} \\ &\xrightarrow{\sim} \dagger\text{Mor}\{e, e'\}^{\neq} \times \dagger\text{Mor}\{e', e''\}^{\neq} \times \dagger\text{Mor}\{e, e''\}^{\neq} \end{aligned}$$

can be reconstructed category-theoretically from the data $(\text{Con}_X, F_{G, e}^+, F_{G', e'}^+, F_{G'', e''}^+)$, i.e., a category and three distinct minimal objects in this category (cf. Proposition 3.5 for the category-theoretic reconstruction of $\dagger\text{Mor}\{-, -\}^{\neq}$).

Proof. The assertion follows from the various definitions involved. \square

Remark 3.8. Let $a \in \text{Mor}(e, e')$ and $b \in \text{Mor}\{e', e''\}$. Then, it follows from the above proposition that, by means of the image of $\text{Com}(e, e', e'')^{\neq} \sqcup \text{Com}(e, e', e'')_{\text{op}}^{\neq}$ via $\xi_{e, e', e''}^{\neq}$, one can recognize whether or not the composite $b \circ a$ can be defined in \bar{X} (i.e., the codomain of a coincides with the domain of b).

Now let us combine the previous two propositions (and Proposition 3.4). Let $G, G', G'', e, e',$ and e'' be as above. Moreover, we will write

$$\begin{aligned} \text{Com}(e, e', e'') &:= \left\{ (u, u', u'') \in \text{Mor}(e, e') \times \text{Mor}(e', e'') \times \text{Mor}(e, e'') \mid u'' = u' \circ u \right\}, \\ \text{Com}(e, e', e'')_{\text{op}} &:= \left\{ (u, u', u'') \in \text{Mor}(e', e) \times \text{Mor}(e'', e') \times \text{Mor}(e'', e) \mid u'' = u \circ u' \right\}. \end{aligned}$$

That is to say, $\text{Com}(e, e', e'')$ and $\text{Com}(e, e', e'')_{\text{op}}$ are defined as the graphs of the composition law defining \bar{X} .

For each $e, e' \in \text{Ob}(\bar{X})$, we write

$$\dagger\text{Mor}\{e, e'\} = \begin{cases} \dagger\text{Aut}(e) \sqcup \dagger\text{Mor}\{e, e\}^{\neq} & \text{if } e = e', \\ \dagger\text{Mor}\{e, e\}^{\neq} & \text{if } e \neq e'. \end{cases}$$

If $e = e'$ (resp., $e \neq e'$) in \bar{X} , then we have $\text{Mor}\{e, e'\} = \text{Aut}(e) \sqcup \text{Mor}\{e, e\}^{\neq}$ (resp., $\text{Mor}\{e, e'\} = \text{Mor}\{e, e\}^{\neq}$). Hence, the bijections $\xi_{(-)}^{\neq}$ (cf. (3.1)) and $\xi_{(-), (-)}^{\neq}$ (cf. (3.2)) together yield a bijection

$$\xi_{e, e'} : \text{Mor}\{e, e'\} \xrightarrow{\sim} \dagger\text{Mor}\{e, e'\}.$$

Propositions 3.4, 3.6, and 3.7 imply the following assertion.

Corollary 3.9. *The image of $\text{Com}(e, e', e'') \sqcup \text{Com}(e, e', e'')_{\text{op}}$ via the bijection*

$$\begin{aligned} \xi_{e, e', e''} &:= \xi_{e, e'} \times \xi_{e', e''} \times \xi_{e, e''} : \text{Mor}\{e, e'\} \times \text{Mor}\{e', e''\} \times \text{Mor}\{e, e''\} \\ &\xrightarrow{\sim} \dagger\text{Mor}\{e, e'\} \times \dagger\text{Mor}\{e', e''\} \times \dagger\text{Mor}\{e, e''\} \end{aligned}$$

can be reconstructed category-theoretically from the data $(\text{Con}_X, F_{G, e}^+, F_{G', e'}^+, F_{G'', e''}^+)$, i.e., of a category and three distinct minimal objects in this category (cf. Propositions 3.4 and 3.5 for the category-theoretic reconstruction of $\dagger\text{Mor}\{-, -\}$).

3.4 Proof of Theorem 1.1

We can now prove Theorem 1.1 by applying the results obtained so far. First, let us prove the implication (a) \Rightarrow (b). To this end, it suffices to show that $\mathcal{S}p_X \cong \mathcal{S}p_{X^{\text{op}}}$, or equivalently $\text{Con}_X \cong \text{Con}_{X^{\text{op}}}$ (cf. (2.3)). But this fact can be verified because the assignment from each constructive functor $F : Y \rightarrow X$ to $F^{\text{op}} : Y^{\text{op}} \rightarrow X^{\text{op}}$ yields an equivalence of categories $\delta_X : \text{Con}_X \xrightarrow{\sim} \text{Con}_{X^{\text{op}}}$.

Next, let us consider the inverse direction (b) \Rightarrow (a). Suppose that there exists an equivalence of categories $\mathcal{S}p_X \xrightarrow{\sim} \mathcal{S}p_{X'}$; this equivalence together with (2.3) determines an equivalence of

categories $\alpha : \text{Con}_X \xrightarrow{\sim} \text{Con}_{X'}$. Let us fix skeletons of X and X' , which we denote by \overline{X} and \overline{X}' respectively. By Proposition 3.3, α induces, via $\xi_{\overline{X}}$ and $\xi_{\overline{X}'}$, a bijection

$$\overline{\alpha} : \text{Ob}(\overline{X}) \xrightarrow{\sim} \text{Ob}(\overline{X}').$$

It follows from Propositions 3.4 and 3.5 that, for each $e, e' \in \text{Ob}(\overline{X})$, the equivalence α induces, via $\xi_{e,e'}$ and $\xi_{\overline{\alpha}(e),\overline{\alpha}(e')}$, a bijection

$$\overline{\alpha}_{e,e'} : \text{Mor}\{e, e'\} \xrightarrow{\sim} \text{Mor}\{\overline{\alpha}(e), \overline{\alpha}(e')\}.$$

Moreover, by applying Corollary 3.9 and using $\xi_{e,e',e''}$ and $\xi_{\overline{\alpha}(e),\overline{\alpha}(e'),\overline{\alpha}(e')}$, we obtain a bijection

$$\begin{aligned} \overline{\alpha}_{e,e',e''} : \text{Com}(e, e', e'') \sqcup \text{Com}(e, e', e'')_{\text{op}} \\ \xrightarrow{\sim} \text{Com}(\overline{\alpha}(e), \overline{\alpha}(e'), \overline{\alpha}(e'')) \sqcup \text{Com}(\overline{\alpha}(e), \overline{\alpha}(e'), \overline{\alpha}(e''))_{\text{op}} \end{aligned}$$

for any $e, e', e'' \in \text{Ob}(\overline{X})$. If \overline{X} has only one object e , then the bijections $\overline{\alpha}_e$, $\overline{\alpha}_{e,e}$, and $\overline{\alpha}_{e,e,e}$ show that $\overline{X} \cong \overline{X}'$, which implies $X \cong X'$. Hence, it suffices to consider the case where $\sharp\text{Ob}(\overline{X}) \geq 2$. Since X is assumed to be connected, there exists a morphism $u_0 : e_0 \rightarrow e'_0$ in \overline{X} with $e_0 \neq e'_0$.

Here, suppose that $\overline{\alpha}_{e_0,e'_0}(u_0) \in \text{Mor}(\overline{\alpha}(e_0), \overline{\alpha}(e'_0))$. By the connectedness of X again, one may verify (cf. Remark 3.8) that $\overline{\alpha}_{e,e'}$ is restricted to a bijection

$$\overline{\alpha}_{e,e'}^{\circledast} : \text{Mor}(e, e') \xrightarrow{\sim} \text{Mor}(\overline{\alpha}(e), \overline{\alpha}(e'))$$

for any $e, e' \in \text{Ob}(\overline{X})$. Moreover, $\overline{\alpha}_{e,e',e''}$ is restricted to a bijection

$$\overline{\alpha}_{e,e',e''}^{\circledast} : \text{Com}(e, e', e'') \xrightarrow{\sim} \text{Com}(\overline{\alpha}(e), \overline{\alpha}(e'), \overline{\alpha}(e''))$$

for any $e, e', e'' \in \text{Ob}(\overline{X})$. The bijections $\overline{\alpha}$, $\overline{\alpha}_{e,e'}^{\circledast}$, and $\overline{\alpha}_{e,e',e''}^{\circledast}$ for various e, e', e'' together yield an equivalence of categories $\overline{X} \xrightarrow{\sim} \overline{X}'$. Thus, we conclude that $X \cong X'$.

On the other hand, suppose that $\overline{\alpha}_{e_0,e'_0}(u_0) \in \text{Mor}(\overline{\alpha}(e'_0), \overline{\alpha}(e_0))$. Write $\alpha' := \delta_{X'} \circ \alpha$, where $\delta_{(-)}$ denotes the equivalence of categories constructed in the proof of (a) \Rightarrow (b). Then, we have $\overline{\alpha}'_{e_0,e'_0}(u_0) \in \text{Mor}_{\overline{X}'^{\text{op}}}(\overline{\alpha}(e_0), \overline{\alpha}(e'_0))$. By applying the above discussion to α' , we conclude that $X \cong X'^{\text{op}}$ (or equivalently, $X^{\text{op}} \cong X'$). This completes the proof of the implication (b) \Rightarrow (a). Thus, we have finished the proof of Theorem 1.1.

Remark 3.10. The above proof shows that, in order to reach the conclusion, it suffices to assume the connectedness only for either X or X' . In other words, the connectedness of X can be characterized category-theoretically by the category $\mathcal{S}p_X$.

Remark 3.11. The reconstruction carried out in Theorem 1.1 is reminiscent of ‘‘Grothendieck conjecture’’-type theorems in anabelian geometry. Anabelian geometry is, roughly speaking, an area of arithmetic geometry that discusses the issue of how much information concerning the geometry of certain arithmetic varieties (e.g., hyperbolic curves over a number field or a p -adic local field) is contained in the knowledge of the étale fundamental groups, or equivalently, the categories of finite étale coverings.

The classical point of view of anabelian geometry centers around a comparison between two arithmetic varieties (or more generally, two geometric objects of the same kind) via their étale fundamental groups and it is referred to as *bi-anabelian geometry*. On the other hand, *mono-anabelian geometry*, being an alternative and relatively new formulation, centers around the task of establishing a group-theoretic algorithm whose input data consists of a single abstract topological group isomorphic to the étale fundamental group of a single geometric object. In particular, it requires us to reconstruct, unlike the bi-anabelian formulation, the desired data without any mention of some ‘‘fixed reference model’’ copy of initial objects. For basic references, we refer the reader to [8], [18].

As explicitly verbalized in [22], § 3, constructive functors on a category X may be regarded, in some sense, as categorical realizations of coverings over X . In fact, if X is a groupoid G , then

the results of Proposition 2.9 enable us to consider the constructive functor $F_{G,e} : Y_{G,e} \rightarrow G$ as if it were the universal covering of G . Accordingly, we might say that $\mathcal{S}p_X$ ($\cong \mathcal{C}on_X$ by (2.3)) is like the Galois category consisting of finite étale coverings. Notwithstanding the fact that Theorem 1.1 is stated in a completely bi-abelian way, the proof of the theorem actually furnishes a mono-abelian algorithm that reconstructs X from $\mathcal{S}p_X$. Here, however, we will omit the details of the formulation, as well as the proof, in that way.

Remark 3.12. C. Lévi-Strauss developed a structuralist theory of mythology which attempted to explain how seemingly fantastical and arbitrary tales could be so similar across cultures (cf. [12], [13]). Because he believed that there was no one authentic version of a myth, rather that they were all manifestations of the same language, he sought to find the fundamental units of myth, namely, the *mytheme*.

The canonical formula of mythical transformation is an expression proposed in 1955 by Lévi-Strauss in order to account for the abstract relations occurring between characters and their attributes in a myth understood as the collection of its variants. According to the canonical formula, a myth is reducible to an expression:

$$f_x(a) : f_y(b) \cong f_x(b) : f_{a^{-1}}(y), \quad (3.5)$$

where each of these four arguments consist of a term variable (a and b), and a function variable (x and y). This formula describes a structural relationship between a set of narrative terms and their transmutative relationships. See [15] for a reference on this formula.

Note that J. Morava tried to develop a truly mathematical argument for the canonical formula (cf. [19], [20]). He proposed to interpret it as the existence of an anti-isomorphism of the quaternion group. On the other hand, as a mathematical study of myths in another direction, we might expect that some kind of symmetry or structure on the whole category of structure species satisfying the condition expressed by (3.12) serves as a metaphorical explanation of some truth that Lévi-Strauss expected from myths. However, at the time of writing the present paper, the author does not have any effective ideas for the development of this argument.

4 Appendix: Category of categories over a category

In this Appendix, we establish, as an analogy of Theorem 1.1, the reconstruction of a category X from the category of categories over X , i.e., the category of functors $F : Y \rightarrow X$ (that are not necessarily a structure species). The conclusion differs from the case of Theorem 1.1 in that we can reconstruct (up to a certain indeterminacy) not only the equivalence class of X but also its isomorphism class. The following Proposition 4.1 (resp., 4.2; resp., 4.3) will be proved by using a much simpler argument than the proof in the corresponding previous assertion, i.e., Proposition 3.3 (resp., 3.5; resp., 3.7). So we will leave their proofs to the reader.

Let X be a \mathcal{U} -small category. Denote by

$$\mathcal{C}at_X$$

the category defined as follows:

- The objects are functors of the form $F : Y \rightarrow X$, where Y is a \mathcal{U} -small category;
- The morphisms from $F : Y \rightarrow X$ to $F' : Y' \rightarrow X$ are functors $\Psi : Y \rightarrow Y'$ satisfying $F = F' \circ \Psi$.

The identity functor $\text{id}_X : X \xrightarrow{\sim} X$ is a terminal object in this category.

For each object e in X , we shall write $Y_e := \mathcal{D}is(\{e\})$ and write

$$F_e : Y_e \hookrightarrow X$$

for the natural functor; this functor specifies an object in $\mathcal{C}at_X$. We denote by

$$\sharp \text{Ob}(X) (\subseteq \text{Ob}(\mathcal{C}at_X))$$

the set of objects in $\mathcal{C}at_X$ of the form F_e for some $e \in \text{Ob}(X)$. The assignment $e \mapsto F_e$ gives a bijection of sets

$$\zeta_X : \text{Ob}(X) \xrightarrow{\sim} \sharp \text{Ob}(X).$$

Proposition 4.1. *Let $F : Y \rightarrow X$ be an object in Cat_X . Then, F is minimal in Cat_X if and only if F is isomorphic to F_e for some $e \in \text{Ob}(X)$. In particular, the subset ${}^\ddagger\text{Ob}(X)$ of $\text{Ob}(Cat_X)$ can be reconstructed category-theoretically from the data Cat_X (i.e., of a category).*

Next, let e and e' be (possibly the same) objects in X . For each morphism $v : e \rightarrow e'$ in X , we shall set Y_v to be the subcategory of X satisfying $\text{Ob}(Y_v) = \{e, e'\}$ and $\text{Mor}(Y_v) = \{\text{id}_e, \text{id}_{e'}, v\}$. Then, we obtain the natural inclusion

$$F_v : Y_v \hookrightarrow X.$$

The inclusions $Y_e \hookrightarrow Y_v$ and $Y_{e'} \hookrightarrow Y_v$ induce the morphism

$$\Upsilon_v : F_e \sqcup F_{e'} \rightarrow F_v$$

in Cat_X .

We shall denote by

$$\mathcal{D}_{e,e'}$$

the category defined as follows:

- The objects in $\mathcal{D}_{e,e'}$ are pair (F, Υ) consisting of an object $F : Y \rightarrow X$ in Cat_X and a non-invertible monomorphism $\Upsilon : F_e \sqcup F_{e'} \hookrightarrow F$;
- The morphisms from (F, Υ) to (F', Υ') are morphisms $\Psi : F \rightarrow F'$ satisfying $\Upsilon' = \Psi \circ \Upsilon$.

For each element $v \in \text{Mor}(e, e')$, the pair (F_v, Υ_v) introduced above specifies an object in $\mathcal{D}_{e,e'}$. The assignment $(F, \Upsilon) \mapsto \Upsilon$ defines a functor

$$\mathcal{D}_{e,e'} \rightarrow Cat_X. \quad (4.1)$$

It follows from Proposition 4.1 that the category $\mathcal{D}_{e,e'}$ together with the functor (4) can be reconstructed category-theoretically from the data $(Cat_X, F_e, F_{e'})$ (i.e., of a category and two minimal objects in this category). Also, as in the case of " $\mathcal{C}_{e,e'}$ " introduced in § 3.2, there exists a natural identification $\mathcal{D}_{e,e'} = \mathcal{D}_{e',e}$, by which we will not distinguish between $\mathcal{D}_{e,e'}$ and $\mathcal{D}_{e',e}$. In particular, this identification allows us to consider (F_w, Υ_w) 's for various $w \in \text{Mor}(e', e)$ as objects in $\mathcal{D}_{e,e'}$. We denote by

$${}^\ddagger\text{Mor}\{e, e'\}$$

the set of morphisms in Cat_X of the form Υ_v for some $v \in \text{Mor}\{e, e'\}$. The assignment $v \mapsto \Upsilon_v$ defines a bijection of sets

$$\zeta_{e,e'} : \text{Mor}\{e, e'\} \xrightarrow{\sim} {}^\ddagger\text{Mor}\{e, e'\}.$$

Proposition 4.2. *Let (F, Υ) be an object in Cat_X . The following two conditions are equivalent to each other:*

- (F, Υ) is minimal in $\mathcal{D}_{e,e'}$ and there is no triple of morphisms

$$(\Upsilon_0 : F_e \rightarrow F_0, \Upsilon'_0 : F_{e'} \rightarrow F'_0, \Psi : F_0 \sqcup F'_0 \rightarrow F)$$

in Cat_X such that Ψ is an isomorphism and satisfies the equality $\Upsilon = \Psi \circ (\Upsilon_0 \sqcup \Upsilon'_0)$.

- (F, Υ) is isomorphic to (F_v, Υ_v) for some $v \in \text{Mor}\{e, e'\}$.

In particular, the subset ${}^\ddagger\text{Mor}\{e, e'\}$ of $\text{Mor}(Cat_X)$ can be reconstructed category-theoretically from the data $(Cat_X, F_e, F_{e'})$ (i.e., of a category and two minimal objects in this category).

Given three (possibly the same) objects e, e', e'' in X , we shall write

$$\text{Com}(e, e', e'')^{Cat} := \left\{ (u, u', u'') \in \text{Mor}(e, e') \times \text{Mor}(e', e'') \times \text{Mor}(e, e'') \mid u'' = u' \circ u \right\},$$

$$\text{Com}(e, e', e'')_{\text{op}}^{Cat} := \left\{ (u, u', u'') \in \text{Mor}(e', e) \times \text{Mor}(e'', e') \times \text{Mor}(e'', e) \mid u'' = u \circ u' \right\}.$$

Then, we have the following assertion.

Proposition 4.3. *Let us choose $u \in \text{Mor}\{e, e'\}$, $u' \in \text{Mor}\{e', e''\}$, and $u'' \in \text{Mor}\{e, e''\}$. Then, the following two conditions are equivalent to each other:*

- *Either $u \circ u'$ or $u' \circ u$ can be defined and one of the equalities $u'' = u \circ u'$, $u'' = u' \circ u$ holds.*
- *The colimit $F_{\text{limit}}^{\text{Cat}}$ of the diagram*

$$\begin{array}{ccc} F_e & \xrightarrow{\text{inclusion}} & F_{u'} \\ \text{inclusion} \downarrow & & \\ & & F_u \end{array}$$

exists in Cat_X , and there exists a monomorphism $F_{u''} \hookrightarrow F_{\text{limit}}^{\text{Cat}}$ in Cat_X via which the morphism $\Upsilon_{u''} : F_e \sqcup F_{e''} \hookrightarrow F_{u''}$ is compatible with the morphism $F_e \sqcup F_{e''} \hookrightarrow F_{\text{limit}}^{\text{Cat}}$ arising naturally from Υ_u and $\Upsilon_{u'}$.

In particular, the image of $\text{Com}(e, e', e'')^{\text{Cat}} \sqcup \text{Com}(e, e', e'')_{\text{op}}^{\text{Cat}}$ via the bijection

$$\begin{aligned} \zeta_{e, e', e''} &:= \zeta_{e, e'} \times \zeta_{e', e''} \times \zeta_{e, e''} : \text{Mor}\{e, e'\} \times \text{Mor}\{e', e''\} \times \text{Mor}\{e, e''\} \\ &\xrightarrow{\sim} \sharp \text{Mor}\{e, e'\} \times \sharp \text{Mor}\{e', e''\} \times \sharp \text{Mor}\{e, e''\} \end{aligned}$$

can be reconstructed category-theoretically from the data $(\text{Cat}_X, F_e, F_{e'}, F_{e''})$ (i.e., of a category and three minimal objects in this category).

The previous three propositions enable us to prove the following assertion. (The proof is entirely similar to the proof of Theorem 1.1.)

Theorem 4.4. *Let X and X' be connected \mathcal{U} -small categories. Then, the following conditions are equivalent to each other:*

- $X \xrightarrow{\text{isom}} X'$ or $X^{\text{op}} \xrightarrow{\text{isom}} X'$.
- $\text{Cat}_X \cong \text{Cat}_{X'}$.

Proof. The implication (a) \Rightarrow (b) follows immediately from the existence of an equivalence of categories $\text{Cat}_X \xrightarrow{\sim} \text{Cat}_{X^{\text{op}}}$ obtained by assigning $F \mapsto F^{\text{op}}$ for each $F \in \text{Cat}_X$.

Next, we shall consider the inverse direction (b) \Rightarrow (a). Suppose that there exists an equivalence of categories $\beta : \text{Cat}_X \xrightarrow{\sim} \text{Cat}_{X'}$. By Proposition 4.1, β induces, via ζ_X and $\zeta_{X'}$, a bijection

$$\bar{\beta} : \text{Ob}(X) \xrightarrow{\sim} \text{Ob}(X').$$

It follows from Proposition 4.2 that, for each $e, e' \in \text{Ob}(X)$, the equivalence β induces, via $\zeta_{e, e'}$ and $\zeta_{\bar{\beta}(e), \bar{\beta}(e')}$, a bijection of sets

$$\bar{\beta}_{e, e'} : \text{Mor}\{e, e'\} \xrightarrow{\sim} \text{Mor}\{\bar{\beta}(e), \bar{\beta}(e')\}.$$

Moreover, by applying Proposition 4.3 and using the bijections $\zeta_{e, e', e''}$ and $\zeta_{\bar{\beta}(e), \bar{\beta}(e'), \bar{\beta}(e'')}$, we obtain a bijection

$$\begin{aligned} \bar{\beta}_{e, e', e''} &: \text{Com}(e, e', e'')^{\text{Cat}} \sqcup \text{Com}(e, e', e'')_{\text{op}}^{\text{Cat}} \\ &\xrightarrow{\sim} \text{Com}(\bar{\beta}(e), \bar{\beta}(e'), \bar{\beta}(e''))^{\text{Cat}} \sqcup \text{Com}(\bar{\beta}(e), \bar{\beta}(e'), \bar{\beta}(e''))_{\text{op}}^{\text{Cat}} \end{aligned}$$

for any $e, e', e'' \in \text{Ob}(X)$. If X has only one object e , then the bijections $\bar{\beta}$, $\bar{\beta}_{e, e}$, and $\bar{\beta}_{e, e, e}$ show that $X \xrightarrow{\text{isom}} X'$. Hence, it suffices to consider the case where $\sharp \text{Ob}(X) \geq 2$. Since X is assumed to be connected, there exists a morphism $u_0 : e_0 \rightarrow e'_0$ in X with $e_0 \neq e'_0$.

Here, suppose that $\bar{\beta}_{e_0, e'_0}(u_0) \in \text{Mor}(\bar{\beta}(e_0), \bar{\beta}(e'_0))$. From the connectedness of X again, one may verify (for the same reason as stated in the comment on Remark 3.8) that $\bar{\beta}_{e, e'}$ is restricted to a bijection

$$\bar{\beta}_{e, e'}^{\circledast} : \text{Mor}(e, e') \xrightarrow{\sim} \text{Mor}(\bar{\beta}(e), \bar{\beta}(e'))$$

for any $e, e' \in \text{Ob}(X)$. Moreover, $\bar{\beta}_{e, e', e''}$ is restricted to a bijection

$$\bar{\beta}_{e, e', e''} : \text{Com}(e, e', e'')^{\text{Cat}} \xrightarrow{\sim} \text{Com}(\bar{\beta}(e), \bar{\beta}(e'), \bar{\beta}(e''))^{\text{Cat}}$$

for any $e, e', e'' \in \text{Ob}(X)$. The bijections $\bar{\beta}$, $\bar{\beta}_{e, e'}^{\circledast}$, and $\bar{\beta}_{e, e', e''}$ for various $e, e', e'' \in \text{Ob}(X)$ yield an isomorphism of categories $X \xrightarrow{\sim} X'$.

On the other hand, if $\bar{\beta}_{e_0, e'_0}(u_0) \in \text{Mor}(\bar{\beta}(e'_0), \bar{\beta}(e_0))$, then the problem, as in the proof of Theorem 1.1, is reduced to the previous case by using the equivalence of categories $\text{Cat}_{X'} \xrightarrow{\sim} \text{Cat}_{X'^{\text{op}}}$ given by $F \mapsto F^{\text{op}}$. At any rate, we conclude that $X \cong^{\text{isom}} X'$ or $X^{\text{op}} \cong^{\text{isom}} X'$, as desired. This completes the implication (b) \Rightarrow (a). \square

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