**Palestine Journal of Mathematics**

**Vol 13(3)(2024)** , 125–132 

# MINIMAL SPECTRUM OF LATTICE MODULES

S. Mitkari, R. Pathak, N. Phadatare and M. Agalave

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 06D10; Secondary 06E10.

Keywords and phrases: Minimal element; Second element; $min$-radical element; $Max$-spectral space.

*The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.*

**Corresponding Author: N. Phadatare**

**Abstract** Let $L$ be a $C$-lattice. The minimal spectrum $\Sigma(M)$ of a lattice module $M$ over $L$ is the collection of all minimal elements of $M$. In this paper, we study the Zariski topology on $\Sigma(M)$ induced by topology of the second spectrum $Spec^s(M)$ on $M$. The linkage between topological assets of $\Sigma(M)$ and the algebraic properties of $M$ are established. Further, we studied various properties under which $\Sigma(M)$ is $Max$-*spectral*.

## 1 Introduction

H. Ansari-Toroghy et. al. in [2], studied algebraic properties of a ring module with the assistance of topological properties of the minimal spectrum of same module. In lattice modules theory, the notion of minimal element is an abstraction of minimal submodule in a module over ring. In [18], N. Phadatare and V. Kharat used minimal elements in a lattice module $M$ over $L$ to study the second radical elements of $M$. In this paper, we extend the study of N. Phadatare and V. Kharat and obtained topological characterizations of minimal spectrum $\Sigma(M)$ of a lattice module $M$.

For further development in the study of ideal theory of commutative ring $R$, a new structure called multiplicative lattices was framed by M. Ward and R. Dilworth( see [23]).

A complete lattice $L$ is called a *multiplicative lattice* if $L$ has least elements $0_L$ and greatest elements $1_L$ along with additional binary operation called multiplication which is denoted by $\cdot$ and defined as $a \cdot b = ab \leq a \wedge b$ for $a, b \in L$ satisfying conditions given below
$i)$ $a.b = b.a$, for all $a, b \in L$.
$ii)$ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for all $a, b, c \in L$.
$iii)$ For $a, b_\alpha \in L$, $a.(\vee_\alpha b_\alpha) = \vee_\alpha (a.b_\alpha)$, $\alpha \in I$ (an index set).
$iv)$ $a \cdot 1_L = a$, for all $a \in L$.
Element $c \in L$ is called *compact*, if $a \leq \vee_i a_i$, $i \in I$( an index set) then for some $n \in \mathbb{Z}^+$ we have $a_{i_1}, a_{i_2}, \cdots a_{i_n}$ such that $a \leq a_{i_1} \vee a_{i_2} \vee \cdots \vee a_{i_n}$. If each member of $L$ is a join of compact elements of $L$, then $L$ is called *compactly generated*.

A $C$-*lattice* is multiplicative lattice $L$ in which $1_L$ is both compact and multiplicative identity and $L$ is generated under joins by a multiplicatively closed subset $C$ of compact elements.
For more details on $C-$lattice $L$, the reader may refer ( [4], [15], [21]-[22]).
Element $p \in L$ such that $p \neq 1_L$ is *prime* if $p_1.p_2 \leq p$ implies $p_1 \leq p$ or $p_2 \leq p$. Element $m \neq 1_L$ of $L$ is *maximal*, if $m \leq a < 1_L$ then $m = a$. Define the set $\sigma(L) = \{p \in L | p \text{is prime}\}$ and $Max(L) = \{m \in L | m \text{ is maximal}\}$. By $D^l(a)$, we mean the collection of all prime elements $p \in L$ such that $a \leq p$ for any $a \in L$. In [12], F. Callialp et. al. introduced a topology over $\sigma(L)$ with $\{D^l(a) | a \in L\}$ as the family of closed sets.

The work of R. Dilworth [4], attracted many researchers in the area of Noetherian ring. In [14], E. Johnson viewed a Noetherian lattice module. Afterward, the study of Noetherian lattice modules have been further broaden for abstract module theory.

A lattice module $M$ over a $C$-lattice $L$ with least element $0_M$ and greatest element $1_M$ is a complete lattice and consisting an operation $\cdot : L \times M \to M$ such that $a \cdot N = aN$ satisfying conditions $(i)$ to $(iv)$ given below.

(i) $(ab) \cdot N = a \cdot (b \cdot N)$, for all $a, b \in L$ and $N \in M$.

(ii) $(\bigvee_{\alpha} a_\alpha) \cdot (\bigvee_{\beta} N_\beta) = (\bigvee_{\alpha,\beta} (a_\alpha \cdot N_\beta))$, for all $a_\alpha \in L$ and $N_\beta \in M$.

(iii) $1_L \cdot N = N$, where $N \in M$.

(iv) $0_L \cdot N = 0_M$, where $N \in M$.

Element $N \in M$ is called *meet principal*, if for each $a \in L$ and $A \in M$, we have $A \wedge aN = (a \wedge (A : N))N$. Element $N \in M$ is called *join principal*, if $((aN \vee A) : N) = (a \vee (A : N))$ for each $a \in L$ and $A \in M$. If $N$ is both meet principal and join principal, then $N$ is called *principal* element of $M$. Note that, if each member of $M$ is join of principal elements then $M$ is called *principally generated* or *$PG$-lattice*.

For $P, Q \in M$ and $a \in L$, $(P : Q)$ is the largest element $c$ of $L$ satisfying $cQ \leq P$ and $(P : a)$ is the largest element $C$ of $M$ satisfying $aC \leq P$, where " : " is called the *residual division* operation.

Throughout the paper $L$ is a $C$-lattice and $M$ is a lattice module over $L$.

In [9], F. Callialp et. al. extend the study of S. Yassemi in [24] and introduced the *second* lattice module $M$. Thereafter, the concept of second lattice module studied by many authors intensively (see [9], [10] and [18]).

Element $N \neq 0_M$ of $M$ is called *second*, for $a \in L$, either $aN = N$ or $aN = 0_M$. Let $Spec^s(M)$ denotes the set $\{P \in M |$ P is second$\}$. Hence, there exists a topology $\tau$ on $Spec^s(M)$ having $\{D^S(N) | N \in M\}$ as the family of closed sets, where $D^S(N) = \{K \in Spec^s(M) | (0_M : N) \leq (0_M : K)\}$.

More definitions and details related to lattice modules and topology, the reader may refer ([3], [6]- [16], [17]-[18]).

Recently, many researchers have contributed a lot in the study of algebraic and topological aspect of commutative ring $R$ and module over a commutative ring $R$ (see [1], [13], [20], [25]).

## 2 Zariski Topology on $\Sigma(M)$

**Definition 2.1** ([10]). A non-zero element $K$ of $M$ is *minimal*, whenever $0_M \leq N < K$ implies $N = 0_M$, $N \in M$.

Denote the set $\Sigma(M) = \{N \in M | N$ is minimal$\}$. Note that, every minimal element of a lattice module $M$ is second. Converse is not true (see [17] for example). Therefore, we have $\Sigma(M) \subseteq Spec^s(M)$. Also, note that for $N \in M$, where $M$ is a lattice module over a $C$-lattice $L$. If $N$ is second then $(0_M : N)$ is a prime element of $L$.

For $N \in M$, define the set $D^{min}(N) = \{K \in \Sigma(M) | (0_M : N) \leq (0_M : K)\}$.

**Theorem 2.2.** *Let $M$ be a lattice module over $L$. Then for $N, K, N_i \in M$ following statements hold.*

*(i) $D^{min}(1_M) = \Sigma(M)$ and $D^{min}(0_M) = \emptyset$, where $\emptyset$ denote the empty set.*

*(ii) $\cap_{i\in I} D^{min}(N_i) = D^{min}(\wedge_{i\in I}(0_M : (0_M : N_i))$, where $I$ is an indexed set.*

*(iii) $D^{min}(N) \cup D^{min}(K) = D^{min}(N \vee K)$.*

*Proof.* (1) By definition $D^{min}(1_M) = \{K \in \Sigma(M) | (0_M : 1_M) = 0_L \leq (0_M : K)\} = \Sigma(M)$ and $D^{min}(0_M) = \{K \in \Sigma(M) | (0_M : 0_M) = 1_L \leq (0_M : K)\} = \emptyset$.
(2) Let $S \in \cap_{i\in I} D^{min}(N_i)$. Then $S \in D^{min}(N_i)$ for each $i \in I$, therefore $(0_M : N_i) \leq (0_M : S)$ for each $i \in I$ and hence $\vee_{i\in I}(0_M : N_i) \leq (0_M : S)$. Therefore by Lemma 2.3 $(iii)$ and $(iv)$ we have $(0_M : \wedge_{i\in I}(0_M : (0_M : N_i))) \leq (0_M : (0_M : (0_M : S))) = (0_M : S)$ and hence $S \in D^{min}(\wedge_{i\in I}(0_M : (0_M : N_i))$.
(3) It is clear that $D^{min}(N) \cup D^{min}(K) \subseteq D^{min}(N \vee K)$, where $N, K \in M$. Now, let $P \in D^{min}(N \vee K)$. Therefore $(0_M : N \vee K) \leq (0_M : P)$ and hence $(0_M : N) \wedge (0_M : K) \leq$

$(0_M : P)$. Since $P$ is minimal and hence second elements of $M$, we have $(0_M : P)$ is a prime elements of $L$, therefore $(0_M : N) \wedge (0_M : K) \leq (0_M : P)$ implies $(0_M : N) \leq (0_M : P)$ or $(0_M : K) \leq (0_M : P)$. This implies that $P \in D^{min}(N)$ or $P \in D^{min}(K)$, consequently, $P \in D^{min}(N) \cup D^{min}(K)$. □

Hence, there exists a topology $\tau^{min}$ on $\Sigma(M)$ having $\{D^{min}(N) | N \in M\}$ as the family of closed sets. In fact, $\tau^{min}$ is the subspace topology induced by $\tau$ on $\Sigma(M)$.
We essentially need following Lemmas throughout the paper.

**Lemma 2.3.** *[16] For $x \in L$ and $A, B, C \in M$, following hold.*

*(i)* $x \leq (0_M : (0_M : x))$.

*(ii)* $A \leq (0_M : (0_M : A))$.

*(iii) If* $A \leq B$ *then* $(C : B) \leq (C : A)$.

*(iv)* $(0_M : A) = (0_M : (0_M : (0_M : A)))$.

*(v)* $(A : B \vee C) = (A : B) \wedge (A : C)$.

**Lemma 2.4.** *[17] Let $M$ be a PG-lattice over $L$. If $K \in M$ is minimal then $(0_M : K)$ is maximal element of $L$.*

**Lemma 2.5.** *[17] For a lattice module $M$ over $L$ and $N, K \in M$ if $(0_M : N) \leq (0_M : K)$, then $D^{min}(K) \subseteq D^{min}(N)$.*

**Lemma 2.6.** *[17] For lattice $M$ over $L$ and $N, K \in M$. Then $D^{min}(N) = D^{min}((0_M : (0_M : N)))$.*

It is well known that $\overline{L} = L/(0_M : 1_M) = \{x \in L \mid (0_M : 1_M) \leq x \leq L\}$ is multiplicative lattice and for $a \in L$, $\overline{a}$ represents the element of $\overline{L}$.

Let $M$ be a $PG$-lattice over $L$ and $Max(\overline{L})$ be the collection of all maximal elements of $\overline{L}$. Let the map $\psi : \Sigma(M) \to Max(\overline{L})$ defined by $\psi(N) = \overline{(0_M : N)}$ is called the natural map of $\Sigma(M)$. A lattice module $M$ is *min-surjective* if either $M = 0_M$ or has a surjective natural map. Also, $M$ is *min-injective* if the natural map $\psi$ is injective.

**Lemma 2.7.** *Let $M$ be a PG-lattice over $L$, and $\psi : \Sigma(M) \to Max(\overline{L})$ be the natural map. Then the below statements $(i)$ and $(ii)$ hold.*

*(i) $\psi$ is a continuous map.*

*(ii) $\psi$ is closed and open, if $M$ is min-surjective.*

*Proof.* $(i)$ Suppose that $S \in \psi^{-1}(D^l(\overline{a}))$. Then there exists $\overline{b} \in D^l(\overline{a})$ such that $S = \psi^{-1}(\overline{b})$. Therefore $\psi(S) = \overline{b}$ and so $\overline{(0_M : S)} = \overline{b}$. This implies that $\overline{a} \leq \overline{(0_M : S)} = \overline{b}$ and hence $a \leq (0_M : S) = b$, therefore by Lemma 2.3 $(iii)$ and $(iv)$, $(0_M : (0_M : a)) \leq (0_M : S)$. Thus $S \in D^{min}((0_M : a))$, consequently, $\psi^{-1}(D^l(\overline{a})) \subseteq D^{min}((0_M : a))$.
Now, suppose that for $a \in L$, $K \in D^{min}((0_M : a))$. Then $(0_M : (0_M : a)) \leq (0_M : K)$. Since $a \leq (0_M : (0_M : a))$, $\overline{a} \leq \overline{(0_M : (0_M : a))} \leq \overline{(0_M : K)}$. Therefore, $K \in \psi^{-1}(D^l(\overline{a}))$. Hence, $D^{min}((0_M : a)) \subseteq \psi^{-1}(D^l(\overline{a}))$. Consequently, $\psi^{-1}(D^l(\overline{a})) = D^{min}((0_M : a))$. That is, inverse image of closed set is closed and so $\psi$ is continuous.
$(ii)$ Suppose that $\psi$ is min-surjective. By part $(i)$, for $N \in M$, $\psi^{-1}(D^l(\overline{(0_M : N)}))$ $= D^{min}((0_M : (0_M : N)))$. By Lemma 2.6, we have $D^{min}((0_M : (0_M : N))) = D^{min}(N)$, and so $\psi^{-1}(D^l(\overline{(0_M : N)})) = D^{min}(N)$. As $\psi$ is surjective, $\psi \circ \psi^{-1}(D^l(\overline{(0_M : N)})) = \psi(D^{min}(N))$ implies that $\psi(D^{min}(N)) = D^l(\overline{(0_M : N)})$. Hence, $\psi$ is closed. Similarly, $\psi(\Sigma(M) - D^{min}(N)) = Max(\overline{L}) - D^l(\overline{(0_M : N)})$, i.e., $\psi$ open. □

Note that, in the cofinite topology closed sets are either finite sets or the whole set itself (see [3]).

**Theorem 2.8.** *If the infimum of any collection of maximal elements of $L$ is zero and $M$ is a $PG$-lattice over $L$. Then $\Sigma(M)$ is a topological space with cofinite topology.*

*Proof.* Suppose that for $N \in M$, $D^{min}(N)$ is a proper closed subset of $\Sigma(M)$ which is infinite. Then for each $K \in D^{min}(N)$, $(0_M : N) \leq (0_M : K)$. By Lemma 2.4, $(0_M : N)$ is less equal infinite number of maximal elements of $L$. Since the meet of any collection of maximal elements of $L$ is zero, we have $(0_M : N) = 0_L$. This implies that $D^{min}(N) = \Sigma(M)$, a contradiction, consequently $\Sigma(M)$ is the cofinite topology. □

Note that a topological space $X$ is said to be *Noetherian*, if and only if every ascending (descending) chain of open (closed) is stationary (see [3] and [6]).

**Corollary 2.9.** *If the infimum of any collection of maximal elements of $L$ is zero and $M$ is a $PG$-lattice over $L$. Then $\Sigma(M)$ is a Noetherian topological space.*

For $X \subseteq \Sigma(M)$, denote the join of all elements of $X$ by $\bigvee X$ and the closure of $X$ by $\overline{X}$.

**Theorem 2.10.** *[17] For a lattice $M$ over $L$ and $Y \subseteq \Sigma(M)$, $D^{min}(\bigvee Y) = \overline{Y}$. Hence, $Y$ is closed if and only if $D^{min}(\bigvee Y) = Y$.*

A topological space $X$ is *irreducible* if for any decomposition $X \subseteq A_1 \cup A_2$ with closed subsets $A_i$ of $X$ with $i = 1, 2$, we have $A_1 = X$ or $A_2 = X$. An irreducible component of a topological space $X$ is a maximal irreducible subset of $X$. A singleton subset and its closure in $X$ are irreducible (see [3] and [6]).
Following Corollary follows from Theorem 2.10.

**Corollary 2.11.** *For $N \in \Sigma(M)$, $D^{min}(N)$ is an irreducible and closed subset of $\Sigma(M)$.*

For a proper element $N \in M$, the $min$-radical of $N$ is denoted by $J^{min}(N)$ and defined as the join of all elements of $D^{min}(N)$ i.e., $J^{min}(N) = \bigvee D^{min}(N)$. In case $D^{min}(N) = \emptyset$ then $J^{min}(N) = 0_M$. An element $N \in M$ is said to be $min$*-radical* element if $N = J^{min}(N)$.

We prove the following characterization.

**Theorem 2.12.** *For a lattice $M$ over $L$ and $N, K \in M$, $D^{min}(K) \subseteq D^{min}(N)$ if and only if $J^{min}(K) \leq J^{min}(N)$.*

*Proof.* Suppose that for $N, K \in M$, $D^{min}(K) \subseteq D^{min}(N)$. Then $\vee D^{min}(K) \leq \vee D^{min}(N)$, that is, $J^{min}(K) \leq J^{min}(N)$.

Conversely, suppose that $J^{min}(K) \leq J^{min}(N)$. Then by Lemma 2.3$(iii)$, $(0_M : J^{min}(N)) \leq (0_M : J^{min}(K))$. Therefore by Lemma 2.5, $D^{min}(J^{min}(K)) \subseteq D^{min}(J^{min}(N))$, that is, $D^{min}(\bigvee D^{min}(K)) \subseteq D^{min}(\bigvee D^{min}(N))$. Consequently, $D^{min}(K) \subseteq D^{min}(N)$ by Corollary 2.11. □

Now, we give the characterization of $\Sigma(M)$ to be a Noetherian topological space.

**Corollary 2.13.** *Let $M$ be a lattice over $L$. Then the following statements are equivalent.*

*(i)* *$\Sigma(M)$ is a Noetherian topological space.*

*(ii)* *The descending chain condition for $min$-radical elements of $M$ holds.*

Let $L$ be a $C$-lattice and $a \in L$. A $J$-radical of $a$ is denoted by $J(a)$ and defined as $J(a) = \wedge\{p \in Max(L) | a \leq p\}$. Note that $a \in L$ is said to be a $J$-radical element if $J(a) = a$.

**Lemma 2.14.** *Let $M(\neq 0_M)$ be a $PG$-lattice over $L$. If $M$ is min-surjective, then the following statements hold.*

*(i)* *For $N \in M$, $J((0_M : N)) = (0_M : J^{min}(N))$.*

*(ii)* *If $q$ is a $J$-radical element of $L$ with $(0_M : 1_M) \leq q$, then there exists $Q \in \Sigma(M)$ with $(0_M : Q) = q$.*

*Proof.* $(i)$ Suppose that $M$ is min-surjective and $D^{min}(N) \neq \emptyset$. By definition, $J((0_M : N)) = \wedge_{x \in D^l((0_M:N))} x$ and $(0_M : J^{min}(N)) = (0_M : \vee_{K \in D^{min}(N)} K) = \wedge_{K \in D^{min}(N)} (0_M : K)$ by Lemma 2.3$(v)$. Since $M$ is min-surjective, for each $x \in D^l((0_M : N))$ there exists $P \in \Sigma(M)$ such that $(0_M : P) = x$ and so $\wedge_{x \in D^l((0_M:N))} x = \wedge_{P \in D^{min}(N)} (0_M : P)$. Therefore, we have $J((0_M : N)) = ((0_M : J^{min}(N))$.
$(ii)$ Suppose that $q$ is a $J$-radical element of $L$ containing $(0_M : 1_M)$. Since $M$ is min-surjective, for every $m \in D^l((0_M : N))$ there exists $K_m \in \Sigma(M)$ such that $(0_M : K_m) = m$. But $q$ is $J$-radical with $(0_M : 1_M) \leq q$, therefore by definition, $q = J(q) = \wedge_{m \in D^l(q)} m = \wedge_{m \in D^l(q)} (0_M : K_m) = (0_M : \vee_{m \in D^l(q)} K_m)$ by Lemma 2.3(5). □

**Theorem 2.15.** *Let $M(\neq 0_M)$ be a PG-lattice over $L$ and $\psi$ be a surjective natural map. Then $\Sigma(M)$ is Noetherian if and only if $Max(\overline{L})$ is Noetherian.*

*Proof.* Suppose that $Max(\overline{L}))$ is Noetherian. To prove $\Sigma(M)$ is Noetherian, in view of Corollary 2.13, it is enough to show that the descending chain condition for $min$-radical elements of $M$ holds. As such, let $N_1 \geq N_2 \geq \cdots \geq N_i \geq \cdots$ be a descending chain for $min$-radical elements of $M$. Then by Lemma 2.3(3) and Lemma 2.14(1), $J((0_M : N_1) \leq J((0_M : N_2)) \leq \cdots \leq J((0_M : N_i)) \leq \cdots$ is an ascending chain of $J$-radical elements of $L$. In fact $\overline{(0_M : N_1)} \leq \overline{(0_M : N_2)} \leq \cdots \leq \overline{(0_M : N_i)} \leq \cdots$ is an ascending chain of $J$-radical elements of $\overline{L}$ because $(0_M : 1_M) \leq (0_M : N_i)$ for $N_i \in M$. Since $Max(\overline{L})$ is Noetherian, there exists $k \in \mathbb{Z}^+$ such that $(0_M : N_i) = (0_M : N_{i+k})$ for all $i \in \mathbb{Z}^+$. Therefore by Lemma 2.10, for all $i \in \mathbb{Z}^+$, $D^{min}(N_i) = D^{min}(0_M : (0_M : N_i)) = D^{min}(0_M : (0_M : N_{i+k})) = D^{min}(N_{i+k})$ and hence for all $i \in \mathbb{Z}^+$, $N_i = J^{min}(N_i) = \bigvee D^{min}(N_i) = \bigvee D^{min}(N_{i+k}) = J^{min}(N_{i+k}) = N_{i+k}$. Consequently, $\Sigma(M)$ is Noetherian.
The necessary part is clear. □

For $K \in M$, element $P$ of $M$ is a $J^{min}$*-component* of $K$, if $(0_M : P)$ is the minimal element of the family of $J$-radical prime elements containing $(0_M : K)$.

We essentially need the following Lemma.

**Lemma 2.16.** *[19] A subset $S$ of $Max(L)$ is irreducible if and only if $\bigwedge S$ is prime.*

**Theorem 2.17.** *Let $M(\neq 0_M)$ be a PG-lattice over $L$ and and $M$ is min-surjective. Then for $N \in M$ below statements hold.*

*(i) A subset $Y$ of $\Sigma(M)$ is an irreducible closed if and only if $Y = D^{min}(N)$ with $(0_M : N)$ is a $J$-radical prime element of $L$.*

*(ii) A subset $W$ of $\Sigma(M)$ is an irreducible component of $D^{min}(N)$ if and only if $W = D^{min}(N')$ for some $J^{min}$-component $N'$ of $N$.*

*Proof.* $(i)$ Suppose that $Y$ is an irreducible and closed subset of $\Sigma(M)$. Then by Corollary 2.11, for some $N \in \Sigma(M)$, we have $Y = D^{min}(N)$. Since $M$ is min-surjective and by Lemma 2.7, we have $\psi(D^{min}(N)) = D^l(\overline{(0_M : N)})$ is an irreducible and closed subset of $Max(\overline{L})$, and therefore by Lemma 2.16, $\wedge D^l(\overline{(0_M : N)})$ is prime, consecuently, $(0_M : N)$ is a $J$-radical prime element of $L$.

Conversely, suppose that $Y = D^{min}(K)$ with $(0_M : K)$ is a $J$-radical prime element of $L$. Note that, $D^{min}(K)$ is a closed subset of $\Sigma(M)$. Now, to show that $D^{min}(K)$ is irreducible. Suppose for $N, P \in M$, $D^{min}(K) \subseteq D^{min}(N) \cup D^{min}(P)$. Then $\psi(D^{min}(K)) \subseteq \psi(D^{min}(N)) \cup \psi(D^{min}(P))$, therefore by Lemma 2.7, $D^l(\overline{(0_M : K)}) \subseteq D^l(\overline{(0_M : N)}) \cup D^l(\overline{(0_M : P)})$. Since $(0_M : K)$ is a $J$-radical prime element of $L$, by Lemma 2.16, $D^l(\overline{(0_M : K)})$ is an irreducible subset of $Max(\overline{L})$ therefore $D^l(\overline{(0_M : K)}) \subseteq D^l(\overline{(0_M : N)})$ or $D^l(\overline{(0_M : K)}) \subseteq D^l(\overline{(0_M : P)})$ and hence by Lemma 2.7, $D^{min}(K) \subseteq D^{min}(N)$ or $D^{min}(K) \subseteq D^{min}(P)$, consequently $D^{min}(K)$ is irreducible.
$(ii)$ Suppose that $W \subseteq \Sigma(M)$ is an irreducible component of $D^{min}(N)$. Note that, irreducible component is closed, therefore by $(i)$, $W = D^{min}(X)$ with $(0_M : X)$ is a $J-$radical prime element of $L$. In order to prove that $(0_M : X)$ is a $J$-component of $(0_M : N)$, let

$(0_M : N) \leq q \leq (0_M : X)$, where $q$ is $J-$radical prime element of $L$. Since $M$ is min-surjective, there exists $Q \in \Sigma(M)$ with $q = (0_M : Q)$ by Lemma 2.14($ii$). Therefore $(0_M : N) \leq (0_M : Q) \leq (0_M : X)$ and so $D^{min}(X) \subseteq D^{min}(Q) \subseteq D^{min}(N)$. Since $W = D^{min}(X)$ is an irreducible component of $D^{min}(N)$, we have $D^{min}(X) = D^{min}(Q)$ and hence by using Lemma 2.12, we have $(0_M : X) = (0_M : Q)$, i.e., $(0_M : X)$ is the minimal element of the family of $J$-radical prime elements, consequently, $(0_M : X)$ is a $J$-component of $(0_M : N)$.

Conversely, suppose that $W = D^{min}(N')$ for some $J^{min}$-component $N'$ of $N$. Then by $(i)$, $W = D^{min}(N')$ is an irreducible and closed subset of $\Sigma(M)$. In order to prove that $W = D^{min}(N')$ is an irreducible component of $D^{min}(N)$. Suppose $D^{min}(N') \subseteq D^{min}(N'') \subseteq D^{min}(N)$ where $N'' \in M$ such that $(0_M : N'')$ is a $J$-radical prime element $L$. Then by Lemma 2.3($v$), $(0_M : J^{min}(N)) \leq (0_M : J^{min}(N'')) \leq (0_M : J^{min}(N'))$ therefore by Lemma 2.14($i$), $J(0_M : N) \leq J(0_M : N'') \leq J(0_M : N')$. Since $(0_M : N')$ and $(0_M : N'')$ are $J$-radical prime elements of $L$ and $(0_M : N) \leq J(0_M : N)$, we have $(0_M : N) \leq (0_M : N'') \leq (0_M : N')$. But $N'$ is a $J^{min}$-component of $N$, therefore $(0_M : N'') = (0_M : N')$ and hence $D^{min}(N') = D^{min}(N'')$ consequently, $W = D^{min}(N')$ is an irreducible component of $D^{min}(N)$. □

**Theorem 2.18.** *Let $M$ be a PG-lattice over $L$ and $M$ is a min-surjective. Then the following statements hold.*

*(i) $\Sigma(M)$ is connected if and only if $Max(\overline{L})$ is connected.*

*(ii) $\Sigma(M)$ is irreducible if and only if $Max(\overline{L})$ is irreducible.*

*(iii) $\Sigma(M)$ is compact if and only if $Max(\overline{L})$ is compact.*

*Proof.* $i$) Suppose that $\Sigma(M)$ is a connected space, then $Max(\overline{L})$ is connected by Lemma 2.7.

Conversely, suppose that $Max(\overline{L})$ is a connected space. If $\Sigma(M)$ is disconnected, then for $N, K \in M$, $\Sigma(M) = D^{min}(N) \cup D^{min}(K)$ with $D^{min}(N) \cap D^{min}(K) = \emptyset$, where $D^{min}(N), D^{min}(K)$ are nonempty closed subsets of $\Sigma(M)$. Therefore by Lemma 2.7, $Max(\overline{L}) = \psi(\Sigma(M)) = \psi(D^{min}(N)) \cup \psi(D^{min}(K)) = D^l(\overline{(0_M : N)}) \cup D^l(\overline{(0_M : K)})$ with $D^l(\overline{(0_M : N)}) \cap D^l(\overline{(0_M : K)}) = \emptyset$, where $D^l(\overline{(0_M : N)})$ and $D^l(\overline{(0_M : K)})$ are nonempty closed subsets of $Max(\overline{L})$. This implies that $Max(\overline{L})$ is disconnected, a contradiction, consequently $\Sigma(M)$ is a connected space.

$ii$) Follows from $i$).

$iii$) Suppose that $\Sigma(M)$ is a compact space. Then by Lemma 2.7, $\psi(\Sigma(M)) = Max(\overline{L})$ is also a compact space.

Conversely, suppose that $Max(\overline{L})$ is a compact space and for $N_i \in M$, $\{D^{min}(N_i) | i \in I\}$ is a family of closed subsets of $\Sigma(M)$ with $\cap_{i \in I} D^{min}(N) = \emptyset$. Then by Lemma 2.7, $\{\psi(D^{min}(N_i)) | i \in I\}$ is a family of closed subsets of $Max(\overline{L})$ with $\cap_{i \in I} \psi(D^{min}(N_i)) = \emptyset$. Since $Max(\overline{L})$ is a compact space, there exists a finite subset $I_1$ of $I$ with $\cap_{i \in I_1} \psi(D^{min}(N_i)) = \emptyset$ and hence $\cap_{i \in I_1} D^{min}(N_i) = \emptyset$ because $M$ is min-surjective, consequently $\Sigma(M)$ is compact. □

**Theorem 2.19.** *Let $M$ be a PG-lattice over $L$. Then the given below statements are equivalent.*

*(i) $M$ is min-injective.*

*(ii) $\Sigma(M)$ is a $T_0$-space.*

*(iii) $\Sigma(M)$ is a $T_1$-space.*

*Proof.* $i) \Rightarrow iii)$ Suppose that $N, K \in \Sigma(M)$ with $N \neq K$. Since $N \in \Sigma(M)$, we have $N \in D^{min}(N)$. We contend that $K \notin D^{min}(N)$. Indeed, if $K \in D^{min}(N)$ then $(0_M : N) \leq (0_M : K)$. Now, since $M$ is principally generated, by Lemma 2.4, we have, $(0_M : N), (0_M : K) \in Max(L)$ therefore $(0_M : N) = (0_M : K)$ and hence $\overline{(0_M : N)} = \overline{(0_M : K)}$. As $M$ is min-injective, we have $N = K$, a contradiction and consequently $\Sigma(M)$ is a $T_1$-space.

$iii) \Rightarrow ii)$ followes from defination.

$ii) \Rightarrow i)$ Suppose that $M$ is not min-injective. Then there exist $P, K \in \Sigma(M)$ such that $\overline{(0_M : P)} = \overline{(0_M : K)}$ and $P \neq K$. Since $\Sigma(M)$ is a $T_0$-space, there exists $N \in M$ such that $P \notin D^{min}(N)$ and $K \in D^{min}(N)$, i.e., $(0_M : N) \nleq (0_M : P)$ and $(0_M : N) \leq (0_M : K)$, a contradiction to the fact that $(0_M : P) = (0_M : K)$, consequently $M$ is min-injective. □

Hochster [5] characterized Max-spectral spaces. A space $X$ is *Max-spectral* if and only if $X$ satisfy the following conditions:

(i) $X$ is a $T_1$-space.

(ii) $X$ is compact.

**Theorem 2.20.** *Let $M$ be a $PG$-lattice over $L$. If $M$ is min-injective then $\Sigma(M)$ is Max-spectral space in each of the following cases.*

*(i) $(0_M : 1_M)$ is a maximal element of $L$.*

*(ii) $\Sigma(M)$ is a finite set.*

*(iii) The meet of any collection of maximal elements of $L$ is zero.*

*(iv) The descending chain condition for $min$-radical elements of $M$ holds.*

*Proof.* $i$) Suppose that $(0_M : 1_M)$ is a maximal element of $L$. Then $|\Sigma(M)| \leq 1$, indeed if $S_1, S_2 \in \Sigma(M)$, then both $(0_M : S_1)$ and $(0_M : S_2)$ are maximal elements of $L$ by Lemma 2.4. Since $(0_M : 1_M)$ is maximal, we have $(0_M : 1_M) = (0_M : S_1) = (0_M : S_2)$ therefore $S_1 = S_2$ because $M$ is Min-injective, and hence $|\Sigma(M)| \leq 1$, consequently, $\Sigma(M)$ is Max-spectral.
$ii$) Follows from the fact that every finite set is compact.
For $iii$) and $iv$) use Corollary 2.9 and Lemma 2.13 respectively and follows from the fact that every Noetherian topological space is compact. □

**Corollary 2.21.** *Let $M$ be a $PG$-lattice over $L$. If $\Sigma(M)$ is a Max-spectral topological space, then $M$ is min-injective.*

*Proof.* Follows from Theorem 2.19. □

## 3 Conclusions

In this paper, we have studied the Zariski topology on $\Sigma(M)$, where $M$ is a lattice module over a $C$-lattice $L$ and discussed the some basic properties (topological) like Compact, Noetherian, Irreducible, Connected, etc. Furthermore, for $\Sigma(M)$ to be the Max-spectral space, we have obtained different conditions on $M$ and $L$.

## References


[1] H. Ansari-Toroghy and F. Farshadifar, *Some generalizations of second submodules*, Palest. J. Math., **8(2)**, 159–168, (2019).

[2] H. Ansari-Toroghy and S.S. Pourmortazavi, *On the minimal submodules of a module,* Taiwanese J. Math., **11(4)**, 1189-1201, (2017).

[3] M. F. Atiyah, I. G. Macdonald, *Introduction to commutative algebra,* Addison-Wesley, (1969).

[4] R. P. Dilworth, *Abstract commutative ideal theory,* Pacific J. Math, **12**, 481-498, (1962).

[5] M. Hochster, *Prime ideal structure in commutative rings,* Trans. Amer. Math. Soc., **1**42, 43-60, (1969).

[6] J. R. Munkres, *Topology,* second ed. Prentice Hall, New Jersey, (1999).

[7] E. A. AL-Khouja, *Maximal elements and prime elements in lattice modules,* Damascus Univ. Basic Sci., **19**, 9-20, (2003).

[8] S. Ballal and V. Kharat, *Zariski topology on lattice modules,* Asian-European J. Math., **8(4)**, 10 pages, (2015).

[9] F. Callialp, U.Tekir, E. A. Ugurlu and K. H. Oral, *Second and Secondary Lattice Modules,* Sci. World J., **2014**, 4 pages, (2014).

[10] F. Callialp, U. Tekir and G. Ulucak, *Comultiplication lattice modules,* Iran. J. Sci. Technol., **39(A2)**, 213-220, (2015).

[11] F. Callialp and U. Tekir, *Multiplication lattice modules,* Iran. J. Sci. Technol., **A4**, 309-313, (2011).

[12] F. Callialp, G. Ulucak and U. Tekir, *On the Zariski topology over an $L$-module $M$,* Turk. J. Math., **41(2)**, 326-336, (2017).

[13] F. Farzalipour and P. Ghiasvand, *On weakly semiprime subsemimodules*, Palest. J. Math., **9(2)**, 620-625, (2020).

[14] E. W. Johnson, *Completions of noetherian lattices,* Unpublished notes.

[15] E. W. Johnson, *A-transforms of noether Lattices,* Ph.D. Dissertation, Univ. of California, Riverside, (1966).

[16] J. A. Johnson, *a-adic completions of noetherian lattice modules,* Fund. Math., **66**, 341-371, (1970).

[17] N. Phadatare, S. Ballal and V. Kharat, *On the second spectrum of lattice modules,* Discuss. Math. Gen. Algebra Appl., **37(1)**, 59-74, (2017).

[18] N. Phadatare and V. Kharat, *On the second radical elements of lattice modules,* Tbilisi Math. J., **11(4)**, 165-173, (2018).

[19] N. Phadatare, S. Ballal and V. Kharat, *On the maximal spectrum of lattice modules,* Southeast Asian Bull. Math., **44(1)**, 105-117, (2020).

[20] M. Rahmatinia and A. Y. Darani, *Noetherian modules with prime nilradical*, Palest. J. Math., **9(1)**, 112–117, (2020).

[21] N. K. Thakare and C. S. Manjarekar, *Abstract spectral theory: Multiplicative lattices in which every character is contained in a unique maximal character,* in: Algebra and Its Applications (Marcel Dekker, New York), 265–276, (1984).

[22] N. K. Thakare, C. S. Manjarekar and S. Maeda, *Abstract spectral theory II: Minimal characters and minimal spectrums of multiplicative lattices,* Acta Sci. Math., **52**, 53-67, (1988).

[23] M. Ward and R. P. Dilworth, *Residuated lattices,* Trans. Amer. Math. Soc., **45**, 335-354, (1939).

[24] S. Yassemi, *The dual notion of prime submodules*, Arch. Math. (Brno), **37**, 273-278, (2001).

[25] N. Zeidi, *On weakly 2-irreducible ideals of commutative rings*, Palest. J. Math., **10(1)**, 1-7, (2021).

## Author information

S. Mitkari, Department of Mathematics, School of Science Sandip Unicersity, Nashik, India.
E-mail: santosh.mitkari@bharatividyapeeth.edu

R. Pathak, Department of Mathematics, School of Science Sandip Unicersity, Nashik, India.
E-mail: renu.pathak@sandipuniversity.edu.in

N. Phadatare, Bharati Vidyapeeth Deemed to be University College of Engineering, Pune-411 043, India.
E-mail: nmphadatare@bvucoep.edu.in

M. Agalave, Department of Mathematics, Fergusson College(Autonomus), Pune-411 004, India.
E-mail: manishagalave@gmail.com

Received: 2022-08-26
Accepted: 2024-05-08