Palestine Journal of Mathematics

Vol 13(3)(2024) , 133–137

Steps in Anderson-Badawi's Conjecture on n-Absorbing and Strongly n-Absorbing Ideals

Matija Delic and Khalid Adarbeh

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: 13A15, 13F05, 13G05..

Keywords and phrases:2-absorbing ideals, n-absorbing ideals, Strongly 2-absorbing ideals, u-rings.

The authors would like to thank the reviewers and editor for their comments and valuable suggestions that improved the quality of our paper.

Acknowledgements. *The authors are grateful to the Undergraduate Research Opportunities program at MIT (UROP) as well as to the J-WEL Grant in Higher Education Innovation, "Educational Innovation in Palestine" for providing and funding this research opportunity. Also, The authors would like to thank Professor Haynes Miller for his crucial role in mentoring this project.*

Corresponding Author: Khalid Adarbeh

Abstract *This article aims to solve positively Anderson-Badawi Conjecture of n-Absorbing and strongly n-absorbing ideals of commutative rings in the class of u-rings. The main result extends and recovers Anderson-Badawi's related result on Prufer domains [3, Corollary 6.9].*

1 Introduction

Throughout this article, R denotes a commutative ring with $1 \neq 0$. In 2007, A. Badawi introduced the concept of 2-absorbing ideals of commutative rings as a generalization of prime ideals. He defined an ideal I of R to be 2-absorbing if whenever $a, b, c \in R$ and $abc \in I$, then ab or ac or bc is in I [4]. As in the case of prime ideals, 2-absorbing ideals have a characterization in terms of ideals. Namely, I is 2-absorbing if whenever I_1, I_2, I_3 are ideals of R and $I_1I_2I_3 \subseteq I$, then I_1I_2 or I_1I_3 or I_2I_3 is contained in I [4, Theorem 2.13].

In 2011, D.F. Anderson, A. Badawi inspired from the definition of 2-absorbing ideals and defined the n-absobing ideals for any positive integer n as follows: An ideal I is called n-absorbing ideal if whenever $x_1 \dots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. Also they introduced the strongly-n-absorbing ideals as another generalization of prime ideals, where an ideal I of R is said to be a strongly n-absorbing ideal if whenever $I_1 \dots I_{n+1} \subseteq I$ for ideals $I_1, \dots, , I_{n+1}$ of R, then the product of some n of the I_j's is contained in I. Obviously, a strongly n- absorbing ideal of R is also an n-absorbing ideal of R, and by the last fact in the previous paragraph, 2-absorbing and strongly 2 absorbing are the same. Moreover, D.F. Anderson and A. Badawi were able to prove that n-absorbing and strongly n-absorbing are equivalent in the class of Prufer domains [3, Corollary 6.9]. Then they conjectured that these two concepts are equivalent in any commutative ring [3, Conjecture 1]. For more about the absorbing concepts, one may refer to [1, 2, 3, 4, 5, 7, 11].

In 1975, Jr. P. Quartararo and H.S. Butts defined the u-rings to be those rings in which if an ideal I is contained in the union of ideals, then it must be contained in one of them. Then, they proved that it suffices to consider the case I is finitely generated ideal of R [10, Proposition 1.1] (i.e., R is a u-ring if each finitely generated ideal I satisfies the condition that when I is contained in the union of ideals, then it must be contained in one of them.). Moreover, in [10, Corollary 1.6], they proved that the class of Prufer domains (domains in which every finitely generated ideal is invertible) is contained in the class of u-rings. So we have the following diagram of implications:

Prufer domains
$\Downarrow$
u-rings

where the implication is irreversible in general; see Example 3.9 for a u-ring which is not a domain, particularly, not a Prufer domain.

In section one of this paper, we provide an alternative proof of [4, Theorem 2.13]. The technique of this proof helps in proving the main result of Section 2, which solves positively Anderson-Badawi's Conjecture of n-Absorbing and strongly n-absorbing ideals in the class of u-rings. The main result (Theorem 3.1) extends and recovers Anderson-Badawi's related result on Prufer domains (Corollary 3.7).

2 Alternative proof of [4, Theorem 2.13].

As we mentioned in the introduction, 2-absorbing ideals and strongly 2-absorbing ideals are the same. This follows trivially from [4, Theorem 2.13]. In this section, we present an alternative proof of [4, Theorem 2.13], which inspires us in solving [3, Conjecture 1] in the class of u-rings. For the seek of completeness, We provide the proof of the following lemma; which can be found as an exercise in the classical ring theory texts.

Lemma 2.1. *Let I be an ideal of R. If $I = I_1 \cup I_2$, where I_1 and I_2 are also ideals, then $I = I_1$ or $I = I_2$.*

Proof. Suppose $I_1 \setminus I_2$ and $I_2 \setminus I_1$ are nonempty. Let $a \in I_1 \setminus I_2$ and $b \in I_2 \setminus I_1$. Since $I_1 \cup I_2$ is ideal, $a+b \in I_1 \cup I_2$. Assume, without loss of generality, that $a+b \in I_1$. Then $b = (a+b)-a \in I_1$, a contradiction. Therefore, either $I_1 \setminus I_2 = \phi$ or $I_2 \setminus I_1 = \phi$; equivalently, $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. So that $I = I_1$ or $I = I_2$. □

Now, we prove a few lemmas in a sequence, finishing with the proof of Theorem 2.4.

Lemma 2.2. *Suppose that I is a 2-absorbing ideal of R, J is an ideal of R and $xyJ \subseteq I$ for some $x, y \in R$. Then $xy \in I$ or $xJ \subseteq I$ or $yJ \subseteq I$.*

Proof. Suppose $xy \notin I$. Denote by $J_x = \{z \in J \mid xz \in I\}$ and $J_y = \{z \in J \mid yz \in I\}$. It is not hard to show that J_x and J_y are ideals. Now, if $a \in J$, then $xya \in I$. But I being 2-absorbing and $xy \notin I$ imply that $xa \in I$ or $ya \in I$. Thus, either $a \in J_x$ or $a \in J_y$, and hence $J = J_x \cup J_y$. Therefore, by Lemma 2.1, either $J = J_x$, and hence $xJ \subseteq I$ or $J = J_y$, and hence $yJ \subseteq I$. □

We generalize the previous lemma as follows:

Lemma 2.3. *Suppose that I is a 2-absorbing ideal of R, I_1 and I_2 are ideals of R, and $xI_1I_2 \subseteq I$ for some $x \in R$. Then $xI_1 \subseteq I$ or $xI_2 \subseteq I$ or $I_1I_2 \subseteq I$.*

Proof. Suppose $xI_2 \nsubseteq I$. By Lemma 2.2, for all $y \in I_1$, either $xy \in I$ or $yI_2 \subseteq I$. Let $N = \{y \in I_1 \mid xy \in I\}$ and $M = \{y \in I_1 \mid yI_2 \subseteq I\}$. Then M and N are ideals of R, and similarly as in the proof of Lemma 2.2, $I_1 = N \cup M$. Thus, again by Lemma 2.1, either $I_1 = N$, and in this case $xI_1 \subseteq I$, or $I_1 = M$, and in this case $I_1I_2 \subseteq I$. □

Finally, we use the above lemmas to prove the main theorem of this section.

Theorem 2.4. *[4, Theorem 2.13] An ideal I of R is 2-absorbing ideal if and only if it is strongly 2-absorbing ideal.*

Proof. Obviously, strongly 2-absorbing ideals are 2-absorbing. Conversely, Assume that I is 2-absorbing and $I_1I_2I_3 \subseteq I$, where I_1, I_2, and I_3 are ideals of R. Further, Suppose $I_2I_3 \nsubseteq I$, and let $N = \{x \in I_1 \mid xI_2 \subseteq I\}$ and $M = \{x \in I_1 \mid xI_3 \subseteq I\}$. Then M and N are ideals. By Lemma 2.3, all $x \in I_1$ are in either N or M, and thus $I_1 = N \cup M$. Therefore by Lemma 2.1, either $I = N$ or $I = M$; which implies that $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$. □

3 The main result.

The following conjecture was announced in [3].
Anderson and Badawi's conjecture: *In every ring, the notions of n-absorbing ideals and strongly n-absorbing ideals are equivalent.*

It is easy to see that strongly n-absorbing ideals are n-absorbing. We aim to find conditions for the converse to hold. We adopt the following terminology from [6] and [10]: If $I_1, ..., I_n$ are ideals of R, then $I_1 \cup ... \cup I_n$ is called an efficient covering of I if $I \subseteq I_1 \cup ... \cup I_n$, but I is not contained in the union of any $n-1$ of these ideals [6]. In view of this definition, an ideal I of R is called a u-ideal if there is no efficient covering of I with more then one ideal.

The following result solves Anderson and Badawi's conjecture to u-rings, generalizing thus Corollary 6.9 from [3].

Theorem 3.1. *In a u-ring, an n-absorbing ideal is strongly n-absorbing.*

In order to prove this main theorem (Theorem 3.1), we prove the following four lemmas:

Lemma 3.2. *A principal ideal is a u-ideal.*

Proof. Say $I \subseteq I_1 \cup ... \cup I_n$, and $I = (x)$. Then for some j, $x \in I_j$ so $I \subseteq I_j$. □

Lemma 3.3. *Let I be an n-absorbing ideal of R, and $I_1, ..., I_{n+1}$ be u-ideals of R. Suppose that the following condition is satisfied:*

whenever $I_1 \cdots I_{n+1} \subseteq I$, and at least $k+1$ of the ideals $I_1, ..., I_{n+1}$ are principal, then I contains a product of some n of them.

Then the same holds when we replace $k+1$ with k. Here $n \geq k \geq 0$.

Proof. Assume the statement is true for I and $k+1$. Let $I_1 \cdots I_{n+1} \subseteq I$, where I_j is principal for $j \leq k$. Assume $\prod_{j \leq n} I_j \not\subseteq I$. For all $i \leq n$, let

$$J_i = \{y \in I_{n+1} \mid \quad y \prod_{j \neq n+1, i} I_j \subseteq I\}$$

Then by our assumption, $I_{n+1} = \cup_{i \leq n} J_i$. Since it is a u-ideal, it is equal to some J_i. But then

$$\prod_{j \neq i} I_j \subseteq I$$

This concludes the proof. □

Lemma 3.4. *Let I be an n-absorbing ideal. If $I_1 \cdots I_{n+1} \subseteq I$, where every I_j is a u-ideal, then I contains the product of some n of these ideals.*

Proof. By the definition of I, and Lemma 3.2, the statement holds when $I_1, ..., I_{n+1}$ are all principal ideals. We use Lemma 3.3 to induct down from the case $k = n$ (where we require $k+1$ ideals to be principle) to $k = 0$ (where we require no ideals to be principle), which is exactly what we want. □

Now, we are ready to introduce the proof of the main theorem of this article (Theorem 3.1).
Proof of Theorem 3.1*: Assume the contrary. Then in some u-ring, there are ideals $I, I_1, ..., I_{n+1}$ such that I is n-absorbing and $I_1 \cdots I_{n+1} \subseteq I$, but I doesn't contain the product of any n of these ideals. But R is a u-ring, and hence $I_1, ..., I_n$ are u-ideals. Lemma 3.4 gives a contradiction.*

Remark 3.5. We can alter the proof of Lemma 3.4 above slightly, to get a more general statement when $n = 2$. Indeed, notice that if $I = I_1 \cup I_2$, then $I = I_1$ or I_2 (well-known). Then we can drop the condition of the ideals needing to be u-ideals from Lemma 3.4, and hence we obtain for arbitrary rings, every 2-absorbing ideal is strongly 2-absorbing. This is Theorem 2.4.

We can use this to give an alternative proof to [3, Corollary 6.9]. To achieve that, we cite the following result first.

Proposition 3.6. *Every invertible ideal is a u-ideal, and a Prüfer domain is a u-ring.*

Proof. See [10, Theorem 1.5 and Corollary 1.6]. □

As a straightforward application of Theorem 3.1, we recover Anderson-Badawi's related result on Prufer domains

Corollary 3.7. *In Prüfer domains, an n-absorbing ideal is strongly n-absorbing.*

Lastly, to ensure that u-rings is strictly larger that the class of Prüfer domains, we prove the following lemma which provides an example of one such family of u-rings. A more general result; which was proved in the same way; can be found in [10].

Lemma 3.8. *Suppose R is a ring with $\mathbb{Q} \subseteq R$. Then R is a u-ring.*

Proof. Let $I = I_1 \cup \cdots \cup I_n$ be an efficient covering of I. Take $a_1 \in I_1$ with $a_1 \notin I_j$ for $j \neq 1$. Choose a_2 analogously. Then for all $k \in \mathbb{Z}$, $a_1 + ka_2 \notin I_1, I_2$. Since there are infinite possibilities for k, there will be $a_1 + ka_2$ and $a_1 + la_2$ in the same I_j. But then $(k-l)a_2 \in I_j$, so $a_2 \in I_j$ for $j \neq 2$, contradiction. □

The following is an example of a u-ring which is not a domain, and hence not a Prüfer domain.

Example 3.9. $\mathbb{Q} \times \mathbb{Q}$ is a ring with zero divisors (not domain) which contains $\mathbb{Q} \cong 0 \times \mathbb{Q}$ as a subring. Consequently, by Lemma 3.8, $\mathbb{Q} \times \mathbb{Q}$ is a u-ring.

References

[1] *Malek Achraf, Hamed Ahmed and Benhissi Ali, 2-Absorbing Ideals in Formal Power Series Rings, Palestine Journal of Mathematics, Vol. 6(2)(2017) , 502–506.*

[2] *I. Akray, Adil K. Jabbar and Shadan A. Othman, Graded n-absorbing I-ideals, Palestine Journal of Mathematics, Vol 13(1)(2024) , 201–213*

[3] *D.F. Anderson, A. Badawi, On n-absorbing ideals of commutative rings, Comm. Algebra, 39 (2011), 1646-1672.*

[4] *A. Badawi, On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc., 75 (2007), 417-429.*

[5] *Yousefian Darani, A., Puczylowski, E.R., On 2-absorbing commutative semigroups and thier applications to rings, Semigroup Forum 86 (2013), 83-91*

[6] *Christian Gottlieb. On finite unions of ideals and cosets. Communications in Algebra. 22 (1994), 3087-3097.*

[7] *A. Laradji, On n-absorbing rings and ideals, Colloquium Mathematicum 147 (2017), 265-273.*

[8] *S. McAdam, Finite coverings by ideals, Ring theory, Proceedings of the Oklahoma conference, Lecture Notes in Pure and Applied Mathematics volume 7, Dekker 1974, 163-171.*

[9] *N. McCoy, A note on finite unions of ideals and subgroups, Proc. Amer.Math. Soc. 8 (1957), 633-637.*

[10] P. Quartararo Jr. & H.S. Butts, Finite unions of ideals and modules, Proc. Amer. Math. Soc. 52 (197W5), 91-96.

[11] Nilofer I. Shaikh and Rajendra P. Deore, ON n-Absorbing Primary Submodules, Palestine Journal of Mathematics Vol. 10(2)(2021) , 845–851.

[12] Nabil Zeidi, Onweakly 2-irreducible ideals of commutative rings, Palestine Journal of Mathematics, Vol. 10(1)(2021), 1–7.

Author information

Matija Delic, Department of Mathematics, Massachusetts Institute of Technology, USA.
E-mail: `mdelic@mit.edu`

Khalid Adarbeh, Department of Mathematics, An-Najah National University, Palestine.
E-mail: `khalid.adarbeh@najah.edu`

Received: 2022-09-08
Accepted: 2024-06-29