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k-Content semimodule

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Abstract The study of semimodules over semirings is an essential tool in characterizing properties of the semirings and plays a central role in many areas of Mathematics. In this paper, inverse semimodules over the semiring R is considered where R is a distributive lattice of rings. Ohm and Rush [8], defined content modules and algebra over commutative ring and studied different interesting properties. The objective of this article is to introduce and investigate several properties of k-content semimodules as a generalization of content modules.

1 Introduction

Semimodules over semirings have an important role and have many applications in structure theory, computer science and cryptography (see [1]). In 1999, J. S. Golan [2] addressed on semimodule over semiring. The concept of inverse semimodule over a semiring was introduced by Yusuf [3], in 1966 and he obtained several results for inverse semimodules which are generalization of the corresponding results in module theory. In 2020, Sen, Bhuniya and Maity [4], studied congruences, specially the R-module congruences, on inverse semimodules. Farzalipour and Ghiasvand [5], studied on weakly semiprime subsemimodules and in [6], the author discussed on colon operations and special types of ideals. Also, some remarks on ideals of commutative semirings has been studied by P. Nashepour [7]. Ohm and Rush [8], defined Content modules and algebra over commutative ring and studied different interesting properties. Later on, in [9], the authors defined content semimodule over a commutative semiring with zero and studied its properties. In this paper, we consider inverse semimodule over an additive and multiplicative commutative semiring R with identity 1 such that R is a distributive lattice of rings. The objective of this paper is to introduce and investigate several properties of k-content semimodule as a generalization of content module and content semimodule. Some basic definitions and preliminaries are discussed in **Section 2**. In **Section 3**, we state some basic properties of k-content semimodule and establish the relation between k-content semimodule and weak multiplication semimodule. Finally, in **Section 4**, k-content semimodules over regular semiring are discussed. In this Section, it is proved that if R is a regular semiring, then every full k-subsemimodule of a k-content R-semimodule is a k-content R-semimodule with restricted k-content function. We also prove that if every k-subsemimodule of a k-content R-semimodule is a k-content R-semimodule with restricted k-content function, then R is a 2-regular semiring. Moreover, if we take R is a semiring such that $E^+(R)$ is a k-ideal and every k-subsemimodule of a k-content R-semimodule is a k-content R-semimodule with restricted k-content function, then R is a regular semiring. Finally, it is proved that if R is a regular semiring and M is an R-semimodule such that $E(M)$ is a k-set, then M is a k-content R-semimodule if and only if $Ann(x)$ is finitely generated for all $x \in M$.

2 Definitions and Preliminaries

A *semiring* $(R, +, \cdot)$ is a type $(2, 2)$ algebra whose semigroup reducts $(R, +)$ and $(R, \cdot)$ are connected by distributivity, i.e., $r(s+t) = rs + rt$ and $(s+t)r = sr + tr$ for all $r, s, t \in R$. A semiring $(R, +, \cdot)$ is said to be *additive regular* if for every element $a \in R$, we have $a+x+a = a$ for some $x \in R$. Additive regular semirings were first studied by J. Zeleznekow [10] in 1981. If for each element a in a semiring R, there exists unique element $a' \in R$ such that $a + a' + a = a$ and $a' + a + a' = a'$, then $(R, +, \cdot)$ is an *additive inverse semiring*. In 1974, Karvellas [11], first studied additive inverse semirings. Throughout the paper, the set of all additive idempotents of the semiring R is denoted by $E^+(R)$. A subsemiring I of a semiring $(R, +, \cdot)$ is called an *ideal* of R if $RI, IR \subseteq I$. For any ideal I of R, if $E^+(R) \subseteq I$, then I is called a full ideal of R. For each ideal I of a semiring R, the k-closure $\overline{I}$ of I is defined by $\overline{I} = \{a \in R : a + a_1 = a_2$ for some $a_1, a_2 \in I\}$ and is an ideal of R satisfying $I \subseteq \overline{I}$ and $\overline{\overline{I}} = \overline{I}$. An ideal I is called a k-ideal of R if and only if $I = \overline{I}$ holds. For a semiring R, let $Id_k(R)$ denotes the set of full k-ideals of R.

We need the following result:

Corollary 2.1. *[12] Let R be an additive commutative semiring. Then R is a distributive lattice of rings if and only if it is an additive inverse semiring satisfying the following conditions:*
(i) $r(s+s') = s+s'$,
(ii) $r(s+s') = (s+s')r$
(iii) $r + r(s+s') = r$, for all $r, s \in R$.

Let $(M, +)$ be a commutative semigroup and $(R, +, \cdot)$ be a semiring with identity. Then M is called a *left R-semimodule* or simply an *R-semimodule* if there exists a mapping $R \times M \to M$, written as $(r, m) \mapsto rm$, for all $r \in R$ and for all $m \in M$, satisfying (i) $r(m+n) = rm + rn$, (ii) $(r+s)m = rm + sm$, (iii) $r(sm) = (rs)m$ and (iv) $1m = m$ for all $r, s \in R$ and $m, n \in M$. If an R-semimodule M is such that $(M, +)$ is an inverse semigroup, then M is said to be an *inverse semimodule* [3]. Any subsemimodule N of M contain the set of all idempotents of the semigroup $(M, +)$, denoted by $E(M)$, is said to be full subsemimodule of M. For any two subsemimodules N and K of M, the set $\{a \in R : aK \subseteq N\}$ is denoted by $(N : K)$. It is easy to verify that $(N : K)$ is an ideal of R. We call a subset Q of M is a *k-set* if $a, a+b \in Q$ implies that $b \in Q$. A subsemimodule N of an R-semimodule M is said to be a k-subsemimodule of M if for $x, x+y \in N$ for some $y \in M$ imply that $y \in N$. For any subsemimodule N of an R-semimodule M, the k-closure of N, denoted by $\overline{N}$, is defined by $\overline{N} = \{x \in M : x + y = z$ for some $y, z \in N\}$. For an R-semimodule M, let $\mathscr{L}(M)$ denotes the set of all full k-subsemimodules of M.

Throughout this paper, all semirings R are assumed to be additive as well as multiplicative commutative which are distributive lattices of rings. This means R denotes an additive commutative and multiplicative commutative additive inverse semiring satisfying the conditions of Corollary 2.1. Also, assume that R contains an identity element 1 such that $1 \notin E^+(R)$ and all semimodules are inverse semimodules with $M \neq E(M)$.

3 k-Content Semimodule

Similar to module theory, here we define k-content semimodule and study their some properties.

Definition 3.1. Let M be an R-semimodule and $x \in M$. Consider $\mathscr{A} = \{$I: I is a full k-ideal of R and $x \in \overline{IM}\}$. We define the k-content of x by, $c_M(x) = \bigcap_{I \in \mathscr{A}} I$. Then c_M is a function from M to the set of all full k-ideals of R.

For any subset N of M, we define $c_M(N) = \sum_{x \in N} c_M(x)$. When no confusion arises, we omit the subscript M and simply write $c(x)$ instead of $c_M(x)$. It is clear that $c(N)$ is a full ideal of R.

Definition 3.2. An R-semimodule M will be called a k-content R-semimodule if for any $x \in M$, $x \in \overline{c(x)M}$.

Lemma 3.3. *Let M be a k-content R-semimodule and $x \in M$. Then $c(x) = \overline{J}$ for some finitely generated full ideal J of R.*

Proof. As M is a k-content R-semimodule, we have $x \in \overline{c(x)M}$. Then $x+x_1 \in c(x)M$ for some $x_1 \in c(x)M$. This implies $x+x_1+x_1' \in c(x)M$ and so $x+(x_1+x_1') = c_1y_1+c_2y_2+\cdots+c_ny_n$, where $c_i \in c(x)$ and $y_i \in M$ for $1 \leq i \leq n$. Then $x + (x_1 + x_1') \in \langle c_1, c_2, \cdots, c_n\rangle M \subseteq (\langle c_1, c_2, \cdots, c_n\rangle + E^+(R))M = JM$ where $J = \langle c_1, c_2, \cdots, c_n\rangle + E^+(R)$. As $E^+(R) \subseteq J$ and $E^+(R) = \langle 1+1'\rangle$, we have J is a finitely generated full ideal of R. Again, $x+(x_1+x_1') \in JM$ and $x_1 + x_1' \in E(M) \subseteq JM$ implies $x \in \overline{JM}$. Then from the definition of the k-content of x, we have $c(x) \subseteq \overline{J}$. Also, $\overline{J} \subseteq c(x)$ implies $c(x) = \overline{J}$, where J is a finitely generated full ideal of R. □

Theorem 3.4. *Let M be a k-content R-semimodule. Then $c(M) = R$ if and only if $\overline{PM} \neq M$ for any full maximal ideal P of R.*

Proof. Let $c(M) = R$ and $\overline{PM} = M$ for some full maximal ideal P of R. Then for all $x \in M$, $x \in \overline{PM}$. Also, $P = \overline{P}$, otherwise $\overline{P} = R$ as P is a maximal ideal of R. So $1 \in \overline{P}$. Then $1 + p \in P$ for some $p \in P$. This implies $1 = 1 + p + p' \in P$, which is a contradiction. Therefore, P is a full k-ideal of R such that $x \in \overline{PM}$. This implies $c(x) \subseteq P$ for all $x \in M$. Then $c(M) \subseteq P$ which is not possible. Thus $\overline{PM} \neq M$ for any full maximal ideal P of R.

For the converse part, assume that $\overline{PM} \neq M$ for any full maximal ideal P of R. Then for any full maximal ideal P of R, there exists an element $x \in M$ such that $x \notin \overline{PM}$. This implies $c(x) \nsubseteq P$, otherwise $x \in \overline{c(x)M} \subseteq \overline{PM}$, which is not possible. Since $c(x) \subseteq c(M)$ and $c(x) \nsubseteq P$, we must have $c(M) \nsubseteq P$. Since $c(M)$ is a full ideal of R and $c(M) \nsubseteq P$ for any full maximal ideal P of R, we must have $c(M) = R$.

□

Theorem 3.5. *Let M be an R-semimodule. Then the following statements are equivalent:*

(i) M is a k-content R-semimodule.

(ii) $\bigcap(\overline{I_iM}) = \overline{(\bigcap I_i)M}$ for any collection of full k-ideals $\{I_i\}$ of R.

(iii) There exists a function $f : M \to Id_k(R)$ such that for all $x \in M$ and for every full k-ideal I of R, $x \in \overline{IM}$ if and only if $f(x) \subseteq I$.

Proof. The proof is similar as content module. □

Theorem 3.6. *Let N be a k-subsemimodule of a k-content R-semimodule M. Then the following statements are equivalent:*

(i) $\overline{IM} \cap \overline{N} = \overline{IN}$ for every full k-ideal I of R.

(ii) $x \in \overline{c_M(x)N}$ for all $x \in N$.

(iii) N is a k-content R-semimodule and c_M restricted to N is c_N.

Proof. The proof is similar as content module. □

Let M be an R-semimodule, N a subsemimodule of M and I be an ideal of R. Let $s \in R$ and consider $(I :_R s) = \{r \in R : rs \in I\}$ and $(N :_M s) = \{x \in M : sx \in N\}$. Then $(I :_R s)$ is an ideal of R and $(N :_M s)$ is a subsemimodule of M. Also, in this paper, $(E^+(R) :_R I) = \{r \in R : rI \subseteq E^+(R)\}$ is denoted by $Ann(I)$ and for any $x \in M$, $Ann(x)$ is defined as $Ann(x) := (E(M) :_R x) = \{r \in R : rx \in E(M)\}$.

Theorem 3.7. *Let M be a k-content R-semimodule and $s \in R$. Then the following statements are equivalent:*

(i) $\overline{sc(x)} \subseteq c(sx)$ for all $x \in M$.

(ii) $\overline{(I :_R s)M} = (\overline{IM} :_M s)$ for every full k-ideal I of R.

(iii) $\overline{(I :_R J)M} = (\overline{IM} :_M J)$ for every pair of full k-ideals I, J of R.

Proof. $(i) \implies (ii)$: Let I be a full k-ideal of R and $x \in (\overline{IM} :_M s)$. Then $sx \in \overline{IM}$. This implies $c(sx) \subseteq I$. Then from (i), $\overline{sc(x)} \subseteq c(sx) \subseteq I$. So $c(x) \subseteq (I :_R s)$. As M is a k-content R-semimodule, we have $x \in \overline{c(x)M} \subseteq \overline{(I :_R s)M}$. For the reverse inclusion, let $x \in \overline{(I :_R s)M}$. Then $x + y \in (I :_R s)M$ for some $y \in (I :_R s)M$. This implies $x + y + y' \in (I :_R s)M$ and so $x+y+y' = r_1m_1 + r_2m_2 + \cdots + r_nm_n$, where $r_i \in (I :_R s)$ and $m_i \in M$ for $1 \leq i \leq n$. Then for all i, $sr_i \in I$ and hence $s(x + y + y') = s(r_1m_1 + r_2m_2 + \cdots + r_nm_n) \in IM$. Therefore, $sx \in \overline{IM}$ as $s(y + y') \in E(M) \subseteq IM$. So $x \in (\overline{IM} :_M s)$. Thus $\overline{(I :_R s)M} \subseteq (\overline{IM} :_M s)$. Hence $\overline{(I :_R s)M} = (\overline{IM} :_M s)$.

$(ii) \implies (iii)$:

$$\begin{aligned}\overline{(I :_R J)M} &= \overline{\{\bigcap_{j\in J}(I :_R j)\}M}\\ &= \bigcap_{j\in J}\overline{(I :_R j)M} \qquad \text{[By condition (ii) of Theorem 3.5]}\\ &= \bigcap_{j\in J}(\overline{IM} :_M j)\\ &= (\overline{IM} :_M J)\end{aligned}$$

$(iii) \implies (i)$: We consider the full k-ideal $J = \overline{\langle s\rangle + E^+(R)}$ of R. Then by the given condition, we have $\overline{(c(sx) :_R J)M} = (\overline{c(sx)M} :_M J)$. Again, M is a k-content R-semimodule implies $sx \in \overline{c(sx)M}$. Now, let $r \in J$ be an arbitrary element. Then $r + e \in \langle s\rangle + E^+(R)$ for some $e \in E^+(R)$. So $r + e = ts + f$ for some $t \in R$ and $f \in E^+(R)$. Therefore, $(r + e)x = tsx + fx \in \overline{c(sx)M}$. Since $ex \in E(M) \subseteq \overline{c(sx)M}$, we must have $rx \in \overline{c(sx)M}$. Since $r \in J$ is arbitrary, so from $rx \in \overline{c(sx)M}$, we have $x \in (\overline{c(sx)M} :_R J) = \overline{(c(sx) :_R J)M}$. Therefore, $c(x) \subseteq (c(sx) :_R J)$. Since $s \in J$, so from $c(x) \subseteq (c(sx) :_R J)$, it follows that $sc(x) \subseteq c(sx)$.

□

Definition 3.8. An R-semimodule M is said to be E-torsionfree if for any $a \notin E^+(R)$, $ax \in E(M)$ for some $x \in M$, implies $x \in E(M)$.

Theorem 3.9. *Let M be a k-content E-torsionfree R-semimodule such that $E(M)$ is a k-set. Then for every $s \in R$ and $x \in M$, $sc(x) \subseteq c(sx)$.*

Proof. We assume that $s \notin E^+(R)$, otherwise the inclusion will be trivial. Now, it is clear that $sx \in \overline{\langle s\rangle + E^+(R)M}$. Then $c(sx) \subseteq \overline{\langle s\rangle + E^+(R)}$. Now, let $r \in c(sx)$. Then $r \in \overline{\langle s\rangle + E^+(R)}$. Thus $r + r_1 \in \langle s\rangle + E^+(R)$ for some $r_1 \in \langle s\rangle + E^+(R)$. Then $r + (r_1 + r_1') = ts + e$ for some $t \in R$ and $e \in E^+(R)$. As $r \in c(sx)$ and $E^+(R) \subseteq c(sx)$, we have $ts + e \in c(sx)$. Since $c(sx)$ is a k-ideal of R and $e \in E^+(R) \subseteq c(sx)$, we have $ts \in c(sx)$ and therefore, $t \in (c(sx) :_R s)$. This leads to, $r + (r_1 + r_1') \in \langle s\rangle(c(sx) :_R s) + E^+(R)$. Also, $r_1 \in \langle s\rangle + E^+(R)$ implies $r_1 = us + f$ for some $u \in R$ and $f \in E^+(R)$. Then $r_1 + r_1' = s(u + u') + f \in \langle s\rangle(c(sx) :_R s) + E^+(R)$. Therefore, $r \in \overline{\langle s\rangle(c(sx) :_R s) + E^+(R)}$. Then $c(sx) \subseteq \overline{\langle s\rangle(c(sx) :_R s) + E^+(R)} \subseteq c(sx)$. Therefore, $c(sx) = \overline{\langle s\rangle(c(sx) :_R s) + E^+(R)}$. Let $J = \overline{(c(sx) :_R s)}$. Then $c(sx) = \overline{\langle s\rangle J + E^+(R)}$. As M is a k-content R-semimodule, we have $sx \in \overline{c(sx)M} \subseteq \overline{\langle s\rangle JM + E(M)}$. Then $sx + sy + m_1 = sz + m_2$ for some $y, z \in JM$ and $m_1, m_2 \in E(M)$. This implies $s(x + y + z') + m_1 \in E(M)$. As $E(M)$ is a k-set, we have $s(x + y + z') \in E(M)$. This implies $x + y + z' \in E(M)$, as M is E-torsionfree R-semimodule. Since $E(M) \subseteq JM$, we have $x + y + z' \in JM$, where $y + z' \in JM$ and so $x \in \overline{JM}$. Then $c(x) \subseteq J$. Hence $sc(x) \subseteq sJ = s(c(sx) :_R s) \subseteq c(sx)$. □

Now, we need the following definitions and lemma from [13].

Definition 3.10. [13] Let M an R-semimodule satisfying the property $(Rm : M) \neq \emptyset$ for all $m \in M$. Then M is said to be a weak multiplication semimodule if for each full subsemimodule N of M there exists a full ideal I of R such that $N = IM$.

Definition 3.11. [13] Let M be an R-semimodule and P be an ideal of R. We define

$$T_P(M) = \{m \in M : \text{ there exists } p \in P \text{ such that } (1 + p')m \in E(M)\}.$$

Then one can easily check that $T_P(M)$ is a subsemimodule of M. An R-semimodule M is said to be faithful if $(E(M) :_R M) = \{r \in R : rM \subseteq E(M)\} = E^+(R)$.

Lemma 3.12. *[13] Let M be a weak-multiplication R-semimodule. Then for any maximal ideal P of R either $M = T_P(M)$ or there exist $q \in P$ and $m \in M$ such that $(1+q')M \subseteq Rm+E(M)$.*

Next theorem establishes a relation between a E-unitary weak multiplication semimodule (i.e., a weak multiplication semimodule whose set of idempotents is a k-set) and a k-content semimodule.

Theorem 3.13. *Let M be a faithful weak-multiplication R-semimodule such that $E(M)$ is a k-set. Then M is a k-content R-semimodule.*

Proof. Let $I_\lambda(\lambda \in \Lambda)$ be any non-empty collection of full k-ideals of R. To show M is a k-content R-semimodule, it is sufficient to show that $\bigcap_{\lambda\in\Lambda} \overline{I_\lambda M} = \overline{(\bigcap_{\lambda\in\Lambda} I_\lambda)M}$. Let $I = \bigcap_{\lambda\in\Lambda} I_\lambda$. Clearly, $IM \subseteq \bigcap_{\lambda\in\Lambda}(I_\lambda M)$. This implies $\overline{IM} \subseteq \bigcap_{\lambda\in\Lambda} \overline{I_\lambda M}$. For the reverse inclusion, let $x \in \bigcap_{\lambda\in\Lambda} \overline{I_\lambda M}$ and $x \notin \overline{IM}$. We consider $K = \{r \in R : rx \in \overline{IM}\}$. Then $1 \notin K$ and thus K is a proper ideal of R. Then there exists a maximal ideal P of R such that $K \subseteq P$. We claim that $x \notin T_P(M)$. Otherwise there exists $p \in P$ such that $(1+p')x \in E(M) \subseteq \overline{IM}$ and so $1+p' \in K \subseteq P$ implies $1 = 1+p+p' \in P$ and thus $P = R$, which is a contradiction. Then by Lemma 3.12, there exist elements $m \in M$ and $p_1 \in P$ such that $(1+p_1')M \subseteq Rm+E(M)$. Now $(1+p_1')x \in (1+p_1')\overline{I_\lambda M} \subseteq \overline{(1+p_1')I_\lambda M} \subseteq \overline{I_\lambda m + E(M)}$ for all $\lambda \in \Lambda$. Then for each $\lambda \in \Lambda$, there exist $r_\lambda, a_\lambda \in I_\lambda$, such that $(1+p_1')x + r_\lambda m + m_\lambda = a_\lambda m + n_\lambda$ for some $m_\lambda, n_\lambda \in E(M)$. Choose $\alpha \in \Lambda$. Then we have $(1+p_1')x + r_\alpha m + m_\alpha = a_\alpha m + n_\alpha$ where $r_\alpha, a_\alpha \in I_\alpha$ and $m_\alpha, n_\alpha \in E(M)$. Then for each $\lambda \in \Lambda$, we have $(1+p_1')x + r_\lambda m + m_\lambda + r_\alpha m + m_\alpha = a_\lambda m + n_\lambda + r_\alpha m + m_\alpha$. Thus $a_\alpha m + n_\alpha + r_\lambda m + m_\lambda = a_\lambda m + n_\lambda + r_\alpha m + m_\alpha$. This implies $(a_\lambda + a_\alpha' + r_\alpha + r_\lambda')m \in E(M)$, as $E(M)$ is a k-set. Again, $(1+p_1')(a_\lambda + a_\alpha' + r_\alpha + r_\lambda')M \subseteq (a_\lambda + a_\alpha' + r_\alpha + r_\lambda')(Rm + E(M)) \subseteq E(M)$. Therefore, $(1+p_1')(a_\lambda + a_\alpha' + r_\alpha + r_\lambda') \in (E(M) :_R M) = E^+(R) \subseteq I_\lambda$ for each $\lambda \in \Lambda$. So $(1+p_1')(a_\alpha' + r_\alpha) + (1+p_1')(a_\lambda + r_\lambda' + a_\lambda' + r_\lambda) \in I_\lambda$ for each $\lambda \in \Lambda$. As each I_λ is a k-ideal, we have $(1+p_1')(a_\alpha' + r_\alpha) \in I_\lambda$ for each $\lambda \in \Lambda$ and hence $(1+p_1')(a_\alpha' + r_\alpha) \in I$. Also, we have $(1+p_1')((1+p_1')x + r_\alpha m + m_\alpha) = (1+p_1')(a_\alpha m + n_\alpha)$. So $(1+p_1')^2 x + \underline{(1+p_1')}(r_\alpha m + r_\alpha' m + m_\alpha) = (1+p_1')(a_\alpha + r_\alpha')m + (1+p_1')n_\alpha \in IM$ and hence $(1+p_1')^2 x \in \overline{IM}$. Then $(1+p_1')^2 \in K \subseteq P$ and hence $1 \in P$ which is a contradiction. Therefore, $\bigcap_{\lambda\in\Lambda} \overline{I_\lambda M} \subseteq \overline{IM}$. Hence $\bigcap_{\lambda\in\Lambda} \overline{I_\lambda M} = \overline{(\bigcap_{\lambda\in\Lambda} I_\lambda)M}$. Hence M is a k-content R-semimodule. □

Definition 3.14. (see [14, Definition 3.8]). Let M be an R-semimodule. An element $a \in R$ is said to be M-vn-regular if $aM + E(M) = a^2M + E(M)$.

Definition 3.15. (see [14, Definition 3.9]). An R-semimodule M is said to be a vn-regular semimodule if for any $m \in M$, $Rm + E(M) = aM + E(M)$ for some M-vn-regular element $a \in R$.

Lemma 3.16. *Let M be a faithful vn-regular R-semimodule such that $E(M)$ is a k-set and $(Rm :_R M) \neq \emptyset$. Then M is a k-content R-semimodule.*

Proof. As M is a vn-regular R-semimodule, for any $m \in M$, $Rm + E(M) = aM + E(M)$ for some M-vn-regular element a of R. Let $I = \langle a \rangle + E^+(R)$. Then we have $Rm + E(M) = IM + E(M)$, where I is a full ideal of R. Then by [13, Theorem 3.6], we have M is a weak-multiplication R-semimodule and hence from Theorem 3.13, M is a k-content R-semimodule. □

Next result establishes sufficient conditions on a k-content R-semimodule for which every element of R is M-vn-regular.

Theorem 3.17. *Let M be a finitely generated k-content R-semimodule such that $E(M)$ is a k-set and every k-subsemimodule of M is a k-content R-semimodule with restricted k-content function. Then every element of R is M-vn-regular.*

Proof. Let $a \in R$ be arbitrary. To show a is M-vn-regular, it is enough to show that $aM + E(M) = a^2M + E(M)$. For this, we consider the k-subsemimodule $N = \overline{\langle a \rangle M}$ of M. From the hypothesis, we have N is a k-content R-semimodule with restricted k-content function and hence applying Theorem 3.6, it follows that $\overline{IM} \cap N = \overline{IN}$, for every full k-ideal I of R. We now consider the full k-ideal $J = \overline{\langle a \rangle + E^+(R)}$ of R. Then $JN = \overline{\langle a \rangle + E^+(R)}\ \overline{\langle a \rangle M} \subseteq \overline{(\langle a \rangle + E^+(R))\langle a \rangle M} \subseteq \overline{\langle a \rangle\langle a \rangle M + E(M)}$. Also, we have $\langle a \rangle M \subseteq \overline{JM} \cap N = \overline{JN} \subseteq \overline{\langle a \rangle\langle a \rangle M + E(M)}$. Since M is finitely generated, it follows that $\langle a \rangle M$ is also finitely generated. Suppose $\langle a \rangle M$ is generated by $x_1, x_2, \ldots, x_n$. Now, for all $i = 1, 2, \ldots, n$; $x_i \in \langle a \rangle M \subseteq \overline{\langle a \rangle\langle a \rangle M + E(M)}$ implies

$$x_i + f_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + g_i,$$

where $a_{ij} \in \langle a \rangle$ and $f_i, g_i \in E(M)$, for all $i, j \in \{1, 2, \ldots, n\}$.

Then similar to the proof of [14, Lemma 3.12], we have $\langle a \rangle + (E(M) :_R M) = \langle a^2 \rangle + (E(M) :_R M)$. Then $a = a + a' + a \in \langle a \rangle + (E(M) : M) = \langle a^2 \rangle + (E(M) : M)$ implies $a = ra^2 + r_1$ for some $r_1 \in (E(M) : M)$ and $r \in R$. Thus $aM + E(M) \subseteq a^2M + E(M)$. Since $a^2M + E(M) \subseteq aM + E(M)$ holds trivially, we have $aM + E(M) = a^2M + E(M)$. Consequently, every element of R is M-vn-regular. □

4 k-Content Semimodule over regular semiring

An element a of a semiring R is regular if there exists an element x of R such that $a = axa$. A semiring R is regular if every element of R is regular. Also, if there is a fixed integer m such that for every element a of R, a^m is regular, then R is said to be m-regular. Here we study about k-content semimodules over regular semiring. First we state a result from [15].

Theorem 4.1. *[15] In a regular semiring R, every finitely generated ideal is generated by an idempotent element of R.*

Theorem 4.2. *Let R be a regular semiring and M be a k-content R-semimodule. Then every full k-subsemimodule of M is a k-content R-semimodule with restricted k-content function.*

Proof. Let N be a full k-subsemimodule of a k-content R-semimodule M. From Theorem 3.6, it is sufficient to show that $x \in \overline{c_M(x)N}$ for all $x \in N$. Let $x \in N$. Since M is a k-content R-semimodule, we have $x \in \overline{c_M(x)M}$. Again, since M is a k-content R-semimodule from Lemma 3.3, it follows that $c_M(x) = \overline{J}$ for some finitely generated full ideal J of R. Also, from Theorem 4.1, we have $J = \langle e \rangle$ for some idempotent element e of R. Then $x \in \overline{c_M(x)M} = \overline{JM} = \overline{\langle e \rangle M}$. Therefore, $x + em_1 = em_2$ for some $m_1, m_2 \in M$. This implies $ex + em_1 = em_2 = x + em_1$. Then $x + e(m_1 + m_1') = e(x + m_1 + m_1') \in c_M(x)N$. So $x \in \overline{c_M(x)N}$. Thus N is a k-content R-semimodule with restricted k-content function. □

Theorem 4.3. *Suppose that R is a semiring. If every k-subsemimodule of a k-content R-semimodule is a k-content R-semimodule with restricted k-content function, then R is a 2-regular semiring.*

Proof. Let $a \in R$ and we consider the full k-ideal $I = \overline{\langle a \rangle + E^+(R)}$ of R. Since R is itself a k-content R-semimodule over R and $\overline{\langle a \rangle}$ is a k-subsemimodule of R, then from Theorem 3.6, it follows that $\overline{IR} \cap \overline{\langle a \rangle} = \overline{I\overline{\langle a \rangle}} = \overline{I\langle a \rangle}$. Now, $I\langle a \rangle = \overline{\langle a \rangle + E^+(R)}\langle a \rangle \subseteq \overline{(\langle a \rangle + E^+(R))\langle a \rangle} \subseteq \overline{\langle a^2 \rangle + E^+(R)}$. Therefore, $\overline{IR} \cap \overline{\langle a \rangle} \subseteq \overline{\langle a^2 \rangle + E^+(R)}$. Since $a \in \overline{IR} \cap \overline{\langle a \rangle}$, we have $a \in \overline{\langle a^2 \rangle + E^+(R)}$ and hence $a + a_1 \in \langle a^2 \rangle + E^+(R)$ for some $a_1 \in \langle a^2 \rangle + E^+(R)$. This leads to, $a + (a_1 + a_1') = ra^2 + e$ for some $r \in R$ and $e \in E^+(R)$. Then $a + f = ra^2 + e$, where $f = a_1 + a_1' \in E^+(R)$. So $a^2 + af = ara^2 + ea$. Again, $a^2 + af = a^2 + a(f + f') = a^2 + f(a + a') = a^2 + fa(a + a') = a^2 + a^2(f + f') = a^2$, implies $a^2 = ara^2 + ea$. Multiplying both sides by ar, we get $ara^2 = a^2r^2a^2 + ea^2r$. Thus $a^2 = a^2r^2a^2 + ea^2r + ea = a^2r^2a^2 + ea$.

Again, we have $a^2 = a^2 + (a^2)' + a^2 = a(a + a') + a^2r^2a^2 + ea = a + a' + a(e + e') + a^2r^2a^2 = a + a' + a^2r^2a^2 = a^2(a + a')a^2 + a^2r^2a^2 = a^2(a + a' + r^2)a^2 = a^2xa^2$, where $x = a + a' + r^2 \in R$. Therefore, for each $a \in R$, there exists an element $x \in R$ such that $a^2 = a^2xa^2$. Hence R is a 2-regular semiring. □

Theorem 4.4. *Suppose that R is a semiring such that $E^+(R)$ is a k-ideal. If every k-subsemimodule of a k-content R-semimodule is a k-content R-semimodule with restricted k-content function, then R is a regular semiring.*

Proof. If we take $M = R$, then from Theorem 3.17, we have $aR + E^+(R) = a^2R + E^+(R)$ for any $a \in R$. Then $a = a + a' + a \in aR + E^+(R) = a^2R + E^+(R)$ implies $a = ra^2 + e$, for some $r \in R$ and $e \in E^+(R)$. This leads to, $ea = era^2 + e^2 = era^2 + e$, as $e^2 = e \cdot e = e(e + e') = e + e' = e$. Thus $ea = e + (e + e')ra^2 = e + e(ra^2 + (ra^2)') = e$. Therefore, we have $a = ra^2 + ea$. Again, $a = a + a' + a = a + a' + ra^2 + ea = a + a' + ra^2 = a(a + a')a + ara = a(a + a' + r)a = axa$, where $x = a + a' + r \in R$. Hence, for each $a \in R$, there exists an element $x \in R$ such that $a = axa$. So R is a regular semiring. □

Lemma 4.5. *Let R be a regular semiring. Then for every finitely generated full ideal A of R, $Ann(A)$ is also finitely generated and $Ann(Ann(A)) = A$.*

Proof. Let A be a finitely generated full ideal of R. Again, from Theorem 4.1, we have $A = \langle e \rangle$ where e is an idempotent element of R. Now, let $r \in Ann(Ann(A))$. Then $rAnn(A) \subseteq E^+(R)$. Also, we have $(1+e')e = e + (e^2)' = e + e' \in E^+(R)$. Thus $(1+e') \in (E^+(R) :_R e)$ and so $(1+e') \in (E^+(R) :_R A) = AnnA$. Therefore, $r(1+e') \in E^+(R)$. This implies, $r(1+e') = f$ for some $f \in E^+(R)$. So $r + re' + re = f + re \in A$. Hence $r \in A$. This leads to, $Ann(Ann(A)) \subseteq A$. Again, the reverse inclusion is clearly holds. Hence $Ann(Ann(A)) = A$.

Now, we show that $Ann(A)$ is also finitely generated. For this let, $r \in Ann(A)$. Then $re \in E^+(R)$, as $A = \langle e \rangle$. Also, we have $r = r(1+e+e') = r(1+e') + re$. Since $re \in E^+(R) = R(1+1')$, we have $r \in \langle 1+e', 1+1' \rangle \subseteq Ann(A)$. Thus $Ann(A) = \langle 1+e', 1+1' \rangle$ and hence $Ann(A)$ is finitely generated. □

Lemma 4.6. *Let R be a regular semiring and M be an R-semimodule. Then $(E(M) :_M a) = (E^+(R) :_R a)M$ for all $a \in R$.*

Proof. Let $a \in R$ and $x \in (E^+(R) :_R a)M$. Then $x = r_1m_1 + r_2m_2 + \cdots + r_nm_n$, where $r_i \in (E^+(R) :_R a)$ and $m_i \in M$ for $1 \leq i \leq n$. So, $ar_i \in E^+(R)$, for all i. Therefore, $ax = a(r_1m_1 + r_2m_2 + \cdots + r_nm_n) \in E(M)$ and hence $x \in (E(M) :_M a)$. Thus $(E^+(R) :_R a)M \subseteq (E(M) :_M a)$. For the reverse inclusion, let $y \in (E(M) :_M a)$. Then $ay \in E(M)$. Now, R is regular implies $a = ra^2$ for some $r \in R$. Then we have $a(1+r'a) \in E^+(R)$ and so $(1+r'a) \in (E^+(R) :_R a)$. Also, $y = (1+r'a+ra)y = ray + (1+r'a)y \in (E^+(R) :_R a)M$, as $(1+r'a)y \in (E^+(R) :_R a)M$ and $ay \in E(M)$ implies $ray \in E(M) \subseteq (E^+(R) :_R a)M$. Therefore, $(E(M) :_M a) \subseteq (E^+(R) :_R a)M$. Hence $(E(M) :_M a) = (E^+(R) :_R a)M$ for all $a \in R$. □

Lemma 4.7. *Let R be a regular semiring and M be an R-semimodule. Then $(I \cap J)M = IM \cap JM$ for any ideal I, J of R.*

Proof. Let $x \in IM \cap JM$. Then $x = r_1m_1 + r_2m_2 + \cdots + r_nm_n = s_1x_1 + s_2x_2 + \cdots + s_kx_k$ where $r_i \in I$, $s_j \in J$ and m_i, $x_j \in M$ for $1 \leq i \leq n$, $1 \leq j \leq k$. Then $x \in \langle r_1, r_2, \cdots, r_n \rangle M \cap \langle s_1, s_2, \cdots, s_k \rangle M$. Again from Theorem 4.1, we have $\langle r_1, r_2, \cdots, r_n \rangle = \langle e \rangle$ and $\langle s_1, s_2, \cdots, s_k \rangle = \langle f \rangle$ for some idempotent element e, f of R. Then $x \in \langle e \rangle M \cap \langle f \rangle M$ and so $x = ey = fz$ for some y, $z \in M$. This implies $x = efx$ and hence $x \in IJM$, as $ef \in IJ$. Thus $(I \cap J)M = IM \cap JM$. □

Theorem 4.8. *Let R be a regular semiring and M be an R-semimodule such that $E(M)$ is a k-set. Then for every $x \in M$, $c(x) = Ann(Ann(x))$.*

Proof. Let $ax \in E(M)$ for some $a \in R$. This implies $x \in (E(M) :_M a) = (E^+(R) :_R a)M$, (by Lemma 4.6). Now, in [16, Corollary 3.23], we have proved that for a regular semiring R, $E^+(R)$ is a k-ideal of R. For the sake of completeness we also prove this here. For this let, $r \in \overline{E^+(R)}$. Then $r + e \in E^+(R)$ for some $e \in E^+(R)$. Thus $r^2 + re \in E^+(R)$. Therefore, $r^2 + (r+r')e \in E^+(R)$ and so $r^2 = r^2 + r(r+r')e \in E^+(R)$. As R is a regular semiring, there exists $x \in R$ such that $r = r^2x$. Hence $r \in E^+(R)$. Therefore, $E^+(R)$ is a k-ideal. This implies $(E^+(R) :_R a)$ is a full k-ideal of R. Therefore, by the definition of k-content of x we have $c(x) \subseteq (E^+(R) :_R a) = Ann(a)$ for all $a \in Ann(x)$. Then $c(x) \subseteq \bigcap_{a \in Ann(x)} Ann(a)$. Therefore, $c(x) \subseteq Ann(Ann(x))$.

Now, we have to show that $Ann(Ann(x)) \subseteq c(x)$. Let I be a finitely generated full ideal of R such that $x \in \overline{IM}$. First, we show that $Ann(I) \subseteq Ann(x)$. For this let $r \in Ann(I)$. Then $rx \in r\overline{IM} \subseteq \overline{rIM} \subseteq \overline{E(M)} = E(M)$. So $r \in Ann(x)$. Therefore, $Ann(I) \subseteq Ann(x)$. This implies $Ann(Ann(x)) \subseteq Ann(Ann(I)) = I$ (by Lemma 4.5). Thus $Ann(Ann(x)) \subseteq \bigcap\{I :$ I is a finitely generated full ideal of R such that $x \in \overline{IM}\} = F(x)$ (say). Now, let A be any full k-ideal of R such that $x \in \overline{AM}$. Then $x + x_1 \in AM$ for some $x_1 \in AM$. This implies $x + (x_1 + x_1') = c_1m_1 + c_2m_2 + \cdots + c_nm_n$, where $c_i \in A$ and $m_i \in M$ for $1 \leq i \leq n$. Then $x + (x_1 + x_1') \in \langle c_1, c_2, \cdots, c_n \rangle M \subseteq (\langle c_1, c_2, \cdots, c_n \rangle + E^+(R))M = JM$, where $J = \langle c_1, c_2, \cdots, c_n \rangle + E^+(R)$. Then J is a finitely generated full ideal of R such that $x \in \overline{JM}$. Then by the definition of $F(x)$, we have $F(x) \subseteq J \subseteq A$. Since A is any full k-ideal of R such that $x \in \overline{AM}$, we have $F(x) \subseteq c(x)$. Thus $Ann(Ann(x)) \subseteq F(x) \subseteq c(x)$. Hence the result.

□

Theorem 4.9. *Let R be a regular semiring and M be an R-semimodule such that $E(M)$ is a k-set. Then M is a k-content R-semimodule if and only if $Ann(x)$ is finitely generated for all $x \in M$.*

Proof. Let $x \in M$ such that $Ann(x)$ is finitely generated. Then we have $Ann(x) = \langle a_1, a_2, \cdots, a_n \rangle$ for some $a_i \in R$. Then for all $i = 1, 2, \cdots, n$; $a_i x \in E(M)$ implies $x \in \bigcap_{i=1}^{n}(E(M) :_M a_i) = \bigcap_{i=1}^{n}(E^+(R) :_R a_i)M = [\bigcap_{i=1}^{n}(E^+(R) :_R a_i)]M$ [by Lemma 4.7]. Then, we have $x \in Ann(Ann(x))M$. Again, by Theorem 4.8, we have $c(x) = Ann(Ann(x))$ and hence $x \in c(x)M$. Therefore, M is a k-content R-semimodule.

Conversely, let M be a k-content R-semimodule and $r \in Ann(x)$ for any $x \in M$. Then $rx \in E(M)$ implies $x \in (E(M) :_M r) = (E^+(R) :_R r)M$. So, $c(x) \subseteq (E^+(R) :_R r)$ and hence $rc(x) \subseteq E^+(R)$. Therefore, we have $r \in Ann(c(x))$. Thus $Ann(x) \subseteq Ann(c(x))$. Again, to show $Ann(c(x)) \subseteq Ann(x)$, let $r_1 \in Ann(c(x))$. This implies $r_1 c(x) \subseteq E^+(R)$ and therefore, $r_1 c(x)M \subseteq E(M)$, i.e., $r_1\left(\overline{c(x)M}\right) \subseteq \overline{r_1 c(x)M} \subseteq \overline{E(M)} = E(M)$, as $E(M)$ is a k-set. Since M is a k-content semimodule, we must have $x \in \overline{c(x)M}$ and this implies $r_1 x \in E(M)$, i.e., $r_1 \in Ann(x)$. Thus $Ann(c(x)) \subseteq Ann(x)$ and hence $Ann(c(x)) = Ann(x)$. Also, from Lemma 3.3, we have $c(x) = \overline{J}$, for some finitely generated full ideal J of R. Therefore, $Ann(\overline{J}) = Ann(x)$. Now, we show that $Ann(\overline{J}) = Ann(J)$. For this let, $s \in Ann(J)$. We have to show that $s \in Ann(\overline{J})$. Now, for any $t \in \overline{J}$, we have $t + t_1 \in J$ for some $t_1 \in J$. This implies $s(t + t_1) \in sJ \subseteq E^+(R)$, where $st_1 \in E^+(R)$. So $st \in E^+(R)$, as $E^+(R)$ is a k-set. This leads to, $s \in Ann(\overline{J})$ and hence $Ann(J) \subseteq Ann(\overline{J})$. Also, it is clear that $Ann(\overline{J}) \subseteq Ann(J)$. So $Ann(\overline{J}) = Ann(J)$. Therefore, we have $Ann(x) = Ann(J)$. Again, from Lemma 4.5, J is finitely generated full ideal of R implies $Ann(J)$ is finitely generated and hence $Ann(x)$ is finitely generated. □

5 Conclusion remarks

This paper aims to introduce and investigate several properties of k-content semimodule as a generalization of content module and content semimodule. Also, some interesting properties of k-content semimodules over regular semiring have been discussed. Therefore, the findings of this study are varied and important, making it intriguing and worth exploring further in the future.

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