Palestine Journal of Mathematics

Vol 13(3)(2024) , 147–157

ON THE SOLUTION OF THE n-DIMENSIONAL OPERATOR $\circledast^k$ RELATED TO THE WAVE OPERATOR

S. Bupasiri

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 46F10.

Keywords and phrases: Laplacian, Diamond operator, Wave operator.

The author would like to thank the referee for his suggestions which enhanced the presentation of the paper. The author was supported by Sakon Nakhon Rajabhat University.

.

Corresponding Author: S. Bupasiri

Abstract In this paper, we study the fundamental solution of the partial differential equation, iterated k-times of the form

$$\circledast^k G(x,m) = (\oplus + m^2)^k \left(\frac{1}{8}\triangle^4 + \frac{1}{8}\square^4 + \frac{6}{8}\diamond^2\right)^k G(x,m) = \delta$$

where m is a non-negative real number, $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, k is a non-negative integer. After that, we apply the fundamental solution related to the operator $\otimes^k$, ultra - hyperbolic operator $\square^k$, Laplace operator $\triangle^k$ and wave operator.

1 Introduction

The diamond operator iterated k-times, first introduced by Kananthai [2], is one of the most well-known partial differential operators in the theory of distribution or the generalized function. Kananthai [2] has studied the fundamental solution of the equation $\diamondsuit^k u(x)=\delta$, we obtain $u(x)=R^H_{2k}(u)*(-1)^k R^e_{2k}(v)$ is the fundamental solution and δ is the Dirac delta function. Later, Kananthai, Suantai and Longani [3] have studied the relationship between the operator $\oplus^k$ and the wave operator, and the relationship between the operator $\oplus^k$ and the Laplacian. Moreover, the equation $\oplus^k K(x)=\delta$ we have $K(x)=[R^H_{2k}(u)*(-1)^k R^e_{2k}(v)]*S_{2k}(w)*T_{2k}(z)$ is the fundamental solution of the operator $\oplus^k$, which is defined by

$$\begin{aligned}\oplus^k &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k\\ &= \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\end{aligned}$$

, δ is the Dirac delta function. Kananthai [2] has studied the diamond operator, which is defined by

$$\diamondsuit^k = \left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \tag{1.1}$$

Otherwise, the operator $\diamondsuit^k$ can also be expressed in the form $\diamondsuit^k = \Box^k \triangle^k = \triangle^k \Box^k$, where $\Box^k$ is the ultra-hyperbolic operator iterated k-times, which is defined by

$$\Box^k = \left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \tag{1.2}$$

$\triangle^k$ is the Laplace operator iterated k-times, which is defined by

$$\triangle^k = \left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, p+q=n. \tag{1.3}$$

By putting $p = k = 1$ and $x_1 = t$ (time) in (1.2), then we obtain the wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}. \tag{1.4}$$

Tariboon and Kananthai [4] have studied the Green's function of the operator

$$(\oplus + m^2)^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 + m^2 \right]^k \tag{1.5}$$

, iterated k-times. Moreover, the operator $(\oplus + m^2)^k$ can be related to the ultra-hyperbolic Klein Gordon operator $(\Box + m^2)^k$, the Helmholtz operator $(\triangle + m^2)^k$ and the diamond Klein - Gordon operator of the form $(\diamondsuit + m^2)^k$. Satsanit [11] has shown that

$$\odot^k = \left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k = \left(\frac{\triangle^2 + \Box^2}{2} \right)^k. \tag{1.6}$$

Therefore, from (1.6), we obtain

$$\otimes^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^4 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k = \left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2 \right)^k, \tag{1.7}$$

where $p+q=n$ is the dimension of the Euclidean space, $\mathbb{R}^n$ and k are a non-negative integer. In 1988, Trione [8] studied the fundamental solution of the ultra-hyperbolic Klein-Gordon operator iterated k-times such that operator $(\Box + m^2)^k$, which is defined by

$$(\Box + m^2)^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) + m^2 \right]^k. \tag{1.8}$$

From (1.5) and (1.7) the operator $\circledast^k$ can be expressed in the form

$$\begin{aligned}
\circledast^k &= \left[\left(\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^4 + \frac{m^2}{2} \right)^2 - \left(\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 - \frac{m^2}{2} \right)^2 \right]^k \\
&= \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 + m^2 \right]^k \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^4 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k \\
&= (\oplus + m^2)^k \left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2 \right)^k = (\oplus + m^2)^k \otimes^k.
\end{aligned} \tag{1.9}$$

For $m = 0$ then (1.9) becomes

$$\circledast^k = \oplus^k \otimes^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^8 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^8 \right]^k. \tag{1.10}$$

Kananthai, Suantai and Longani [3] have studied the relationship between L_1^k and L_2^k are defined by

$$L_1^k = \left[\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \tag{1.11}$$

and

$$L_2^k = \left[\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k. \tag{1.12}$$

Following that

$$L^k = L_1^k L_2^k = L_2^k L_1^k = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k. \tag{1.13}$$

Bupasiri [9] has studied the operator $\oplus_m^k$, iterated k-times of the equation $\oplus_m^k H(x,m) = \delta$, $H(x,m) = W_{2k}(x,m) * Y_{2k}(x,m) * M_{2k}(x,m) * N_{2k}(x,m)$, δ is the Dirac delta function, k is a non-negative integer and m is a non-negative real number. From (1.9) with $q = m = 0$ and $k = 1$, we obtain the Laplace operator of p-dimension

$$\circledast = \triangle_p^8,$$

where

$$\triangle_p = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}. \tag{1.14}$$

In this paper, we study the fundamental solution of the equation $\circledast^k G(x,m) = \delta$, where $G(x,m)$ is the fundamental solution, δ is the Dirac delta function, k is a non-negative integer and m is a non-negative real number. In particular, for $m = 0$ and $m = q = 0$ the fundamental solution related to the operator $\otimes^k$, $\Box^k$ and $\triangle^k$.

2 Preliminary Notes

We have studied some properties of the *ultra-hyperbolic kernel* and the *elliptic kernel of Marcel Riesz* which will be used as follows.

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.1}$$

where $p + q = n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki (see [12], p.72) that

$$R_\alpha^H(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.2}$$

$R_\alpha^H(u)$ is called the *ultra-hyperbolic kernel of Marcel Riesz*. Here α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$, which is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \tag{2.3}$$

and p is the number of positive terms of

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n$$

and let supp $R_\alpha^H(x) \subset \overline{\Gamma}_+$. Now $R_\alpha^H(x)$ is an ordinary function if Re $\alpha \geq n$ and is a distribution of α if Re $\alpha < n$.

Now, if $p = 1$ then (2.2) reduces to the function $M_\alpha(u)$ say, and is defined by

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if } x \in \Gamma_+; \\ 0, & \text{if } x \notin \Gamma_+ \end{cases} \tag{2.4}$$

where $u = x_1^2 - x_2^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} 2^{\alpha-1} \Gamma(\frac{\alpha-n+2}{2})$. The function $M_\alpha(u)$ is called the *hyperbolic kernel of Marcel Riesz.*

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,

$$v = x_1^2 + x_2^2 + \cdots + x_p^2 + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2. \tag{2.5}$$

Define the function

$$R_\alpha^e(v) = \frac{v^{\frac{\alpha-n}{2}}}{H_n(\alpha)} \tag{2.6}$$

where α is any complex number and the constant $H_n(\alpha)$ is given by the formula

$$H_n(\alpha) = \frac{\pi^{\frac{1}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.7}$$

Now the function $R_\alpha^e(v)$ is called the *Elliptic Kernel of Marcel Riesz.*

Lemma 2.3. *[2] Given the equation $\triangle^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\triangle^k$ is the Laplace operator iterated k-times, which is defined by (1.3). Then $u(x) = (-1)^k R_{2k}^e(v)$ is the fundamental solution of the operator $\triangle^k$ where*

$$R_{2k}^e(v) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k} \pi^{\frac{n}{2}} \Gamma(k)} |v|^{2k-n}. \tag{2.8}$$

Lemma 2.4. *[8] If $\square^k u(x) = \delta$ for $x \in \Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0$ and $u > 0\}$, where $\square^k$is the ultra-hyperbolic operator iterated k-times, which is defined by (1.2). Then $u(x) = R_{2k}^H(u)$ is the unique fundamental solution of the operator $\square^k$ where*

$$R_{2k}^H(u) = \frac{u^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} = \frac{(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\left(\frac{2k-n}{2}\right)}}{K_n(2k)} \tag{2.9}$$

for

$$K_n(2k) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+2k-n}{2}\right) \Gamma\left(\frac{1-2k}{2}\right) \Gamma(2k)}{\Gamma\left(\frac{2+2k-p}{2}\right) \Gamma(\frac{p-2k}{2})}. \tag{2.10}$$

Lemma 2.5. *[2] Given the equation $\diamondsuit^k u(x) = \delta$ for $x \in \mathbb{R}^n$, then $u(x) = (-1)^k R_{2k}^e(v) * R_{2k}^H(u)$ is the unique fundamental solution of the operator $\diamondsuit^k$, where $\diamondsuit^k$ is the diamond operator iterated k- times, which is defined by (1.1), $R_{2k}^e(v)$ and $R_{2k}^H(u)$ are defined by (2.8) and (2.9), respectively. Moreover, $(-1)^k R_{2k}^e(v) * R_{2k}^H(u)$ is a tempered distribution.*

It is not difficult to show that $R_{-2k}^e(v) * R_{-2k}^H(u) = (-1)^k \diamondsuit^k \delta$, for k is a non-negative integer.

Lemma 2.6. *[3] Given the equation $L_1^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where L_1^k is the operator, which is defined by (1.11), then $u(x) = (-1)^k (-i)^{\frac{q}{2}} S_{2k}(w)$ is the fundamental solution of the operator L_1^k, where*

$$S_{2k}(w) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k} \pi^{\frac{n}{2}} \Gamma(k)} [x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + \cdots + x_{p+q}^2)]^{\left(\frac{2k-n}{2}\right)}, i = \sqrt{-1}, \tag{2.11}$$

$$w = x_1^2 + x_2^2 + \cdots + x_p^2 - i(x_{p+1}^2 + \cdots + x_{p+q}^2).$$

Lemma 2.7. *[3] Given the equation $L_2^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where L_2^k is the operator, which is defined by (1.12), then $u(x) = (-1)^k (i)^{\frac{q}{2}} T_{2k}(z)$ is the fundamental solution of the operator L_2^k, where*

$$T_{2k}(z) = \frac{\Gamma\left(\frac{n-2k}{2}\right)}{2^{2k}\pi^{\frac{n}{2}}\Gamma(k)}[x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + \cdots + x_{p+q}^2)]^{\left(\frac{2k-n}{2}\right)}, i = \sqrt{-1}. \tag{2.12}$$

$$z = x_1^2 + x_2^2 + \cdots + x_p^2 + i(x_{p+1}^2 + \cdots + x_{p+q}^2).$$

Lemma 2.8. *[3] Given the equation $L^k u(x) = \delta$ for $x \in \mathbb{R}^n$, then $u(x) = S_{2k}(w) * T_{2k}(z)$ is the fundamental solution of the operator L^k, which is defined by (1.13), $S_{2k}(w)$ and $T_{2k}(z)$ are defined by (2.11) and (2.12), respectively.*

Lemma 2.9. *[11] (Convolution of $R_\alpha^e(v)$ and $R_\alpha^H(u)$). If $R_\alpha^e(v)$ and $R_\alpha^H(u)$ are defined by (2.8) and (2.9) respectively, then*

(i) $R_\alpha^e(v) * R_\beta^e(v) = R_{\alpha+\beta}^e(v)$ *where α and β are complex parameters;*

(ii) $R_\alpha^H(u) * R_\beta^H(u) = R_{\alpha+\beta}^H(u)$ *where α and β are both integers and except only the case both α and β are both integers.*

Lemma 2.10. *The function $R_{-2k}^H(u)$ and $(-1)^k R_{-2k}^e(v)$ are the inverse in the convolution algebra of $R_{2k}^H(u)$ and $(-1)^k R_{2k}^e(v)$ respectively. That is,*

$$R_{-2k}^H(u) * R_{2k}^H(u) = R_{-2k+2k}^H(u) = R_0^H(u) = \delta,$$

$$(-1)^k R_{-2k}^e(v) * (-1)^k R_{2k}^e(v) = (-1)^{2k} R_{-2k+2k}^e(v) = R_0^e(v) = \delta.$$

Proof. For proof of the this Lemma is given (see [6, 7, 10]). □

Lemma 2.11. *[5] (Convolution of $S_\gamma(w)$ and $T_\gamma(z)$). If $S_\gamma(w)$ and $T_\gamma(z)$ are defined by (2.11) and (2.12), respectively., then*

(i) $S_\gamma(w) * S_{\gamma'}(w) = (i)^{q/2} S_{\gamma+\gamma'}(w)$;

(ii) $T_\gamma(z) * T_{\gamma'}(z) = (-i)^{q/2} T_{\gamma+\gamma'}(z)$ *where γ and γ' are complex parameters.*

Moreover, $S_0(w) = (i)^{q/2}\delta$ and $T_0(w) = (-i)^{q/2}\delta$.

Lemma 2.12. *[4] Given the equation*

$$(\oplus + m^2)^k u(x) = \delta, \tag{2.13}$$

where $(\oplus + m^2)^k$ is the operator iterated k-times, which is defined by (1.5), δ is the Dirac delta function, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, m is a non-negative real number and k is a non-negative integer, we obtain

$$\begin{aligned} u(x) &= Y_{2k,2k,2k,2k}(u, v, w, z, m) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} m^{2r} R_{2k+2r}^H(u) * (-1)^{k+r} R_{2k+2r}^e(v) * S_{2k+2r}(w) * T_{2k+2r}(z) \end{aligned} \tag{2.14}$$

is the fundamental solution of (2.13). Since

$$\begin{aligned} &Y_{2k,2k,2k,2k}(u, v, w, z, m) \\ &= \binom{-k}{0} m^{2(0)} R_{2k+2(0)}^H(u) * (-1)^{k+0} R_{2k+2(0)}^e(v) * S_{2k+2(0)}(w) * T_{2k+2(0)}(z) \\ &+ \sum_{r=1}^{\infty} \binom{-k}{r} m^{2r} R_{2k+2r}^H(u) * (-1)^{k+r} R_{2k+2r}^e(v) * S_{2k+2r}(w) * T_{2k+2r}(z). \end{aligned} \tag{2.15}$$

The second summand of the right-hand member of (2.15) vanishes for $m = 0$ and then, we have $Y_{2k,2k,2k,2k}(u, v, w, z, 0) = R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * S_{2k}(w) * T_{2k}(z)$ which is the fundamental solution of the operator $\oplus^k$.

3 Main Results

Theorem 3.1. *Given the equation*

$$\otimes^k G(x) = \left(\frac{\triangle^4 + \Box^4 + 6\diamond^2}{8}\right)^k G(x) = \delta \tag{3.1}$$

for $x \in \mathbb{R}^n$, where $\otimes^k$ is the operator iterated k-times, which is defined by (1.7) . Then we obtain $G(x)$ is the fundamental solution of the equation (3.1), where

$$G(x) = (R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v)) * (H^{*k}(x))^{*-1} \tag{3.2}$$

or

$$G(x) = (R^H_{12k}(u) * R^e_{12k}(v)) * (H^{*k}(x))^{*-1} \tag{3.3}$$

and

$$H(x) = \frac{1}{8}\left(R^H_{12}(u) * (-1)^2 R^e_4(v)\right) + \frac{1}{8}\left(R^H_4(u) * (-1)^6 R^e_{12}(v)\right) + \frac{6}{8}\left(R^H_8(u) * (-1)^4 R^e_8(v)\right) \tag{3.4}$$

or

$$H(x) = \frac{1}{8}\left(R^H_{12}(u) * R^e_4(v)\right) + \frac{1}{8}\left(R^H_4(u) * R^e_{12}(v)\right) + \frac{6}{8}\left(R^H_8(u) * R^e_8(v)\right).$$

Here $H^{*k}(x)$ denotes the convolution of $H(x)$ itself k-times, $(H^{*k}(x))^{*-1}$ denotes the inverse of $H^{*k}(x)$ in the convolution algebra. Moreover, $G(x)$ is a tempered distribution.

Proof. We have

$$\otimes^k G(x) = \left(\frac{\triangle^4 + \Box^4 + 6\diamond^2}{8}\right)^k G(x) = \delta$$

or we can write

$$\left(\frac{\triangle^4 + \Box^4 + 6\diamond^2}{8}\right)\left(\frac{\triangle^4 + \Box^4 + 6\diamond^2}{8}\right)^{k-1} G(x) = \delta.$$

Convolving both sides of the above equation by $R^H_{12}(u) * (-1)^6 R^e_{12}(v)$,

$$\begin{aligned}&\left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)\left(R^H_{12}(u) * (-1)^6 R^e_{12}(v)\right)\left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)^{k-1} G(x)\\ &= \delta * \left(R^H_{12}(u) * (-1)^6 R^e_{12}(v)\right)\end{aligned}$$

or

$$\begin{aligned}&\left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)\left(R^H_{12}(u) * (-1)^6 R^e_{12}(v)\right)\left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)^{k-1} G(x)\\ &= \frac{1}{8}\triangle^4\left(R^H_8(u) * (-1)^4 R^e_8(v) * (R^H_4(u) * (-1)^2 R^e_4(v))\right)\\ &\quad + \frac{1}{8}\Box^4\left(R^H_8(u) * (-1)^4 R^e_8(v) * (R^H_4(u) * (-1)^2 R^e_4(v))\right)\\ &\quad + \frac{6}{8}\diamond^2\left(R^H_8(u) * (-1)^4 R^e_8(v) * (R^H_4(u) * (-1)^2 R^e_4(v))\right)\\ &\quad \times\left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)^{k-1} G(x) \quad \text{(by Lemma 2.9)}\\ &= \delta * \left(R^H_{12}(u) * (-1)^6 R^e_{12}(v)\right).\end{aligned}$$

By Lemma 2.3, Lemma 2.4 and Lemma 2.5, we obtain

$$\frac{1}{8}\delta * \left(R_8^H(u) * (R_4^H(u) * (-1)^2 R_4^e(v))\right) + \frac{1}{8}\delta * \left((-1)^4 R_8^e(v) * (R_4^H(u) * (-1)^2 R_4^e(v))\right)$$
$$+ \frac{6}{8}\delta * \left(R_8^H(u) * (-1)^4 R_8^e(v)\right) \left(\frac{1}{8} \triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)^{k-1} G(x)$$
$$= \delta * \left(R_{12}^H(u) * (-1)^6 R_{12}^e(v)\right).$$

By properties of convolutions and Lemma 2.9,

$$\frac{1}{8}\left(R_{12}^H(u) * (-1)^2 R_4^e(v)\right) + \frac{1}{8}\left(R_4^H(u) * (-1)^6 R_{12}^e(v)\right)$$
$$+ \frac{6}{8}\left(R_8^H(u) * (-1)^4 R_8^e(v)\right) \left(\frac{1}{8} \triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)^{k-1} G(x)$$
$$= \left(R_{12}^H(u) * (-1)^6 R_{12}^e(v)\right).$$

Keeping on convolving both sides of the above equation by $R_{12}^H(u) * (-1)^6 R_{12}^e(v)$, up to $k-1$ times, we obtain

$$H^{*k}(x) * G(x) = \left(R_{12}^H(u) * (-1)^6 R_{12}^e(v)\right)^{*k} \tag{3.5}$$

the symbol $*k$ denotes the convolution of itself k-times. By properties of $R_{2k}^H(u)$ and $R_{2k}^e(v)$ in Lemma 2.9, we have

$$\left(R_{12}^H(u) * (-1)^6 R_{12}^e(v)\right)^{*k}(x) = R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v).$$

Thus (3.5) becomes,

$$H^{*k}(x) * G(x) = R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v),$$
$$G(x) = \left(R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v)\right) * \left(H^{*k}(x)\right)^{*-1} \tag{3.6}$$

or

$$G(x) = \left(R_{12k}^H(u) * R_{12k}^e(v)\right) * \left(H^{*k}(x)\right)^{*-1} \tag{3.7}$$

is the fundamental solution of (3.1). We consider the function $H^{*k}(x)$, since $R_{12}^H(u) * (-1)^6 R_{12}^e(v)$ is a tempered distribution. Thus $H(x)$ defined by (3.4) is tempered distribution, we obtain $H^{*k}(x)$ is tempered distribution.

Now, $R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v) \in S'$, the space of tempered distribution. Choose $S' \subset D'_R$, where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v) \in D'_R$. It follows that $R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v)$ is an element of convolution algebra, since D'_R is a convolution algebra. Hence Zemanian [1], (3.3) has a unique solution

$$G(x) = \left(R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v)\right) * \left(H^{*k}(x)\right)^{*-1},$$

or

$$G(x) = \left(R_{12k}^H(u) * R_{12k}^e(v)\right) * \left(H^{*k}(x)\right)^{*-1},$$

where $\left(H^{*k}(x)\right)^{*-1}$ is an inverse of $H^{*k}(x)$ in the convolution algebra. $G(x)$ is called the fundamental solution of the operator $\otimes^k$.

Since $R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v)$ and $\left(H^{*k}(x)\right)^{*-1}$ are lies in S', then by (see [1], p.152) again, we have $\left(R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v)\right) * \left(H^{*k}(x)\right)^{*-1} \in S'$. Hence, $G(x)$ is a tempered distribution. □

Theorem 3.2. *Given the equation*

$$\circledast^k G(x,m) = (\oplus + m^2)^k \otimes^k G(x,m) = \delta \tag{3.8}$$

where $(\oplus + m^2)^k$ and $\otimes^k$ are the operators iterated k-times, which is defined by (1.5) and (1.7), respectively, δ is the Dirac delta function, $x \in \mathbb{R}^n$, m is a non-negative real number and k is a non-negative integer. Then we obtain

$$G(x,m) = Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R_{12k}^H(u) * (-1)^{6k} R_{12k}^e(v) * (H^{*k}(x))^{*-1}\right] \tag{3.9}$$

or

$$G(x,m) = Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * R^e_{12k}(v) * (H^{*k}(x))^{*-1}\right] \tag{3.10}$$

is the fundamental solution for the operator $\circledast^k$ *iterated* k*-times, which is defined by* (1.9)*. In particular,* $m = 0$ *then* (3.8) *becomes*

$$\circledast^k G(x,0) = \oplus^k \otimes^k G(x,0) = \left[\left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2}\right)^8 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^8\right]^k G(x,0) = \delta, \tag{3.11}$$

we obtain

$$G(x,0) = (-1)^{7k} R^e_{14k}(v) * R^H_{14k}(u) * S_{2k}(w) * T_{2k}(z) * (H^{*k}(x)^{*-1} \tag{3.12}$$

is the fundamental solution of the (3.11)*, for* $q = m = 0$ *then* (3.8) *becomes*

$$\triangle_p^{8k} G(x,0) = \delta, \tag{3.13}$$

we obtain

$$G(x,0) = R^e_{16k}(v) \tag{3.14}$$

is the fundamental solution of (3.13)*, where* $\triangle_p^{8k}$ *is the Laplace operator of* p*-dimension, iterated* $8k$*-times, which is defined by* (1.14)*. Moreover, from* (3.12) *we obtain*

$$\big(R^H_{-12k}(u) * (-1)^{7k} R^e_{-14k}(v) * S_{-2k}(w) * T_{-2k}(z)\big) * \big(H^{*k}(x)\big) * G(x,0) = R^H_{2k}(u) \tag{3.15}$$

is the fundamental solution of the ultra-hyperbolic operator $\square^k$ *iterated* k*-times, which is defined by* (1.2)*,*

$$\big(R^H_{-14k}(u) * (-1)^{6k} R^e_{-12k}(v) * S_{-2k}(w) * T_{-2k}(z)\big) * \big(H^{*k}(x)\big) * G(x,0) = (-1)^k R^e_{2k}(v) \tag{3.16}$$

or

$$\big(R^H_{-14k}(u) * R^e_{-12k}(v) * S_{-2k}(w) * T_{-2k}(z)\big) * \big(H^{*k}(x)\big) * G(x,0) = (-1)^k R^e_{2k}(v) \tag{3.17}$$

is the fundamental solution of the Laplace operator $\triangle^k$ *iterated* k*-times, which is defined by* (1.3) *and*

$$\big(R^H_{-14k}(u) * (-1)^{7k} R^e_{-14k}(v)\big) * \big(H^{*k}(x)\big) * G(x,0) = S_{2k}(w) * T_{2k}(z) \tag{3.18}$$

is the fundamental solution of the operator $L^k = L_1^k L_2^k$ *iterated* k*-times, which is defined by* (1.13)*, where* $R^e_{-14k}(v)$*,* $R^H_{-14k}(u)$*,* $S_{-2k}(w)$*, and* $T_{-2k}(z)$ *are the inverse of* $R^e_{14k}(v)$*,* $R^H_{14k}(u)$*,* $S_{2k}(w)$*, and* $T_{2k}(z)$*, respectively. From* (3.12) *and* (3.15) *with* $p = 1, q = n-1, k = 1$*,* $m = 0$ *and* $x_1 = t$ *(time), we obtain*

$$\big((-1)^7 R^e_{-14}(v) * M^H_{-12}(u) * S_{-2}(w) * T_{-2}(z) * (H^*(x))\big) * G(x,0) = M^H_2(u) \tag{3.19}$$

or

$$\big(-R^e_{-14}(v) * M^H_{-12}(u) * S_{-2}(w) * T_{-2}(z) * (H^*(x))\big) * G(x,0) = M^H_2(u) \tag{3.20}$$

is the fundamental solution of the wave operator is defined by (1.4)*, where* $M_2(u)$ *is defined by* (2.4) *with* $\alpha = 2$*.*

Proof. From (1.9) and (3.8), we have

$$\circledast^k G(x,m) = (\oplus + m^2)^k \left(\frac{1}{8}\triangle^4 + \frac{1}{8}\square^4 + \frac{6}{8}\diamond^2\right)^k G(x,m) = \delta. \tag{3.21}$$

Convolving both sides of (3.21) by $Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * (H^{*k}(x))^{*-1}\right]$, we obtain

$$\begin{aligned}
&\big(Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * (H^{*k}(x))^{*-1}\right]\big) \\
&* (\oplus + m^2)^k \left(\frac{1}{8}\triangle^4 + \frac{1}{8}\square^4 + \frac{6}{8}\diamond^2\right)^k G(x,m) \\
&= \big(Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * (H^{*k}(x))^{*-1}\right]\big) * \delta.
\end{aligned}$$

By properties of convolution

$$\begin{aligned}&(\oplus + m^2)^k (Y_{2k,2k,2k,2k}(u,v,w,z,m))\\&*\left(\frac{1}{8}\triangle^4 + \frac{1}{8}\Box^4 + \frac{6}{8}\diamond^2\right)^k \left(\left[R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * (H^{*k}(x))^{*-1}\right]\right) * G(x,m)\\&\quad = Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * (H^{*k}(x))^{*-1}\right].\end{aligned}$$

By Lemma 2.12 and Theorem 3.1, we obtain,

$$\delta * \delta * G(x,m) = G(x,m) = Y_{2k,2k,2k,2k}(u,v,w,z,m) * \left[R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v) * (H^{*k}(x))^{*-1}\right] \tag{3.22}$$

is the fundamental solution of the operator $\circledast^k$. In particular, $m = 0$ then (3.8) becomes

$$\oplus^k \otimes^k G(x,0) = \delta, \tag{3.23}$$

from Lemma 2.12, Lemma 2.9 , (3.22) and by properties of convolution, we obtain

$$\begin{aligned}G(x,0) &= \left((-1)^k R^e_{2k}(v) * R^H_{2k}(u) * S_{2k}(w) * T_{2k}(z)\right) * \left((R^H_{12k}(u) * (-1)^{6k} R^e_{12k}(v)) * (H^{*k}(x))^{*-1}\right)\\&= (-1)^{7k} R^e_{14k}(v) * R^H_{14k}(u) * S_{2k}(w) * T_{2k}(z) * (H^{*k}(x)^{*-1}\end{aligned} \tag{3.24}$$

is the fundamental solution of (3.11), for $q = m = 0$ then (3.8) becomes

$$\triangle_p^{8k} G(x,0) = \delta, \tag{3.25}$$

where $\triangle_p^{8k}$ is the Laplace operator of p-dimension iterated $8k$-times. By Lemma 2.3, we have

$$G(x,0) = (-1)^{8k} R^e_{16k}(v) = R^e_{16k}(v)$$

is the fundamental solution of (3.25). Convolving both sides of (3.24) by

$$\left(R^H_{-12k}(u) * (-1)^{7k} R^e_{-14k}(v) * S_{-2k}(w) * T_{-2k}(z)\right) * \left(H^{*k}(x)\right),$$

we obtain

$$\begin{aligned}&\left(R^H_{-12k}(u) * (-1)^{7k} R^e_{-14k}(v) * S_{-2k}(w) * T_{-2k}(z)\right) * \left(H^{*k}(x)\right) * G(x,0)\\&= (R^H_{12k}(u) * R^H_{-12k}(u)) * ((-1)^{7k} R^e_{14k}(v) * (-1)^{7k} R^e_{-14k}(v))\\&* (S_{-2k}(w) * S_{2k}(w)) * (T_{2k}(z) * T_{-2k}(z)) * \left(\left(H^{*k}(x)\right) * \left(H^{*k}(x)\right)^{*-1}\right) * R^H_{2k}(u))\end{aligned}$$

or

$$\begin{aligned}&\left(R^H_{-12k}(u) * (-1)^{7k} R^e_{-14k}(v) * S_{-2k}(w) * T_{-2k}(z)\right) * \left(H^{*k}(x)\right) * G(x,0)\\&= \delta * \delta * \delta * \delta * \delta * R^H_{2k}(u) = R^H_{2k}(u)\end{aligned}$$

by Lemma 2.9, Lemma 2.10, Lemma 2.11, Theorem 3.1 and properties of convolution. It follows that

$$\left(R^H_{-12k}(u) * (-1)^{7k} R^e_{-14k}(v) * S_{-2k}(w) * T_{-2k}(z)\right) * \left(H^{*k}(x)\right) * G(x,0) = R^H_{2k}(u) \tag{3.26}$$

as the fundamental solution of the ultra-hyperbolic operator $\Box^k$ iterated k-times, which is defined by (1.2). Similarly,

$$\begin{aligned}&\left(R^H_{-14k}(u) * (-1)^{6k} R^e_{-12k}(v) * S_{-2k}(w) * T_{-2k}(z)\right) * \left(H^{*k}(x)\right) * G(x,0)\\&= \left(R^H_{14k}(u) * R^H_{-14k}(u)\right) * \left((-1)^{7k} R^e_{14k}(v) * (-1)^{6k} R^e_{-12k}(v)\right)\right)\\&* (S_{-2k}(w) * S_{2k}(w)) * (T_{2k}(z) * T_{-2k}(z)) * \left(\left(H^{*k}(x)\right) * \left(H^{*k}(x)\right)^{*-1}\right) * (-1)^{13k} R^e_{2k}(v))\end{aligned}$$

or

$$\big(R^H_{-14k}(u) * (-1)^{6k} R^e_{-12k}(v) * S_{-2k}(w) * T_{-2k}(z)\big) * \big(H^{*k}(x)\big) * G(x,0)$$
$$= \delta * \delta * \delta * \delta * \delta * (-1)^{13k} R^e_{2k}(v)) = (-1)^k R^e_{2k}(v)).$$

It follows that

$$\big(R^H_{-14k}(u) * (-1)^{6k} R^e_{-12k}(v) * S_{-2k}(w) * T_{-2k}(z)\big) * \big(H^{*k}(x)\big) * G(x,0) = (-1)^k R^e_{2k}(v)$$

is the fundamental solution of the Laplace operator $\triangle^k$ iterated k-times, which is defined by (1.3), and

$$\big(R^H_{-14k}(u) * (-1)^{7k} R^e_{-14k}(v)\big) * \big(H^{*k}(x)\big) * G(x,0)$$
$$= \big(R^H_{14k}(u) * R^H_{-14k}(u)\big) * \big((-1)^{7k} R^e_{14k}(v) * (-1)^{7k} R^e_{-14k}(v)\big)\big)$$
$$* \Big(\big(H^{*k}(x)\big) * \big(H^{*k}(x)\big)^{*-1}\Big) * S_{2k}(w) * T_{2k}(z)$$

or

$$\big(R^H_{-14k}(u) * (-1)^{7k} R^e_{-14k}(v)\big) * \big(H^{*k}(x)\big) * G(x,0)$$
$$= \delta * \delta * \delta * S_{2k}(w) * T_{2k}(z) = S_{2k}(w) * T_{2k}(z).$$

It follows that

$$\big(R^H_{-14k}(u) * (-1)^{7k} R^e_{-14k}(v)\big) * \big(H^{*k}(x)\big) * G(x,0) = S_{2k}(w) * T_{2k}(z)$$

is the fundamental solution of the operator L^k iterated k-times, which is defined by (1.13). In particular, if we put $p = 1, q = n-1, k = 1, m = 0$ and $x_1 = t$ (time) in (3.26) then $R^H_{-12}(u)$ reduces to $M^H_{-12}(u)$ and $R^H_2(u)$ reduce to $M^H_2(u)$ where $M^H_{-12}(u)$ and $M^H_2(u)$ are defined by (2.4) with $\alpha = -12$, $\alpha = 2$ respectively. Thus, (3.26) becomes

$$\big(M^H_{-12}(u) * (-1)^7 R^e_{-14}(v) * S_{-2}(w) * T_{-2}(z)\big) * (H^*(x)) * G(x,0) = M^H_2(u) \tag{3.27}$$

or

$$\big(M^H_{-12}(u) * (-R^e_{-14}(v)) * S_{-2}(w) * T_{-2}(z)\big) * (H^*(x)) * G(x,0) = M^H_2(u) \tag{3.28}$$

as the fundamental solution of the wave operator, which is defined by (1.4) and $R^e_{-14}(v)$ which is defined by (2.8). This completes the proof. □

References

[1] A. H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, (1964).

[2] A. Kananthai, *On the solutions of the n-dimensional diamond operator*, Appl. Math. Comput., **88**, 27-37, (1997).

[3] A. Kananthai, S. Suantai, V. Longani, *On the operator $\oplus^k$ related to the wave equation and Laplacian*, Appl. Math. Comput., **132**, 219-229, (2002).

[4] J. Tariboon, A. Kananthai, *On the Green function of the $(\oplus + m^2)^k$ operator*, Integral Transforms and Special Functions, **18**, 297-304, (2007).

[5] A. Kananthai, S. Suantai, *The convolution product of the distributional kernel $K_{\alpha,\beta,\gamma,v}$*, IJMMS., **3**, 153-158, (2003).

[6] M.A. Tellez, S.E. Trione, *The distributional convolution products of Marcel Riesz's ultra-hyperbolic Kernel*, Ravista de la Union Mathematica Argentina, **39**, 115-124, (1995).

[7] S.E. Trione, *On Marcel Riesz's Ultra-hyperbolic Kernel*, Studies in applied mathematics, **79**, 185-191, (1988).

[8] S.E. Trione, *On the elementary retarded, ultra-hyperbolic solution of the Klein-Gordon operator, iterated k-times*, Studies in Applied Mathematics, **79**, 127-141, (1988).

[9] S. Bupasiri, *On the Operator $\oplus_m^k$ Related to the Wave Equation and Laplacian*, European Journal of Pure and Applied Mathematics, **14**, 881-894, (2021).

[10] W.F. Donoghue, *Distribution and Fourier Transform*, Academic Press, New York, (1969).

[11] W. Satsanit, *Green function and Fourier transform for o-plus operator*, Electronic Journal of Differential Equation, **2010**, 1-14, (2010).

[12] Y. Nozaki, *On Riemann-Liouville integral of ultra-hyperbolic type*, Kodai Mathematical Seminar Reports, **16**, 69-87, (1964).

Author information

S. Bupasiri, Faculty of Education, Sakon Nakhon Rajabhat University, Thailand.
E-mail: sudprathai@gmail.com

Received: 2022-09-23
Accepted: 2024-05-03