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On an open problem concerning trees with equal independent *ve*-domination and domination numbers

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Abstract. In this note, we address an open problem posed in [Boutrig, Chellali, Haynes and Hedetniemi, Vertex-edge domination in graphs. Aquaestiones Mathematicae 90 (2016) 355-366] concerning the characterization of nontrivial trees T with equal independent *ve*-domination number $i_{ve}(T)$ and domination number $\gamma(T)$. We provide four equivalent conditions for trees T with $i_{ve}(T) = \gamma(T)$ involving, among others, the domination number, the total domination number $\gamma_t(T)$ and the *ve*-domination number $\gamma_{ve}(T)$.

1 Introduction

In a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$.

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S, and is a *total dominating set* if every vertex in V has a neighbor in S. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (respectively, total dominating set) of G.

A set S of vertices in a graph G is a *packing* if the vertices in S are pairwise at distance at least 3 apart in G, or equivalently, for every vertex $v \in V$, $|N[v] \cap S| \leq 1$. It is well-known [3] that for every graph G and packing S of G, $|S| \leq \gamma(G)$. As defined in Bange et al. [1], a dominating set S for which $|N[v] \cap S| = 1$ for all $v \in V$ is an *efficient dominating set*. Equivalently, a set S is an efficient dominating set if S is both a dominating set and a packing of G. Note that not every graph has an efficient dominating set, however as shown in [1], if a graph G has such a set, then every efficient dominating set is a minimum dominating set.

A vertex $u \in V$ is said to *ve-dominate* an edge $vw \in E$ if: (i) $u = v$ or $u = w$, that is, u is incident to vw, or (ii) uv or uw is an edge in G, that is, u is incident to an edge that is adjacent to vw. In other words, a vertex u *ve*-dominates all edges incident to any vertex in $N[u]$.

A set $S \subseteq V$ is a *vertex-edge dominating set* (or simply a *ved-set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. The minimum cardinality of a *ved*-set of G is called the *ve-domination number* $\gamma_{ve}(G)$.

A set $S \subset V$ is *independent* if no two vertices in S are adjacent. A set $S \subseteq V$ is an *independent vertex-edge dominating set* (or simply an *independent ved-set*) if S is both independent and *ved*-set. The *independent ve-domination number,* $i_{ve}(G)$, of G is the minimum cardinality of an independent *ved*-set of G. Clearly for every graph G, $\gamma_{ve}(G) \leq i_{ve}(G)$ (see [6]). Also, since any dominating set of a nontrivial connected graph G is a *ved*-set, we therefore have $\gamma_{ve}(G) \leq \gamma(G)$. Moreover, if a nontrivial connected graph G has an efficient dominating set S, then S is an independent *ved*-set of G leading to $\gamma_{ve}(G) \leq i_{ve}(G) \leq |S| = \gamma(G)$.

2 Main result

Restricted to the class of trees, Boutrig et al. [2] have shown that the domination number is an upper bound for the independent *ve*-domination number.

Theorem 2.1 ([2]). *For every nontrivial tree T, $\gamma_{ve}(T) \leq i_{ve}(T) \leq \gamma(T)$.*

Lewis et al. [7] were interested in the characterization of trees T such that $\gamma_{ve}(T) = \gamma(T)$ where both a descriptive and a constructive characterization of those trees were given. Let $\mathcal{T}$ be the family of trees that can be obtained from r disjoint stars, each of order at least three, by first adding $r-1$ edges so that they are incident only with leaves of the stars and the resulting graph is a tree in which every center vertex of a star remains a support vertex.

Theorem 2.2 ([7]). *For any tree T of order at least three, the following statements are equivalent:*

(1) $\gamma_{ve}(T) = \gamma(T)$.

(2) T has an efficient dominating set S where each vertex in S is a support vertex in T.

(3) $T \in \mathcal{T}$.

The characterization of trees T such that $i_{ve}(T) = \gamma(T)$ remained open and this problem was the first on the list of open problems presented in [2].

Our aim in this paper is to address this problem, the proof of which is based on the next result, which establishes a relationship between the independent *ve*-domination and the total domination numbers in bipartite graphs, thereby extending and strengthening the inequality chain in Theorem 2.1. It is well-known (see [3]) that for every graph G with no isolated vertex, $\gamma_t(G) \le 2\gamma(G)$.

Theorem 2.3. *For every nontrivial connected bipartite graph G, $\gamma_{ve}(G) \le i_{ve}(G) \le \gamma_t(G)/2 \le \gamma(G)$.*

Proof. We only need to show that $i_{ve}(G) \le \gamma_t(G)/2$. Let A and B be the partite sets of G with $|A| \le |B|$, and let D be a minimum total dominating set of G. Also, let $A' = D \cap A$ and $B' = B \cap D$. Note that A' and B' are both nonempty independent sets, and every edge of G is incident with either a vertex of $A' \cup B'$ or a vertex dominated by $A' \cup B'$. Now, since every vertex of B has a neighbor in A' we deduce that A' *ve*-dominates all edges of G. Therefore $\gamma_{ve}(G) \le i_{ve}(G) \le |A'| \le |D|/2 = \gamma_t(G)/2 \le \gamma(G)$. □

It is worth noting that the characterization of the graphs G with $\gamma_t(G) = 2\gamma(G)$ is considered in [5] as one of the most important open problems in total domination in graphs. However, for the class of trees a characterization was given by Henning [4] as follows.

Theorem 2.4 ([4]). *A tree T of order at least 3 satisfies $\gamma_t(T) = 2\gamma(T)$ if and only if T has a dominating set S such that the following two conditions hold:*

(1) Every vertex of S is a support vertex of T.

(2) The set S is a packing in T.

We note that conditions (1) and (2) in Theorem 2.4 immediately imply that the set S is an efficient dominating set where each vertex in S is a support vertex in T. We therefore deduce from Theorems 2.2 and 2.4 the following corollary.

Corollary 2.5. *For any tree T of order at least three, $\gamma_{ve}(T) = \gamma(T)$ if and only if $\gamma_t(T) = 2\gamma(T)$.*

Corollary 2.6. *For any tree T of order at least three, $i_{ve}(T) = \gamma(T)$ if and only if $\gamma_t(T) = 2\gamma(T)$.*

Proof. Assume that $i_{ve}(T) = \gamma(T)$. Since $\gamma_t(T) \le 2\gamma(T)$, Theorem 2.3 implies that $\gamma(T) = i_{ve}(T) \le \gamma_t(T)/2 \le \gamma(T)$ and therefore $\gamma_t(T) = 2\gamma(T)$.

Conversely, assume that $\gamma_t(T) = 2\gamma(T)$. By Corollary 2.5, $\gamma_{ve}(T) = \gamma(T)$ and by Theorem 2.1 $i_{ve}(T) = \gamma(T)$. □

We are now ready to state the main result which follows directly from the previous results.

Theorem 2.7. *For any tree T of order at least three, the following statements are equivalent:*

(1) $i_{ve}(T) = \gamma(T)$.

(2) $\gamma_{ve}(T) = \gamma(T)$.

(3) $\gamma_t(T) = 2\gamma(T)$.

(4) T *has an efficient dominating set* S *where each vertex in* S *is a support vertex in* T.

(5) $T \in \mathcal{T}$.

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