Palestine Journal of Mathematics

Vol 13(3)(2024) , 158–173

Some Simpson's type inequalities via fractional integral with respect to another function and its applications

Henok Desalegn Desta, Deepak B. Pachpatte, Jebessa B. Mijena and Tadesse Abdi

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 26A51; Secondary 26A33, 25D10, 26D15.

Keywords and phrases: Simpson's inequality, s-convex function, Hölder's inequality, Fractional integral operator.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Henok Desalegn Desta

Abstract *In this article, we obtained a new fractional integral identity, and using it, we established some new Simpson's type inequalities for s-convex functions in the second sense through the fractional integral operators of a function with respect to another function. Some applications of the obtained results to special means are also provided. Our results are more generalized in nature.*

1 Introduction

Integral inequalities have been more prevalent in many areas of mathematics, applied sciences, and engineering because they are essential in approximation theory and numerical analysis [1, 2]. Many researchers have studied and generalized the classical integral inequalities like Simpson's inequality, the Hermite-Hadamard inequality, Ostrowski's inequality, and Chebyshev inequality to fractional integral inequalities, for instance [3]-[5],[8],[12]-[14],[21]-[23] and the reference cited therein. In this paper, our attention will be given to Simpson's type inequality.

Let $\vartheta : [\varrho_1, \varrho_2] \to \mathbb{R}$ *be a four times continuously differentiable function on* (ϱ_1, ϱ_2) *and* $\|\vartheta^{(4)}\|_\infty = \sup_{x\in(\varrho_1,\varrho_2)} \left|\vartheta^{(4)}(x)\right| < \infty$. *Then the Simpson's inequality is given by:*

$$\left|\frac{1}{3}\left[\frac{\vartheta(\varrho_1)+\vartheta(\varrho_2)}{2}+2\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\right]-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\vartheta(x)dx\right| \leq \frac{(\varrho_2-\varrho_1)^4}{2880}\|\vartheta^{(4)}\|_\infty.$$

We cannot apply the classical Simpson quadrature formula when either f is not four times differential or when its fourth derivative is not bounded on the interval $[\varrho_1, \varrho_2]$. Alomari in [6] established new Simpson's type inequalities in terms of the first derivative for s-convex functions and provided some numerical quadrature rules. Recently, many authors studied similar inequalities where a regular integral in the Simpson's inequality is replaced by fractional integral of different forms. This inequality has been studied and generalized by many scholars; see for instance, [7, 10],[15]-[17],[20, 26] and the reference cited therein.

Definition 1.1. The function $\vartheta : [0,\infty) \to \mathbb{R}$ is said to be a convex function if the inequality

$$\vartheta\left(\rho x+(1-\rho)y\right) \leq \rho\vartheta(x)+(1-\rho)\vartheta(y),$$

holds for all $x, y \in [0,\infty)$ and $\rho \in [0,1]$.

Definition 1.2. The function $\vartheta : [0, \infty) \to \mathbb{R}$ is said to be s-convex function (in the second sense) or ϑ belongs to the class K_s^2, if

$$\vartheta(\rho x + (1-\rho)y) \leq \rho^s \vartheta(x) + (1-\rho)^s \vartheta(y),$$

for all $x, y \in [0, \infty)$, $\rho \in [0, 1]$ and $s \in (0, 1]$.

Remark 1.3. If $s = 1$ in Definition 1.2, then we have the Definition 1.1 of a convex function.

Theorem 1.4 ((Hölder's inequality for integral)[19]). *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ϑ and ψ are real function defined on $[\varrho_1, \varrho_2]$ and if $|\vartheta(x)|^p$ and $|\psi(x)|^p$ are integrable on $[\varrho_1, \varrho_2]$, then the following inequality holds:*

$$\int_{\varrho_1}^{\varrho_2} |\vartheta(x)\psi(x)| dx \leq \left(\int_{\varrho_1}^{\varrho_2} |\vartheta(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\varrho_1}^{\varrho_2} |\psi(x)|^q dx \right)^{\frac{1}{q}}, \tag{1.1}$$

with equality if and only if $K_1|\vartheta(x)|^p = K_2|\psi|^q$ almost everywhere, where K_1 and K_2 are constants.

Theorem 1.5 ((Power-mean integral inequality)). *Let $q \geq 1$. If ϑ and ψ be real function defined on $[\varrho_1, \varrho_2]$ and if $|\vartheta(x)|$ and $|\vartheta(x)||\psi(x)|^q$ are integrable on $[\varrho_1, \varrho_2]$, then the following inequality holds:*

$$\int_{\varrho_1}^{\varrho_2} |\vartheta(x)\psi(x)| dx \leq \left(\int_{\varrho_1}^{\varrho_2} |\vartheta(x)| dx \right)^{1-\frac{1}{q}} \left(\int_{\varrho_1}^{\varrho_2} |\vartheta(x)||\psi(x)|^q dx \right)^{\frac{1}{q}}, \tag{1.2}$$

which is the variant of the Hölder's inequality for integral.

Theorem 1.6 ([11]). *Suppose that $\vartheta : [0, \infty) \to [0, \infty)$ is an $s-$convex function in the second sense, where $s \in (0, 1)$ and let $\varrho_1, \varrho_2 \in [0, \infty)$, $\varrho_1 < \varrho_2$. If $\vartheta \in L_1([\varrho_1, \varrho_2])$, then the following inequality holds:*

$$2^{s-1}\vartheta\left(\frac{\varrho_1 + \varrho_2}{2}\right) \leq \frac{1}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \vartheta(x) dx \leq \frac{\vartheta(\varrho_1) + \vartheta(\varrho_2)}{s+1}. \tag{1.3}$$

Definition 1.7 ([18]). Suppose that the function ϑ is integrable on $[\varrho_1, \varrho_2]$ and $\varrho_1 \geq 0$. Then for all $\beta > 0$, we have

$$\mathcal{J}_{\varrho_1^+}^{\beta} \vartheta(x) = \frac{1}{\Gamma(\beta)} \int_{\varrho_1}^{x} (x-\xi)^{\beta-1} \vartheta(\xi) d\xi, \quad x > \varrho_1, \tag{1.4}$$

and

$$\mathcal{J}_{\varrho_2^-}^{\beta} \vartheta(x) = \frac{1}{\Gamma(\beta)} \int_{x}^{\varrho_2} (\xi - x)^{\beta-1} \vartheta(\xi) d\xi, \quad x < \varrho_2, \tag{1.5}$$

where $\Gamma(\beta) = \int_0^\infty e^{-x} x^{\beta-1} dx$ is the gamma function. The notations $\mathcal{J}_{\varrho_1^+}^{\beta}\vartheta(x)$ and $\mathcal{J}_{\varrho_2^-}^{\beta}\vartheta(x)$ are respectively called the right- and left-sided Riemann–Liouville fractional integral of a function ϑ of order β.

Definition 1.8 ([18, 24]). Let $\psi : [\varrho_1, \varrho_2] \to \mathbb{R}$ be an increasing and positive function on $[\varrho_1, \varrho_2]$, having a continuous derivative $\psi'(x)$ on $[\varrho_1, \varrho_2]$. The right and left-sided fractional integrals ${}^{\psi}\mathcal{J}_{\varrho_1^+}^{\beta}\vartheta(y)$ and ${}^{\psi}\mathcal{J}_{\varrho_2^-}^{\beta}\vartheta(y)$ of ϑ with respect to the function ψ on $[\varrho_1, \varrho_2]$ of order $\beta > 0$ are defined by

$${}^{\psi}\mathcal{J}_{\varrho_1^+}^{\beta}\vartheta(y) = \frac{1}{\Gamma(\beta)} \int_{\varrho_1}^{y} \Big[\psi(y) - \psi(u)\Big]^{\beta-1} \psi'(u)\vartheta(u) du, \quad y > \varrho_1, \tag{1.6}$$

and

$${}^{\psi}\mathcal{J}_{\varrho_2^-}^{\beta}\vartheta(y) = \frac{1}{\Gamma(\beta)} \int_{y}^{\varrho_2} \Big[\psi(u) - \psi(y)\Big]^{\beta-1} \psi'(u)\vartheta(u) du, \quad y < \varrho_2, \tag{1.7}$$

respectively.

Remark 1.9. If $\psi(x) = x$ in Definition 1.8, then we obtain the classical Riemann-Liouville fractional integral(Definition 1.7).

Motivated by the above work, we introduce some Simpson's type inequalities for s-convex functions in the second sense via the fractional integral of a function with respect to another function and its application. The paper is organized as follows: In Section 2, we state our main results on inequalities of Simpson's type for s-convex functions via a generalized fractional integral given by Definition 1.8. Section 3 is devoted to some application of newly established inequalities to special means. Finally, Section 4 is devoted to the conclusion of our work.

2 Main results

In the accompanying lemma, we introduce a new integral identity which will serve as a pillar for the forthcoming results. Throughout this paper, we assume that

$\psi : [\varrho_1, \varrho_2] \to \mathbb{R}$ is an increasing and positive function on $[\varrho_1, \varrho_2]$, having a continuous derivative $\psi'(x)$ on $[\varrho_1, \varrho_2]$.

Lemma 2.1. *Let $\vartheta : I \subset [0, \infty) \to \mathbb{R}$ be an absolutely continuous mapping on I^0 such that $\vartheta' \in L_1([\varrho_1, \varrho_2])$, where I^0 denotes the interior of an interval I, $\varrho_1, \varrho_2 \in I^0$ with $\varrho_1 < \varrho_2$. Then the following identity holds:*

$$\begin{aligned}
&\frac{1}{6\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^\beta}\left[\vartheta(\varrho_1)\omega_1^\beta + 2\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\left(\omega_1^\beta+\omega_2^\beta\right)+\vartheta(\varrho_2)\omega_2^\beta\right] \\
&-\frac{2^{\beta-1}\Gamma(\beta+1)}{\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^\beta}\left[{}^{\psi}\mathcal{J}^\beta_{\varrho_1^+}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)+{}^{\psi}\mathcal{J}^\beta_{\varrho_2^-}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\right] \\
&=\frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^\beta}\left[\int_0^1\left(\frac{\omega_3^\beta}{2}-\frac{\omega_2^\beta}{3}\right)\vartheta'\Big(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\Big)d\xi\right. \\
&\left.+\int_0^1\left(\frac{\omega_1^\beta}{3}-\frac{\omega_4^\beta}{2}\right)\vartheta'\Big(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\Big)d\xi\right],
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
\omega_1 &= \psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1), \\
\omega_2 &= \psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right), \\
\omega_3 &= \psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right), \\
\omega_4 &= \psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right).
\end{aligned}$$

Proof. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ represent the first and second integral on the right side of equation (2.1).

Using integration by parts and change of variables, we get

$$\begin{aligned}
\mathfrak{L}_1 &= \int_0^1 \left[\frac{1}{2}\left(\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}-\frac{1}{3}\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}\right] \\
&\quad\times \vartheta'\Big(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\Big)d\xi \\
&= \frac{2}{\varrho_2-\varrho_1}\Bigg\{\left[\frac{1}{2}\left(\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}-\frac{1}{3}\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}\right] \\
&\quad\times \vartheta\Big(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\Big)\Bigg|_0^1 \\
&\quad-\frac{\beta}{2}\int_0^1\left(\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta-1}\psi'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right) \\
&\quad\times \vartheta\Big(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\Big)\frac{\varrho_2-\varrho_1}{2}d\xi\Bigg\} \\
&= \frac{2}{\varrho_2-\varrho_1}\Bigg\{\frac{1}{2}\vartheta(\varrho_2)\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}-\frac{1}{3}\vartheta(\varrho_2)\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta} \\
&\quad+\frac{1}{3}\vartheta(\frac{\varrho_1+\varrho_2}{2})\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}-\frac{\beta}{2}\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_2}\left(\psi(u)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta-1}\psi'(u)\vartheta(u)du\Bigg\} \\
&=\frac{2}{\varrho_2-\varrho_1}\Bigg\{\frac{1}{6}\vartheta(\varrho_2)\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}+\frac{1}{3}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta} \\
&\quad-\frac{\Gamma(\beta+1)}{2}{}^{\psi}\mathcal{J}^{\beta}_{\varrho_2^-}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\Bigg\}.
\end{aligned} \tag{2.2}$$

and similarly

$$\begin{aligned}
\mathfrak{L}_2 &= \int_0^1\left[\frac{1}{3}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}-\frac{1}{2}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right)^{\beta}\right] \\
&\quad\times\vartheta'\Big(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\Big)d\xi \\
&= \frac{2}{\varrho_1-\varrho_2}\Bigg\{\left[\frac{1}{3}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}-\frac{1}{2}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right)^{\beta}\right] \\
&\quad\times\vartheta\Big(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\Big)\Bigg|_0^1 \\
&\quad-\frac{\beta}{2}\int_0^1\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right)^{\beta-1}\psi'\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right) \\
&\quad\times\vartheta\Big(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\Big)\frac{\varrho_1-\varrho_2}{2}d\xi\Bigg\}
\end{aligned}$$

$$= \frac{2}{\varrho_2 - \varrho_1}\left\{\frac{1}{2}\vartheta(\varrho_1)\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta} - \frac{1}{3}\vartheta(\varrho_1)\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}\right.$$

$$\left. + \frac{1}{3}\vartheta(\frac{\varrho_1+\varrho_2}{2})\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta} - \frac{\beta}{2}\int_{\varrho_1}^{\frac{\varrho_1+\varrho_2}{2}}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(u)\right)^{\beta-1}\psi'(u)\vartheta(u)du\right\}$$

$$= \frac{2}{\varrho_2 - \varrho_1}\left\{\frac{1}{6}\vartheta(\varrho_1)\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_2)\right)^{\beta} + \frac{1}{3}\vartheta(\frac{\varrho_1+\varrho_2}{2})\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}\right.$$

$$\left. - \frac{\Gamma(\beta+1)}{2}\,{}^{\psi}\mathcal{J}^{\beta}_{\varrho_1^+}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\right\}. \tag{2.3}$$

From equation (2.2) and (2.3), we have

$$\frac{1}{6\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\left[\vartheta(\varrho_1)\omega_1^{\beta} + 2\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\left(\omega_1^{\beta}+\omega_2^{\beta}\right) + \vartheta(\varrho_2)\omega_2^{\beta}\right]$$
$$- \frac{2^{\beta-1}\Gamma(\beta+1)}{\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\beta}}\left[{}^{\psi}\mathcal{J}^{\beta}_{\varrho_1^+}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right) + {}^{\psi}\mathcal{J}^{\beta}_{\varrho_2^-}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\right]$$
$$= \frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\left[\mathfrak{L}_1 + \mathfrak{L}_2\right],$$

which is equation (2.1). □

Remark 2.2. If we choose $\psi(x) = x$, then the identity (2.1) becomes the identity equation 7 in [9].

Remark 2.3. If we choose $\psi(x) = x$ and $\beta = 1$, then the identity (2.1) becomes the identity equation 1.3 in [25].

Throughout this paper, let

$$\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2) := \frac{1}{6\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\left[\vartheta(\varrho_1)\omega_1^{\beta} + 2\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\left(\omega_1^{\beta}+\omega_2^{\beta}\right) + \vartheta(\varrho_2)\omega_2^{\beta}\right]$$
$$- \frac{2^{\beta-1}\Gamma(\beta+1)}{\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\beta}}\left[{}^{\psi}\mathcal{J}^{\beta}_{\varrho_1^+}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right) + {}^{\psi}\mathcal{J}^{\beta}_{\varrho_2^-}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\right].$$

and

$$\mathfrak{L}^*(\vartheta,\psi,\varrho_1,\varrho_2) := \frac{1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\left[\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\left(\omega_1^{\beta}+\omega_2^{\beta}\right)\right]$$
$$- \frac{2^{\beta-1}\Gamma(\beta+1)}{\left(\psi(\varrho_2)-\psi(\varrho_1)\right)^{\beta}}\left[{}^{\psi}\mathcal{J}^{\beta}_{\varrho_1^+}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right) + {}^{\psi}\mathcal{J}^{\beta}_{\varrho_2^-}\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)\right].$$

Now, our next results are stated as follows:

Theorem 2.4. *Let the assumptions of Lemma 2.1 hold and assume that $|\vartheta'|$ is s-convex on $[\varrho_1, \varrho_2]$,* *for* $s \in (0,1]$, $|\psi'| \le L$ *on* $[\varrho_1, \varrho_2]$, *then the following inequality holds:*

$$\begin{aligned}&\Big|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\Big|\\&\le \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2^{s+1}\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\Big[|\vartheta'(\varrho_1)|+|\vartheta'(\varrho_2)|\Big]\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|\Big[(1+\xi)^s+(1-\xi)^s\Big]d\xi\right).\end{aligned} \tag{2.4}$$

Proof. Since ψ is increasing, differentiable and $|\psi'| \le L$ on $[\varrho_1, \varrho_2]$, ψ is the Lipschitzian function.

$$\begin{aligned}&\left[\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right]\le L\xi\left(\frac{\varrho_2-\varrho_1}{2}\right),\\&\left[\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right]\le L\left(\frac{\varrho_2-\varrho_1}{2}\right).\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\left[\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right]\le L\xi\left(\frac{\varrho_2-\varrho_1}{2}\right),\\&\left[\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right]\le L\left(\frac{\varrho_2-\varrho_1}{2}\right).\end{aligned} \tag{2.6}$$

From Lemma 2.1, inequalities (2.5) and (2.6) and $|\vartheta'|$ is $s-$convex on $[\varrho_1, \varrho_2]$, we get

$$\begin{aligned}&\Big|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\Big|\le\frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\\&\left\{\int_0^1\left[\left|\frac{1}{2}\left(\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi(\frac{\varrho_1+\varrho_2}{2})\right)^{\beta}-\frac{1}{3}\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}\right|\right]\right.\\&\times\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|d\xi\\&+\int_0^1\left[\left|\frac{1}{3}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}-\frac{1}{2}\left(\psi(\frac{\varrho_1+\varrho_2}{2})-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right)^{\beta}\right|\right]\\&\left.\times\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right|d\xi\right\}\\&\le\frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\\&\left\{\int_0^1\left|\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}-\frac{1}{3}\left(L(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\right|\left[\left(\frac{1+\xi}{2}\right)^s|\vartheta'(\varrho_2)|+\left(\frac{1-\xi}{2}\right)^s|\vartheta'(\varrho_1)|\right]d\xi\right.\\&\left.+\int_0^1\left|\frac{1}{3}\left(L(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}-\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\right|\left[\left(\frac{1+\xi}{2}\right)^s|\vartheta'(\varrho_1)|+\left(\frac{1-\xi}{2}\right)^s|\vartheta'(\varrho_2)|\right]d\xi\right\}\\&=\frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\times\frac{L^{\beta}(\frac{\varrho_2-\varrho_1}{2})^{\beta}}{2^s}\left\{\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|\Big[(1+\xi)^s|\vartheta'(\varrho_2)|+(1-\xi)^s|\vartheta'(\varrho_1)|\Big]d\xi\right.\\&\left.+\int_0^1\left|\frac{1}{3}-\frac{\xi^{\beta}}{2}\right|\Big[(1+\xi)^s|\vartheta'(\varrho_1)|+(1-\xi)^s|\vartheta'(\varrho_2)|\Big]d\xi\right\},\end{aligned}$$

which is the required inequality (2.4). □

Corollary 2.5. *In Theorem 2.4 if we choose $s = 1$, then we obtain the following inequality for the convex function.*

$$
\begin{aligned}
&\Big|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\Big| \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta}\Big[|\vartheta'(\varrho_1)|+|\vartheta'(\varrho_2)|\Big]\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|d\xi\right) \\
&= \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta}\Big[|\vartheta'(\varrho_1)|+|\vartheta'(\varrho_2)|\Big]\left(\frac{\beta}{\beta+1}\left(\frac{2}{3}\right)^{\frac{\beta+1}{\beta}}+\frac{1}{2(\beta+1)}-\frac{1}{3}\right).
\end{aligned}
$$

The generalized midpoint inequality in terms of the first derivative is given as follows:

Corollary 2.6. *If we have $\vartheta(\varrho_1) = \vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right) = \vartheta(\varrho_2)$ in Theorem 2.4, then we get the following midpoint inequality:*

$$
\begin{aligned}
&\Big|L^{*}(\vartheta,\psi,\varrho_1,\varrho_2)\Big| \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2^{s+1}\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta}\Big[|\vartheta'(\varrho_1)|+|\vartheta'(\varrho_2)|\Big]\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|\Big[(1+\xi)^s+(1-\xi)^s\Big]d\xi\right).
\end{aligned}
$$

Remark 2.7. The special cases are given below:

a. In theorem 2.4, if we put $\psi(x) = x$, then inequality (2.4) coincides with inequality 11 in [9].

b. In theorem 2.4, if we put $\psi(x) = x$ and $\beta = 1$, then the inequality (2.4) coincides with inequality 2.1 in [25].

c. In Corollary 2.5, if $\psi(x) = x$, $\beta = 1$ and $\vartheta(\varrho_1) = \vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right) = \vartheta(\varrho_2)$, then the inequality reduces to

$$
\left|\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\vartheta(u)du\right| \leq \frac{5(\varrho_2-\varrho_1)}{72}\Big[|\vartheta'(\varrho_1)|+|\vartheta'(\varrho_2)|\Big],
$$

which coincide with equation 2.6 in [25].

Theorem 2.8. *Let the assumptions of Lemma 2.1 hold and assume that $|\vartheta'|^q$ is s-convex on $[\varrho_1, \varrho_2]$, for $s \in (0, 1]$, $|\psi'| \leq L$ on $[\varrho_1, \varrho_2]$ and $q > 1$, then the following inequality holds:*

$$
\begin{aligned}
&\Big|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\Big| \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\frac{|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(\frac{|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right],
\end{aligned}
\tag{2.7}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1, using inequalities (2.5) and (2.6) and Hölder's inequality, we get

$$\left|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\right| \le \frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}$$

$$\Bigg\{\int_0^1 \left[\left|\frac{1}{2}\left(\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\Big(\frac{\varrho_1+\varrho_2}{2}\Big)\right)^{\beta}-\frac{1}{3}\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}\right|\right]$$

$$\times\left|\vartheta'\Big(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\Big)\right|d\xi$$

$$+\int_0^1 \left[\left|\frac{1}{3}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}-\frac{1}{2}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right)^{\beta}\right|\right]$$

$$\times\left|\vartheta'\Big(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\Big)\right|d\xi\Bigg\}$$

$$\le \frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\Bigg\{\int_0^1\left|\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}-\frac{1}{3}\left(L(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\right|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|d\xi$$

$$+\int_0^1\left|\frac{1}{3}\left(L(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}-\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\right|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_2\right)\right|d\xi\Bigg\}$$

$$\le \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2(\psi(\varrho_2)-\psi(\varrho_1))^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right|^q d\xi\right)^{\frac{1}{q}}\right],$$

by using the change of variable, equation (1.3) and $|\vartheta'|^q$ is s-convex, we have

$$\int_0^1\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|^q d\xi=\frac{2}{\varrho_2-\varrho_1}\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_2}|\vartheta'(x)|^q dx$$

$$=\frac{1}{\varrho_2-\frac{(\varrho_1+\varrho_2)}{2}}\int_{\frac{\varrho_1+\varrho_2}{2}}^{\varrho_2}|\vartheta'(x)|^q dx\le\left[\frac{|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right]. \tag{2.8}$$

and

$$\int_0^1\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right|^q d\xi=\frac{2}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\frac{\varrho_1+\varrho_2}{2}}|\vartheta'(x)|^q dx$$

$$=\frac{1}{(\frac{\varrho_1+\varrho_2}{2})-\varrho_1}\int_{\varrho_1}^{\frac{\varrho_1+\varrho_2}{2}}|\vartheta'(x)|^q dx\le\left[\frac{|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right]. \tag{2.9}$$

Hence, by using inequalities (2.8) and (2.9), we obtain

$$\left|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\right|\le\frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\frac{|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right].$$

Thus, the proof is complete. □

Corollary 2.9. *In Theorem 2.8 if we choose $s = 1$, then we obtain the following inequality for a convex function.*

$$\begin{aligned}&\left|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\right|\\ &\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^\beta}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\frac{|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 2.10. *If we have $\vartheta(\varrho_1) = \vartheta\Big(\frac{\varrho_1+\varrho_2}{2}\Big) = \vartheta(\varrho_2)$ in Theorem 2.8, then we get the following midpoint inequality:*

$$\begin{aligned}&\left|L^*(\vartheta,\psi,\varrho_1,\varrho_2)\right|\\ &\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^\beta}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\frac{|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{s+1}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Remark 2.11. The special cases are given below:

a. In theorem 2.8, if we put $\psi(x) = x$, then inequality (2.7) coincides with inequality 14 in [9].

b. In theorem 2.8, if we put $\psi(x) = x$ and $\beta = 1$, then inequality (2.7) coincides with inequality 2.7 in [25].

c. In Corollary 2.9, if $\psi(x) = x$, $\beta = 1$ and $\vartheta(\varrho_1) = \vartheta\Big(\frac{\varrho_1+\varrho_2}{2}\Big) = \vartheta(\varrho_2)$, then the inequality reduces to

$$\begin{aligned}&\left|\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\vartheta(u)du\right|\\ &\leq \frac{\varrho_2-\varrho_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)\left[\left(\frac{|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 2.12. *Let the assumptions of Lemma 2.1 hold and assume that $|\vartheta'|^q$ is s-convex on $[\varrho_1, \varrho_2]$, for $s \in (0, 1]$, $|\psi'| \leq L$ on $[\varrho_1, \varrho_2]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\right|\\ &\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^\beta}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\\ &\times\left[\left(\frac{(2^{s+1}-1)|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right],\end{aligned} \tag{2.10}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1, s-convexity of $|\vartheta'|^q$ and Hölder's inequality, we have

$$\begin{aligned}
&\left|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\right| \leq \frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}} \\
&\left\{\int_0^1 \left[\left|\frac{1}{2}\left(\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\Big(\frac{\varrho_1+\varrho_2}{2}\Big)\right)^{\beta}-\frac{1}{3}\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}\right|\right]\right. \\
&\times\left|\vartheta'\Big(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\Big)\right|d\xi \\
&+\int_0^1\left[\left|\frac{1}{3}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}-\frac{1}{2}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right)^{\beta}\right|\right] \\
&\left.\times\left|\vartheta'\Big(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\Big)\right|d\xi\right\} \\
&\leq \frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\left\{\int_0^1\left|\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}-\frac{1}{3}\left(L(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\right|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|d\xi\right. \\
&\left.+\int_0^1\left|\frac{1}{3}\left(L(\frac{\varrho_1-\varrho_1}{2})\right)^{\beta}-\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\right|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_2\right)\right|d\xi\right\} \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2(\psi(\varrho_2)-\psi(\varrho_1))^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|^q d\xi\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right|^q d\xi\right)\right] \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2(\psi(\varrho_2)-\psi(\varrho_1))^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\left[\left(\int_0^1\left(\frac{1+\xi}{2}\right)^s|\vartheta'(\varrho_2)|^q+\left(\frac{1-\xi}{2}\right)^s|\vartheta'(\varrho_1)|^q d\xi\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(\frac{1+\xi}{2}\right)^s|\vartheta'(\varrho_1)|^q+\left(\frac{1-\xi}{2}\right)^s|\vartheta'(\varrho_2)|^q d\xi\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Since $\int_0^1\left(\frac{1+\xi}{2}\right)^s d\xi=\frac{2^{s+1}-1}{2^s(s+1)}$ and $\int_0^1\left(\frac{1-\xi}{2}\right)^s d\xi=\frac{1}{2^s(s+1)}$ the proof is complete. □

Corollary 2.13. *In Theorem 2.12 if we choose $s=1$, then we obtain the following inequality for a convex function.*

$$\begin{aligned}
&\left|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\right| \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{3|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 2.14. *If we have $\vartheta(\varrho_1)=\vartheta\Big(\frac{\varrho_1+\varrho_2}{2}\Big)=\vartheta(\varrho_2)$ in Theorem 2.12, then we get the follow-*

ing midpoint inequality:

$$
\begin{aligned}
&\left|L^*(\vartheta,\psi,\varrho_1,\varrho_2)\right|\\
&\le \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\\
&\times\left[\left(\frac{(2^{s+1}-1)|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Remark 2.15. The special cases are given below:

a. In theorem 2.12, if we put $\psi(x)=x$, then inequality (2.7) coincides with inequality 16 in [9].

b. In theorem 2.12, if we put $\psi(x)=x$ and $\beta=1$, then inequality (2.7) coincides with inequality 2.9 in [25].

c. In Corollary 2.13, if $\psi(x)=x$, $\beta=1$ and $\vartheta(\varrho_1)=\vartheta\Big(\frac{\varrho_1+\varrho_2}{2}\Big)=\vartheta(\varrho_2)$, then the inequality reduces to

$$
\begin{aligned}
&\left|\vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right)-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\vartheta(u)du\right|\\
&\le\frac{\varrho_2-\varrho_1}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)\left[\left(\frac{3|\vartheta'(\varrho_2)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\vartheta'(\varrho_1)|^q+|\vartheta'(\frac{\varrho_1+\varrho_2}{2})|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Theorem 2.16. *Let the assumptions of Lemma 2.1 hold and assume that $|\vartheta'|^q$ is s-convex on $[\varrho_1,\varrho_2]$ for $s\in(0,1]$ and $q\ge 1$, $|\psi'|\le L$ on $[\varrho_1,\varrho_2]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\right|\\
&\le \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right| d\xi\right)^{1-\frac{1}{q}}\\
&\left[\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|\left(\left(\frac{1+\xi}{2}\right)^s|\vartheta'(\varrho_2)^q|+\left(\frac{1-\xi}{2}\right)^s|\vartheta'(\varrho_1)|^q\right)d\xi\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{\xi^{\beta}}{2}\right|\left(\left(\frac{1+\xi}{2}\right)^s|\vartheta'(\varrho_1)^q|+\left(\frac{1-\xi}{2}\right)^s|\vartheta'(\varrho_2)|^q\right)d\xi\right)^{\frac{1}{q}}\right],
\end{aligned}
\tag{2.11}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, s-convexity of $|\vartheta'|^q$, inequalities (2.5), (2.6) and power mean inequality, we have

$$\Big|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\Big| \leq \frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}$$

$$\Bigg\{\int_0^1\Bigg[\Bigg|\frac{1}{2}\left(\psi\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)-\psi\Big(\frac{\varrho_1+\varrho_2}{2}\Big)\right)^{\beta}-\frac{1}{3}\left(\psi(\varrho_2)-\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)\right)^{\beta}\Bigg|\Bigg]$$

$$\times\left|\vartheta'\Big(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\Big)\right|d\xi$$

$$+\int_0^1\Bigg[\Bigg|\frac{1}{3}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi(\varrho_1)\right)^{\beta}-\frac{1}{2}\left(\psi\left(\frac{\varrho_1+\varrho_2}{2}\right)-\psi\left(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\right)\right)^{\beta}\Bigg|\Bigg]$$

$$\times\left|\vartheta'\Big(\frac{1+\xi}{2}\varrho_1+\frac{1-\xi}{2}\varrho_2\Big)\right|d\xi\Bigg\}$$

$$\leq \frac{\varrho_2-\varrho_1}{2\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\Bigg\{\int_0^1\Bigg|\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}-\frac{1}{3}\left(L(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\Bigg|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|d\xi$$

$$+\int_0^1\Bigg|\frac{1}{3}\left(L(\frac{\varrho_1-\varrho_1}{2})\right)^{\beta}-\frac{1}{2}\left(L\xi(\frac{\varrho_2-\varrho_1}{2})\right)^{\beta}\Bigg|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_2\right)\right|d\xi\Bigg\}$$

$$\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2(\psi(\varrho_2)-\psi(\varrho_1))^{\beta}}L^{\beta}\Bigg\{\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_1\right)\right|d\xi$$

$$+\int_0^1\left|\frac{1}{3}-\frac{\xi^{\beta}}{2}\right|\left|\vartheta'\left(\frac{1+\xi}{2}\varrho_2+\frac{1-\xi}{2}\varrho_2\right)\right|d\xi\Bigg\}$$

$$\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2(\psi(\varrho_2)-\psi(\varrho_1))^{\beta}}L^{\beta}\Bigg\{\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|\left(\left(\frac{1+\xi}{2}\right)^{s}|\vartheta'(\varrho_2)|^{q}+\left(\frac{1-\xi}{2}\right)^{s}|\vartheta'(\varrho_1)|^{q}d\xi\right)\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\frac{1}{3}-\frac{\xi^{\beta}}{2}\right|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1}{3}-\frac{\xi^{\beta}}{2}\right|\left(\left(\frac{1+\xi}{2}\right)^{s}|\vartheta'(\varrho_1)|^{q}+\left(\frac{1-\xi}{2}\right)^{s}|\vartheta'(\varrho_2)|^{q}d\xi\right)\right)^{\frac{1}{q}}\Bigg\},$$

which is the desired inequality. □

Corollary 2.17. *In Theorem 2.16 if we choose $s=1$, then we obtain the following inequality for a convex function.*

$$\begin{aligned}&\Big|\mathfrak{L}(\vartheta,\psi,\varrho_1,\varrho_2)\Big|\\&\leq\frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|d\xi\right)^{1-\frac{1}{q}}\\&\times\left[\left(\int_0^1\left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right|\left(\left(\frac{1+\xi}{2}\right)|\vartheta'(\varrho_2)^{q}|+\left(\frac{1-\xi}{2}\right)|\vartheta'(\varrho_1)|^{q}\right)\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{\xi^{\beta}}{2}\right|\left(\left(\frac{1+\xi}{2}\right)|\vartheta'(\varrho_1)^{q}|+\left(\frac{1-\xi}{2}\right)|\vartheta'(\varrho_2)|^{q}\right)\right)^{\frac{1}{q}}\right].\end{aligned}\tag{2.12}$$

Corollary 2.18. *If we have* $\vartheta(\varrho_1) = \vartheta\left(\frac{\varrho_1+\varrho_2}{2}\right) = \vartheta(\varrho_2)$ *in Theorem 2.16, then we get the following midpoint inequality:*

$$\begin{aligned}
&\left|L^*(\vartheta, \psi, \varrho_1, \varrho_2)\right| \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta} \left(\int_0^1 \left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right| d\xi\right)^{1-\frac{1}{q}} \\
&\left[\left(\int_0^1 \left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right| \left(\left(\frac{1+\xi}{2}\right)^s |\vartheta'(\varrho_2)^q| + \left(\frac{1-\xi}{2}\right)^s |\vartheta'(\varrho_1)|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 \left|\frac{1}{3}-\frac{\xi^{\beta}}{2}\right| \left(\left(\frac{1+\xi}{2}\right)^s |\vartheta'(\varrho_1)^q| + \left(\frac{1-\xi}{2}\right)^s |\vartheta'(\varrho_2)|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 2.19. In theorem 2.16, if we put

a. $\psi(x) = x$, then inequality (2.11) coincides with inequality 20 in [9].

b. $\psi(x) = x$ and $\beta = 1$, then inequality (2.7) coincides with inequality 2.10 in [25].

3 Application to special means

Now we give some applications of our results:

The Arithmetic mean is defined as:

$$\mathcal{A}(\varrho_1, \varrho_2) = \frac{\varrho_1+\varrho_2}{2}, \qquad \varrho_1, \varrho_2 \geq 0.$$

Proposition 3.1. *Let* $\varrho_1, \varrho_2 \in \mathbb{R}$ *and* $0 < \varrho_1 < \varrho_2$ *then, we have*

$$\begin{aligned}
&\frac{1}{6\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}} \left[\varrho_1^s\omega_1^{\beta} + 2\mathcal{A}^s\Big(\varrho_1, \varrho_2\Big)\Big(\omega_1^{\beta}+\omega_2^{\beta}\Big) + \varrho_2^s\omega_2^{\beta}\right] \\
&-\frac{2^{\beta-1}\Gamma(\beta+1)}{\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} \left[\psi\mathcal{J}^{\beta}_{\varrho_1^+}\mathcal{A}^s\Big(\varrho_1, \varrho_2\Big) + \psi\mathcal{J}^{\beta}_{\varrho_2^-}\mathcal{A}^s\Big(\varrho_1, \varrho_2\Big)\right] \\
&\leq \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2^s\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}} L^{\beta} s \left(\int_0^1 \left|\frac{\xi^{\beta}}{2}-\frac{1}{3}\right| \Big[(1+\xi)^s + (1-\xi)^s\Big] d\xi\right) \mathcal{A}(\varrho_1^{s-1}, \varrho_2^{s-1}).
\end{aligned} \tag{3.1}$$

Proof. This can be proved by applying $\vartheta(x) = x^s$ in Theorem 2.4, $s \in (0, 1]$. □

The P-logarithmic mean is defined as :

$$\mathcal{L}_p = \mathcal{L}_p(\varrho_1, \varrho_2) = \begin{cases} \left[\varrho_2^{p+1} - \varrho_1^{p+1}\right]^{\frac{1}{p}} & \text{if } \varrho_1 = \varrho_2 \\ \varrho_1 & \text{if } \varrho_1 = \varrho_2, \qquad p \in \mathbb{R}/\{-1, 0\}; \qquad \varrho_1, \varrho_2 > 0 \end{cases}$$

Remark 3.2. If we put $\psi(x) = x$ and $\beta = 1$ in Proposition 3.1, then we obtain

$$\begin{aligned}
&\left|\frac{1}{3}\mathcal{A}(\varrho_1^s, \varrho_2^s) + \frac{2}{3}\mathcal{A}^s(\varrho_1, \varrho_2) - \mathcal{L}_s^s(\varrho_1, \varrho_2)\right| \\
&\leq 2s(\varrho_2-\varrho_1)\frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)} \mathcal{A}(\varrho_1^{s-1}, \varrho_2^{s-1}),
\end{aligned}$$

which is proved in Sarikaya et al. [25].

Proposition 3.3. *Let $\varrho_1, \varrho_2 \in \mathbb{R}$ and $0 < \varrho_1 < \varrho_2$. Then for all $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned}
&\frac{1}{6\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\left[\varrho_1^s\omega_1^\beta + 2\mathcal{A}^s\Big(\varrho_1,\varrho_2\Big)\Big(\omega_1^\beta+\omega_2^\beta\Big)+\varrho_2^s\omega_2^\beta\right] \\
&-\frac{2^{\beta-1}\Gamma(\beta+1)}{\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}\left[\psi\mathcal{J}^{\beta}_{\varrho_1^+}\mathcal{A}^s\Big(\varrho_1,\varrho_2\Big)+\psi\mathcal{J}^{\beta}_{\varrho_2^-}\mathcal{A}^s\Big(\varrho_1,\varrho_2\Big)\right] \\
&\le \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^\beta}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\frac{s}{(s+1)^{\frac{1}{q}}}\left[\Big(\varrho_2^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right., \\
&\left.+\Big(\varrho_1^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.2}$$

Proof. This can be proved by applying $\vartheta(x) = x^s$ in Theorem 2.8, $s \in (0, 1]$. □

Remark 3.4. If we put $\psi(x) = x$ and $\beta = 1$ in Proposition 3.3, then we obtain

$$\begin{aligned}
&\left|\frac{1}{3}\mathcal{A}(\varrho_1^s,\varrho_2^s)+\frac{2}{3}\mathcal{A}^s(\varrho_1,\varrho_2)-\mathcal{L}_s^s(\varrho_1,\varrho_2)\right| \\
&\le \frac{(\varrho_2-\varrho_1)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\frac{s}{(s+1)^{\frac{1}{q}}}\left[\Big(\varrho_2^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right. \\
&\left.+\Big(\varrho_1^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right].
\end{aligned}$$

which is established in Sarikaya et al. [25].

Proposition 3.5. *Let $\varrho_1, \varrho_2 \in \mathbb{R}$ and $0 < \varrho_1 < \varrho_2$. Then for all $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned}
&\frac{1}{6\left(\frac{\psi(\varrho_2)-\psi(\varrho_1)}{2}\right)^{\beta}}\left[\varrho_1^s\omega_1^\beta + 2\mathcal{A}^s\Big(\varrho_1,\varrho_2\Big)\Big(\omega_1^\beta+\omega_2^\beta\Big)+\varrho_2^s\omega_2^\beta\right] \\
&-\frac{2^{\beta-1}\Gamma(\beta+1)}{\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}\left[\psi\mathcal{J}^{\beta}_{\varrho_1^+}\mathcal{A}^s\Big(\varrho_1,\varrho_2\Big)+\psi\mathcal{J}^{\beta}_{\varrho_2^-}\mathcal{A}^s\Big(\varrho_1,\varrho_2\Big)\right] \\
&\le \frac{(\varrho_2-\varrho_1)^{\beta+1}}{2\Big(\psi(\varrho_2)-\psi(\varrho_1)\Big)^{\beta}}L^{\beta}\left(\int_0^1\left|\frac{\xi^\beta}{2}-\frac{1}{3}\right|^p d\xi\right)^{\frac{1}{p}}\frac{s}{(2^s(s+1))^{\frac{1}{q}}}\left[\Big((2^{s+1}-1)\varrho_2^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right., \\
&\left.+\Big((2^{s+1}-1)\varrho_1^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.3}$$

Proof. This can be proved by applying $\vartheta(x) = x^s$ in Theorem 2.12, $s \in (0, 1]$. □

Remark 3.6. If we put $\psi(x) = x$ and $\beta = 1$ in Proposition 3.5, then we obtain

$$\begin{aligned}
&\left|\frac{1}{3}\mathcal{A}(\varrho_1^s,\varrho_2^s)+\frac{2}{3}\mathcal{A}^s(\varrho_1,\varrho_2)-\mathcal{L}_s^s(\varrho_1,\varrho_2)\right| \\
&\le \frac{(\varrho_2-\varrho_1)}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\frac{s}{(2^s(s+1))^{\frac{1}{q}}}\left[\Big((2^{s+1}-1)\varrho_2^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right. \\
&\left.+\Big((2^{s+1}-1)\varrho_1^{q(s-1)}+\mathcal{A}^{q(s-1)}(\varrho_1,\varrho_2)\Big)^{\frac{1}{q}}\right].
\end{aligned}$$

which is shown by Sarikaya et al. [25].

4 Conclusion

In the present article, we introduced a new integral identity. Using this identity, we generate a new version of fractional Simpson's inequality via the $s-$convex function in the second sense with respect to another function. Moreover, we have shown that our result generalizes the inequalities introduced by Sarikaya et al.[25] and we applied the newly established Simpson's inequality to some special means. Further research could aim to generalize these findings by using different types of convex function classes or fractional integral operators.

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Author information

Henok Desalegn Desta, Department of Mathematics, Addis Ababa University, Addis Ababa 1176, Ethiopia.
E-mail: henokddesta@gmail.com

Deepak B. Pachpatte, Department of Mathematics, Dr. Babasaheb Ambedekar Marathwada University,Aurangabad, Maharashtra 431004, India.
E-mail: pachpatte@gmail.com

Jebessa B. Mijena, Department of Mathematics, Georgia College & State University, Milledgeville, GA 31061, USA.
E-mail: jebessa.mijena@gcsu.edu

Tadesse Abdi, Department of Mathematics, Addis Ababa University, Addis Ababa 1176, Ethiopia.
E-mail: tadesse.abdi@aau.edu.et

Received: 2022-10-08
Accepted: 2024-05-21