Palestine Journal of Mathematics

Vol 13(3)(2024) , 174–185

ON PRINCIPALLY SS-LIFTING MODULES

Figen ERYILMAZ and Burcu NİŞANCI TÜRKMEN

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 16D10; Secondary 16D99.

Keywords and phrases: principally $ss-$supplemented module, principally $ss-$lifting module, principally $\delta_{ss}-$supplemented module, principally $\delta_{ss}-$lifting module.

Abstract In this paper, modules with weak and strong notions of principally lifting modules have been studied and four different module concepts have been defined. The relationship of these modules with the modules in the literature has been established.

1 Introduction

Throughout this paper, each ring R is associate with identity and each module is an unitary left $R-$module. Let M be such a module. By the notation $A \leq M$, we mean that A is a submodule of M. A submodule A of M is said to be *small* in M if $M \neq A + B$ for any proper submodule B of M, denoted by $A \ll M$, and we point with $Rad(M)$, the sum of whole small submodules of M. Dual to this concept, a submodule A of M is said to be *essential* in M, denoted by $A \trianglelefteq M$, if the intersection of A is non-zero with the other submodules of M, except for $\{0\}$. A supplement submodule T of A in M is a minimal element of the set $\{B \leq M | \ M = A + B\}$ that equivalents $M = A + T$ and $A \cap T \ll T$. A module M is said to be *supplemented* if each submodule of M has a supplement in M [16]. On the other hand, the module M is *amply supplemented* if, for any submodules A,B of M with $M = A + B$ there is a supplement T of A such that $T \leq A$ [16]. In [9], a module M is said to be $\oplus-$*supplemented*, if each submodule of M has a supplement which is a direct summand. A study conducted in recent years as a proper generalization of supplemented modules is included in [2].

Small submodules are generalized to $\delta-$small submodules in [18]. According to Zhou, a submodule A of M is said to be $\delta-$*small* in M (denoted by $A \ll_{\delta} M$) if for any submodule B of M with $\frac{M}{B}$ is singular, $M = A + B$ implies that $M = B$ [18]. The sum of $\delta-$small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is $\delta-$small in M, so $Rad\,(M) \subseteq \delta\,(M)$ and $Rad\,(M) = \delta\,(M)$ if M is singular. Also any non-singular semisimple submodule of M is $\delta-$small in M and $\delta-$small submodules of a singular module are small submodules. For more detailed discussion on $\delta-$small submodules we refer to [18]. Let A, B be submodules of a module M, then B is said to be a $\delta-$*supplement* of A in M, if $M = A + B$ and $A \cap B \ll_{\delta} B$. A module M is said to be $\delta-$*supplemented,* if each submodule of M has a $\delta-$-supplement in M.

In [9], a module M is said to be *lifting* if for each submodule A of M lies over a direct summand, that is, there is a decomposition $M = M_1 \oplus M_2$ provided that $M_1 \leq A$, $A \cap M_2 \ll M_2$. By [16], M is lifting if and only if M is amply supplemented and each supplement submodule of M is a direct summand of M. A module M is said to be $\delta-$*lifting*, if for each submodule A of M, there is a direct summand K of M with $K \subseteq A$ and $\frac{A}{K} \ll_{\delta} \frac{M}{K}$. Equivalently, for any $A \leq M$, there exists a decomposition $M = K \oplus B$ provided that $K \leq A$ and $A \cap B \ll_{\delta} B$. A submodule A of M is said to be a *fully invariant* provided that if $\zeta(A) \subseteq A$ for each $\zeta \in S = End(_{R}M)$. In [13], the concept of FI-δ-lifting modules is studied as a generalization of δ-lifting modules. An R-module M is said to be *FI-δ-lifting* provided that each fully invariant submodule A of M contains a direct summand B of M with $\frac{A}{B} \ll_{\delta} \frac{M}{B}$. Also in [13], the concept of strongly FI-δ-lifting modules is defined. M is said to be *strongly FI-δ-lifting* provided that each fully invariant submodule A of M contains a fully invariant direct summand B of M with $\frac{A}{B} \ll_{\delta} \frac{M}{B}$.

Following [19], whole simple submodules of M which are small in M is named $Soc_s\left(M\right)$, that is, $Soc_s\left(M\right) = \sum\left\{A \ll M \mid A \text{ is simple}\right\}$. Note that $Soc_s\left(M\right) \subseteq Rad\left(M\right)$ and $Soc_s\left(M\right) \subseteq Soc\left(M\right)$. In [8], a module M is said to be *strongly local* providing that M is local and $Rad\left(M\right) \subseteq Soc\left(M\right)$. In the same paper, a ring R is said to be *left strongly local ring* if ${}_RR$ is a strongly local module.

Besides, $ss-$supplemented and semisimple lifting modules are introduced in [8] and [3] respectively, as follows. Let M be a module, A,$B \leq M$. If $M = A + B$ and $A \cap B \subseteq Soc_s\left(B\right)$, then B is an *$ss-$supplement* of A in M. Any module M is said to be *$ss-$supplemented* if each submodule A of M has an $ss-$supplement B in M. As a result of this definition, finitely generated module M is $ss-$supplemented if and only if it is supplemented and $Rad\left(M\right) \subseteq Soc\left(M\right)$. According to [3], a module M is said to be *semisimple lifting* or briefly *$ss-$lifting* if for each submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ provided that $M_1 \leq A$, $A \cap M_2 \ll M$ and $A \cap M_2$ is semisimple. Some fundamental properties of $ss-$lifting modules were be examined in this paper.

The concept of principally supplemented modules is a generalization of semiregular modules and the concept of principally lifting modules is a generalization of lifting modules, which are introduced in [1]. A module M is said to be *principally supplemented* if for each cyclic submodule A of M, there is a submodule B of M provided that $M = A + B$ with $A \cap B \ll B$ and a module M is said to be *principally lifting* if for each cyclic submodule A of M, there is a decomposition $M = K \oplus B$ provided that $K \leq A$ and $A \cap B \ll M$. Similarly, principally $\delta-$supplemented and principally $\delta-$lifting modules are studied and some features are obtained in [5]. A module M is said to be *principally $\delta-$ supplemented* if for each cyclic submodule A of M, there is a submodule B of M provided that $M = A + B$ with $A \cap B \ll_\delta B$ and a module M is said to be *principally $\delta-$lifting* if for each cyclic submodule A of M, there is a decomposition $M = K \oplus B$ provided that $K \leq A$ and $A \cap B \ll_\delta M$.

We combine the above motivations by defining the following four types of modules:

(1) We say M is principally $ss-$supplemented if each cyclic submodule has a $ss-$supplement in M.

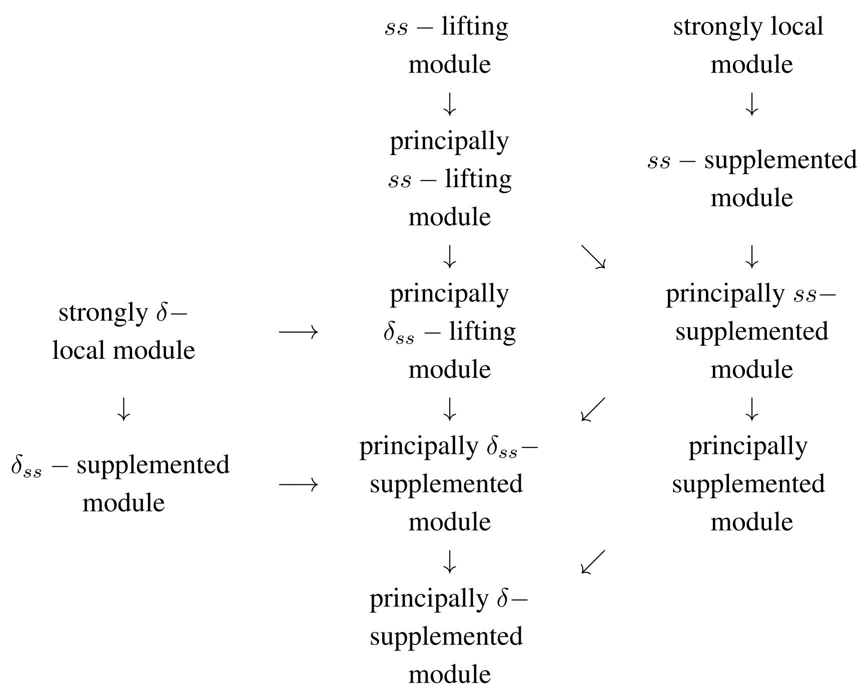
(2) We say M is principally $ss-$lifting if each cyclic submodule Rm of M, M has a decomposition $M = N \oplus K$ provided that $N \subseteq Rm$ and $Rm \cap K \subseteq Soc_s(K)$.

(3) We say M is principally $\delta_{ss}-$supplemented if each cyclic submodule has a $\delta_{ss}-$supplement in M.

(4) We say M is principally $\delta_{ss}-$lifting if each cyclic submodule Rm of M, M has a decomposition $M = N \oplus K$ provided that $N \subseteq Rm$ and $Rm \cap K \subseteq Soc_\delta(K)$.

In Section 2,we are researching the main features of the modules which contained in the first two definitions above. We show that principally $ss-$supplemented modules are closed under extension with some special conditions. We prove the notion of principally $ss-$lifting is inherited by direct summands. We obtained the decomposition as principally $ss-$lifting modules with the help of composition series and semisimple submodules.

In Section 3, our Theorem 3.1 generalizes and extends the main result and we compare the principally $\delta_{ss}-$lifting and principally $\delta_{ss}-$supplemented classes of modules that related to the class of principally lifting modules in Theorem 3.11. In particular, we show that the following implications hold between the various concepts:



2 Principally $ss-$supplemented modules and lifting property

In this part, we give the basic algebraic properties of the principally $ss-$lifting and principally $ss-$supplemented modules. These properties will be used in Section 3.

Firstly we give an example which shows that each principally supplemented module may not be principally $ss-$supplemented.

Example 2.1. Consider the $\mathbb{Z}-$module $M = \mathbb{Z}_8$. Follows from [1, Example 7(3)], M is principally supplemented. Since every submodule of $\mathbb{Z}_8$ is cyclic, M is not principally $ss-$supplemented by [8, Example 2.17].

Recall from [16] that a submodule A of M is said to be *fully invariant* if for each endomorphism φ of M, $\varphi(A) \subseteq A$ and the module M is said to be *duo module* if each submodule of M is fully invariant by [12].

Let us recall the fundamental lemma in [12] that we will use in the following.

Let M be a module which is a direct sum of submodules M_i $(i \in I)$ and A be a fully invariant submodule of M then $A = \bigoplus_{i \in I} (A \cap M_i)$.

Proposition 2.2. *Let M be a direct sum of principally $ss-$supplemented modules M_1 and M_2. If M is a duo module, then M is principally $ss-$supplemented.*

Proof. Let $M = M_1 \oplus M_2$ be a duo module and $A = Rm$ be a cyclic submodule of M. Then $A = (A \cap M_1) \oplus (A \cap M_2)$. Let $m = m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$. We have $A_1 = Rm_1 = A \cap M_1$ and $A_2 = Rm_2 = A \cap M_2$. As A_1 and A_2 are cyclic submodules of M_1 and M_2 respectively, there is a submodule $B_1 \subseteq M_1$ provided that $M_1 = A_1 + B_1$, $A_1 \cap B_1 \subseteq Soc_s(B_1)$ and $B_2 \subseteq M_2$ with $M_2 = A_2 + B_2$, $A_2 \cap B_2 \subseteq Soc_s(B_2)$. Then

$$M = M_1 + M_2 = A_1 + B_1 + A_2 + B_2 = (A \cap M_1) + B_1 + (A \cap M_2) + B_2 = A + B_1 + B_2$$

and

$$A \cap (B_1 + B_2) \subseteq [(A \cap M_1) \cap (B_1 + M_2)] + [(A \cap M_2) \cap (B_2 + M_1)].$$

Here $(A \cap M_1) \cap (B_1 + M_2) \subseteq A_1 \cap (B_1 + M_2)$. So we have $A_1 \cap (B_1 + M_2) = B_1 \cap (A_1 + M_2) = A_1 \cap B_1$. Similarly we can obtain that $A_2 \cap (B_2 + M_1) = B_2 \cap (A_2 + M_1) = A_2 \cap B_2$. Since $A_1 \cap B_1 \subseteq Soc_s(B_1)$ and $A_2 \cap B_2 \subseteq Soc_s(B_2)$, we have $(A_1 \cap B_1) + (A_2 \cap B_2) \subseteq Soc_s(B_1 + B_2)$ by [8, Corollary 2.4]. Since $A \cap (B_1 + B_2) \subseteq (A_1 \cap B_1) + (A_2 \cap B_2)$, it is clear that $A \cap (B_1 + B_2) \subseteq Soc_s(B_1 + B_2)$. Therefore $B_1 + B_2$ is an $ss-$supplement of A in M. Hence M is principally $ss-$ supplemented. □

Now we show that principally $ss-$supplemented modules are closed under direct summands.

Theorem 2.3. *Let M be a duo module. If M is principally $ss-$supplemented, then each direct summand of M is so.*

Proof. Let $M = N \oplus K$ and $m \in N$. Then there is a submodule L of M provided that $M_1 = Rm + L$ and $Rm \cap L \subseteq Soc_s(L)$ by the hypothesis. We have $N = Rm + (N \cap L)$. As M is a duo module, then $L = (N \cap L) \oplus (K \cap L)$. We have to prove that $Rm \cap (N \cap L) \subseteq Soc_s(L)$. It is obvious that $Rm \cap (N \cap L)$ is semisimple because $Rm \cap (N \cap L) \subseteq Rm \cap L$. If we show that $Rm \cap (N \cap L) \ll N \cap L$, as desired. Let V be a submodule of $N \cap L$ with $N \cap L = [Rm \cap (N \cap L)] + V$. Then $L = (N \cap L) \oplus (K \cap L) = [Rm \cap (N \cap L)] + V + (K \cap L)$. It follows from $Rm \cap L \ll L$ that $L = V \oplus (K \cap L)$. Thus $V = N \cap L$. So the desired is achieved. □

Corollary 2.4. *Let $M = M_1 \oplus M_2$ be a duo module. Then M is principally $ss-$supplemented if and only if M_1 and M_2 are principally $ss-$supplemented.*

Proof. ($\Rightarrow$) Clear by Theorem 2.3.
($\Leftarrow$) Clear by Proposition 2.2. □

Recall that a module M is *distributive* provided that $A \cap (B + C) = (A \cap B) + (A \cap C)$ for submodules A,B and C.

With the similar method to the above theorem, the following corollary is obtained.

Corollary 2.5. *Let M be a distributive module. If each direct summand of principally $ss-$supplemented module is principally $ss-$supplemented.*

We prove that notion of principally $ss-$supplemented preserves in factor modules in the following.

Proposition 2.6. *If M is a principally $ss-$supplemented module, then each factor module of M is so.*

Proof. Let M be a principally $ss-$supplemented module and $\frac{M}{N}$ be a factor module of M. By the hypothesis, for any cyclic submodule Rm of M which includes N, there is a submodule B of M provided that $M = Rm + B$ and $Rm \cap B \subseteq Soc_s(B)$. Let $\varphi : M \to \frac{M}{N}$ be the canonicial projection. Then
$\frac{M}{N} = \frac{Rm}{N} + \frac{B+N}{N}$ and $\frac{Rm}{N} \cap \frac{B+N}{N} \ll \frac{B+N}{N}$ by [16, 19.3(4)]. Since $Rm \cap B$ is semisimple, it follows from [7] that $\pi(Rm \cap B) = \frac{(Rm \cap B)+N}{N} = \frac{Rm}{N} \cap \frac{B+N}{N}$ is semisimple. Thus $\frac{Rm}{N} \cap \frac{B+N}{N} \subseteq Soc_s\left(\frac{B+N}{N}\right)$, as required. □

Now we show that the class of principally $ss-$supplemented modules is closed under extensions if we take certain conditions.

Theorem 2.7. *Let $0 \to A \overset{\varphi}{\to} M \overset{\psi}{\to} B \to 0$ be a short exact sequence and M be a duo module. If A and B are principally $ss-$supplemented, so does M. If the sequence splits, the converse holds.*

Proof. Without losing the generality, we assume that $A \subseteq M$. Since $\frac{M}{A} \cong B$ and A principally $ss-$supplemented, then we have M is principally $ss-$supplemented by Proposition 2.6. On the other hand, suppose that the sequence splits. Then $M \cong A \oplus B$. If M is principally $ss-$supplemented, then A and B are so by Theorem 2.3. □

Proposition 2.8. *Let N be a submodule of the duo module M. If $\frac{M}{N}$ is principally $ss-$supplemented, then M is so.*

Proof. Let Rm be a cyclic submodule of M. Then $\frac{Rm+N}{N}$ is a cyclic submodule of $\frac{M}{N}$. By the assumption, there is a submodule $\frac{L}{N}$ of $\frac{M}{N}$ provided that $\frac{M}{N} = \frac{Rm+N}{N} + \frac{L}{N}$ and $\frac{(Rm+N)}{N} \cap \frac{L}{N} = \frac{(Rm+N) \cap L}{N} = \frac{(Rm \cap L) \cap N}{N} \subseteq Soc_s\left(\frac{L}{N}\right)$. □

Let M and N be modules with M is projective. M is said to be a *projective cover* of a module N if there is an epimorphism $f : M \to N$ provided that $Kerf \ll M$. A ring R is said to be *semiperfect* if each simple $R-$module has a projective cover [16]. Also, a module M is said to be *principally semiperfect* if each factor module of M by a cyclic submodule has a projective cover. R is said to be a *principally semiperfect ring* in case the ${}_RR-$module is principally semiperfect [1].

Recall from [10] that a ring R is semiregular if and only if for any $a \in R$, $\frac{R}{Ra}$ has a projective cover. Then, Tuganbaev defines semiregular modules in [14]. The definition is the same as principally lifting modules in [6]. It is clear that a ring R is semiregular if and only if it is principally lifting.

We obtained the concide of principally $ss-$lifting with principally semiperfect in the following.

Theorem 2.9. *Let M be a projective module with $Rad(M) \subseteq Soc(M)$. Then the following statements are equivalent:*
(i) M is principally semiperfect.
(ii) M is principally $ss-$lifting.

Proof. Using [16] and [8, Lemma 39], the proof is clearly obtained. □

Corollary 2.10. *Let R be a ring with $Rad(R) \subseteq Soc({}_RR)$. Then the following statements are equivalent:*
(i) R is principally semiperfect.
(ii) ${}_RR$ is semiregular.
(iii) ${}_RR$ is principally lifting.
(iv) ${}_RR$ is principally $ss-$lifting.

Proof. It can be seen clearly by the same techniques of proofs of [15, Corollary 1.29] and [3, Theorem 2.12]. □

Recall from [6] that a non-zero module M is said to be *$p-$hollow* if each proper cyclic submodule is small in M.

Let's continue the section by giving the properties of principally $ss-$lifting modules.

Proposition 2.11. *The condition principally $ss-$lifting is inherited by direct summands.*

Proof. Let M be principally $ss-$lifting and N be a direct summand of M. Take any cyclic submodule Rm of N. Since $Rm \subseteq M$, M has a decomposition $M = K \oplus L$ with $K \subseteq Rm$ and $Rm \cap L \subseteq Soc_s(L) \subseteq Soc_s(M)$. Then $N = K \oplus (N \cap L)$ and $Rm \cap (N \cap L) \subseteq Soc_s(M)$, so $Rm \cap (N \cap L) \subseteq Soc_s(N)$, since N is a direct summand of M. Hence N is principally $ss-$lifting. □

The following lemmas are important to give us the characterization of principally $ss-$lifting modules with their cyclic submodules.

Recall that a submodule N of M is a direct summand of M if there is some other submodule N' of M provided that M is the direct sum of N and N'. In this case, N and N' are said to be complementary submodules.

Lemma 2.12. *If M is a principally $ss-$lifting module, then each cyclic submodule A has an $ss-$supplement B that is a direct summand and A contains a complementary direct summand of B in M.*

Proof. Obvious from the defining of principally $ss-$lifting and the fact that a small submodule of M is small in any direct summand of M. □

Lemma 2.13. *The following statements are equivalent for a module M:*
(i) M is principally $ss-$lifting.
(ii) Each cyclic submodule A of M can be written as $A = N \oplus K$ with N is a direct summand of M and $K \subseteq Soc_s(M)$.
(iii) For each $m \in M$, there is principal ideals K and J of R provided that $R = Km \oplus Jm$, where Km is a direct summand of M and $Jm \subseteq Soc_s(M)$.

Proof. (i) ($\Rightarrow$)(ii) Clear.

(ii) ($\Rightarrow$)(i) Let A be a cyclic submodule of M, then by the hypothesis, $A = N \oplus K$ with N is a direct summand of M and $K \subseteq Soc_s(M)$. We consider that $M = N \oplus N_1$. Then $A = N \oplus (A \cap N_1)$. Let $\varphi : N \oplus N_1 \to N_1$ be the canonicial projection, we have $A \cap N_1 = \varphi(A) = \varphi(N \oplus K) = \varphi(K) \subseteq Soc_s(M)$. Hence M is principally $ss-$lifting.

(ii) ($\Rightarrow$)(iii) Obvious. □

Proposition 2.14. *Let M be a principally $ss-$lifting module. If $M = A + B$ provided that B is a direct summand of M and $A \cap B$ is cyclic, then B contains a $ss-$supplement of A in M.*

Proof. Since M is principally $ss-$lifting and $A \cap B$ is cyclic, $A \cap B = N \oplus K$, where B is a direct summand of M and $K \subseteq Soc_s(M)$ by Lemma 2.13. Since $K \ll M$, then $K \ll B$. Write $B = N \oplus N_1$. We have $A \cap B = N \oplus (A \cap B \cap N_1) = N \oplus (A \cap N_1)$. Let $\varphi : N \oplus N_1 \to N_1$ be the canonicial projection. It follows that $A \cap N_1 = \varphi(N \oplus (A \cap N_1)) = \varphi(A \cap B) = \varphi(N \oplus K) = \varphi(K)$, thus $A \cap N_1 \subseteq Soc_s(N_1)$. In addition $M = A + B = A + N + N_1 = A + N_1$. So N_1 is an $ss-$supplement of A in M that is included in B. □

Corollary 2.15. *Let M be a principally $ss-$lifting module over a principally ideal ring R. If $M = A + Rm$, then Rm contains an $ss-$supplement of A in M.*

Proof. By Lemma 2.13, we have $Rm = N \oplus K$, where N is a direct summand of M and $K \subseteq Soc_s(M)$. From here, $M = A + N$ where N is a cyclic direct summand of M. Thus $X \cap N$ is a cyclic submodule of M. By applying Proposition 2.11, the proof is complete. □

We need the following proposition to classify indecomposable cyclic submodule of principally $ss-$lifting modules.

Proposition 2.16. *Let M be a principally $ss-$lifting module. Then each indecomposable cyclic submodule A of M is either in $Soc_s(M)$ or a direct summand of M.*

Proof. By Lemma 2.13, there exists $A = N \oplus K$ where N is a direct summand of M and $K \subseteq Soc_s(M)$. As A is indecomposable, we obtain that either $A = N$ or $A = K$. So A is a direct summand of M or $A \subseteq Soc_s(M)$. □

As each cyclic module over a local ring is a local module, we have the following.

Corollary 2.17. *Let M be a module over a local ring R. If M is principally $ss-$lifting, then each cyclic submodule of M is either in $Soc_s(M)$ or a direct summand of M.*

The following theorem is important for the indecomposition of principally $ss-$ lifting modules.

Theorem 2.18. *Let $M = N \oplus K$, where N is simple and K has a composition series $0 \leq A \leq K$ and semisimple, then M is principally $ss-$lifting.*

Proof. Let B be a non-zero proper cyclic submodule of M. Without loss of generality, we can take that $B = R(n + k)$, where $0 \neq n \in N$ and $0 \neq k \in K$. It is obvious that $B + N = N \oplus Rk$ and that Rk is either A or K. Since N is simple, we have either $N \leq B$ or $N \cap B = 0$. If $N \leq B$, then $Rk = A$ and $B = N \oplus A$, where N is a direct summand of M and $A \subseteq Soc_s(M)$. By Lemma 2.13, the proof is completed. If $N \cap B = 0$ then $B \oplus N = N \oplus Rk$. If $Rk = K$, then B is a direct summand of M, and if $Rk = A$, then $B \cong A$ is a simple module. But $M = B + K$ with $B \nleq K$; and hence $B \cap K = 0$ which yields B is a direct summand of M. Therefore M is principally $ss-$lifting. □

Recall from [14] that a module M is said to be *regular* if each cyclic submodule of M is a direct summand of M.

Lemma 2.19. *Let M be a principally $ss-$lifting module. Then $\frac{M}{Soc_s(M)}$ is a regular module.*

Proof. Let $\frac{Rm}{Soc_s(M)}$ be any cyclic submodule of $\frac{M}{Soc_s(M)}$. Then Rm is a cyclic submodule of M. By the hypothesis, there is a decomposition $M = M_1 \oplus M_2$ provided that $M_1 \subseteq Soc_s(M)$ and $Rm \cap M_2 \subseteq Soc_s(M_2)$. Thus $Rm \cap M_2 \subseteq Soc_s(M)$ because $M_2 \subseteq M$. Then we have

$$\frac{M}{Soc_s(M)} = \frac{Rm}{Soc_s(M)} + \frac{M_2 + Soc_s(M)}{Soc_s(M)}.$$

Here

$$\begin{aligned} \frac{Rm}{Soc_s(M)} \cap \frac{M_2 + Soc_s(M)}{Soc_s(M)} &= \frac{[Rm \cap (M_2 + Soc_s(M))]}{Soc_s(M)} \\ &= \frac{[(Rm \cap M_2) + Soc_s(M)]}{M} = \{Soc_s(M)\}. \end{aligned}$$

Therefore $\frac{M}{Soc_s(M)}$ is regular. □

Proposition 2.20. *Let M be a principally $ss-$lifting module. Then $M = M_1 \oplus M_2$ where M_1 is regular and M_2 is a module with $Soc_s(M)$ is essential in M_2.*

Proof. Let M_1 be a submodule of M such that $Soc_s(M) + M_1$ is essential in M and $m \in M_1$. By the hypothesis, there is a decomposition $M = M_2 \oplus M_2'$ provided that $M_2 \subseteq Rm$ and $Rm \cap M_2' \subseteq Soc_s(M_2')$. Thus $Rm \cap M_2' \subseteq Soc_s(M)$. Then $Rm \cap M_2' = 0$. So $m \in M_2$ and $Rm = M_2$. Since $M_2 \cap Soc_s(M) = 0$, there exists a submodule X of M such that $M_2 \cong \frac{X}{Soc_s(M)}$. By using Lemma 2.19, M_2 is regular. In addition, $Soc_s(M) = Soc_s(M_2')$ is essential in M_2, that it is obvious by the construction of M_2'. □

Example 2.21. Let K be the quotient field of a local Dedekind domain R. Consider the left $R-$module K. It is clear that K is principally lifting. Since $Soc_s(K) = 0$, K is not principally $ss-$lifting.

Recall from [17] that, a module M is said to be *refinable if for any* submodules N and K of M with $M = N + K$, there is a direct summand N' of M provided that $N' \subseteq N$ and $M = N' + K$.

Proposition 2.22. *Let M be a refinable module with $Soc_s(M) \ll M$. If $\frac{M}{Soc_s(M)}$ is regular, then M is principally $ss-$supplemented.*

Proof. Let Rm be a cyclic submodule of M. By the hypothesis, there is a submodule $\frac{A}{Soc_s(M)}$ such that $\frac{M}{Soc_s(M)} = \frac{Rm+Soc_s(M)}{Soc_s(M)} \oplus \frac{A}{Soc_s(M)}$. Then $M = Rm + A$. Since

$$\frac{Rm + Soc_s(M)}{Soc_s(M)} \cap \frac{A}{Soc_s(M)} = \frac{[(Rm \cap A) + Soc_s(M)]}{Soc_s(M)} = \{Soc_s(M)\},$$

$Rm \cap A \subseteq Soc_s(M)$. As M is refinable, there is a direct summand B of M with $B \subseteq A$ and $M = Rm + B$. Thus $Rm \cap B \subseteq Soc_s(M)$. Since $Soc_s(M) \ll M$, $Rm \cap B \ll M$. Here $Rm \cap B \ll B$, because B is a direct summand of M. Therefore $Rm \cap B \subseteq Soc_s(B)$. □

Proposition 2.23. *Let M be a $p-$hollow module. Then M is principally $ss-$ supplemented if and only if principally $ss-$lifting.*

Proof. Obvious by [8, Proposition 16] and [3, Corollary 2.4]. □

Before completing this section, let's give the reverse transition of principally $ss-$supplemented and principally $ss-$lifting module concepts in the diagram we have given in the introduction with a special condition.

Theorem 2.24. *Let M be a $\pi-$projective and principally $ss-$supplemented module. Then M is principally $ss-$lifting.*

Proof. Let Rm be a cyclic submodule of M. By [8, Proposition 37], there is a submodule K of M such that $M = Rm + K$, $Rm \cap K \subseteq Soc_s(K)$ and a submodule N of M provided that $N \subseteq Rm$, $M = N + K$ and $N \cap K \subseteq Soc_s(N)$. Hence $N \cap K = 0$ by [16, 41.14(2)]. Thus $M = N \oplus K$. Therefore M is principally $ss-$lifting. □

3 Principally $\delta_{ss}-$supplemented module and lifting property

In this part, we present properties, characterizations and decompositions of principally $\delta_{ss}-$supplemented modules and principally $\delta_{ss}-$lifting modules.

Theorem 3.1. *Let M be a duo module. If M is principally $\delta_{ss}-$supplemented module, then each direct summand of M is so.*

Proof. Let M be a principally $\delta_{ss}-$supplemented duo module. Let $M = N_1 \oplus N_2$ and $m \in N_1$. By the assumption, there is a submodule K of M provided that $M = Rm + K$ and $Rm \cap K \subseteq Soc_\delta(K)$. If we take an intersection the equality $M = Rm + K$ with N_1, we have $N_1 = Rm + (N_1 \cap K)$. We have to prove that $Rm \cap (N_1 \cap K) \subseteq Soc_\delta(N_1 \cap K)$. Since $Rm \cap K$ is semisimple, $Rm \cap (N_1 \cap K)$ is semisimple. To complete this proof, we indicate $Rm \cap (N_1 \cap K) \ll_\delta N_1 \cap K$. Suppose that $N_1 \cap K = [Rm \cap (N_1 \cap K)] + A$ and $\frac{N_1 \cap K}{A}$ is singular. We have $K = (K \cap N_1) \oplus (K \cap N_2)$ because M is a duo module. From here, $K = (N_1 \cap K) \oplus (N_2 \cap K) = [Rm \cap (N_1 \cap K)] + A + (N_2 \cap K)$. Now, $\frac{K}{A \oplus (N_2 \cap K)} = \frac{(N_1 \cap K) \oplus (N_2 \cap K)}{A \oplus (N_2 \cap K)} = \frac{N_1 \cap K}{A}$ is singular. Therefore $K = A \oplus (N_2 \cap K)$. Since $K = (N_1 \cap K) \oplus (N_2 \cap K)$ and $A \leq N_1 \cap K$, it is implies that $A = N_1 \cap K$. Finally, $Rm \cap (N_1 \cap K) \ll_\delta N_1 \cap K$ and so the proof ends. □

Corollary 3.2. *Let M be a principally $\delta_{ss}-$supplemented distributive module. Then each direct summand of M is principally $\delta_{ss}-$supplemented.*

For to show that notion of principally $\delta_{ss}-$supplemented modules is conserved in a finite sum in a distributive module, we need the following Lemma.

Lemma 3.3. *Let M be a principally $\delta_{ss}-$supplemented module and L be a submodule of M. If each cyclic submodule Rm has a $\delta_{ss}-$supplement T with $L \leq T$, then $\frac{M}{L}$ is principally $\delta_{ss}-$supplemented.*

Proof. Suppose that $\frac{T}{L}$ be a cyclic submodule $\frac{M}{L}$. For some $m \in M$, we can write $T = Rm + L$. By the hypothesis and since M is principally $\delta_{ss}-$supplemented, there is a submodule N of M provided that $M = Rm + N$ and $Rm \cap N \subseteq Soc_\delta(N)$ for any cyclic submodule Rm of M which contains L. We consider $\alpha : M \to \frac{M}{L}$ be the canonicial projection. From here, we obtain that $\frac{M}{L} = \frac{(Rm+L)}{L} + \frac{N}{L} = \frac{T}{L} + \frac{N}{L}$ and $\frac{(Rm+L)}{L} \cap \frac{N}{L} = \frac{L+(N \cap Rm)}{L} = \alpha(Rm \cap N) \ll_\delta \alpha(N) = \frac{N}{L}$ by [18, Lemma 2.1]. If we consider [7, Chapter 8.1.5], we say that $\alpha(Rm \cap N) = \frac{(Rm \cap B)+L}{L} = \frac{Rm+L}{L} \cap \frac{N}{L}$ is semisimple because $Rm \cap N$ is semisimple. Finally, $\frac{T=Rm+L}{L} \cap \frac{N}{L} \subseteq Soc_\delta\left(\frac{N}{L}\right)$, as required. □

Proposition 3.4. *Let $M = M_1 \oplus M_2$ be a module where M_1 and M_2 are principally $\delta_{ss}-$supplemented module. If M is a distributive module, then M is principally $\delta_{ss}-$supplemented.*

Proof. Assume that $M = M_1 \oplus M_2$ be a distributive module and $A = Rm$ be a submodule of M. Taking the intersection of the equality with A, we get that $Rm = (Rm \cap M_1) \oplus (Rm \cap M_2)$. It is clear that $A_1 = Rm \cap M_1$ and $A_2 = Rm \cap M_2$ are cyclic submodules of M_1 and M_2. By the assumption, there are submodules $B_1 \leq M_1$ such that $M_1 = A_1 + B_1$, $A_1 \cap B_1 \subseteq Soc_\delta(B_1)$ and $B_2 \leq M_2$ with $M_2 = A_2 + B_2$, $A_2 \cap B_2 \subseteq Soc_\delta(B_2)$. From here

$$\begin{aligned} M &= M_1 + M_2 = A_1 + B_1 + A_2 + B_2 = (Rm \cap M_1) + B_1 + (Rm \cap M_2) + B_2 \\ &= Rm + B_1 + B_2. \end{aligned}$$

We will prove that

$$Rm \cap (B_1 + B_2) \leq (Rm \cap B_1) + (Rm \cap B_2).$$

It is clear that $(Rm \cap B_1) + (Rm \cap B_2) \leq Rm \cap (B_1 + B_2)$. On the other hand,

$$\begin{aligned} Rm \cap (B_1 + B_2) &\leq B_1 \cap (Rm + B_2) + B_2 \cap (Rm + B_1) \\ &= (B_1 \cap [(Rm \cap M_1) + M_2]) + (B_2 \cap [M_1 + (Rm \cap M_2)]). \end{aligned}$$

Here

$$\begin{aligned} B_1 \cap [(Rm \cap M_1) + M_2] &\leq (Rm \cap M_1) \cap (B_1 + M_2) + (M_2 \cap (Rm \cap M_1)) + B_1 \\ &= Rm \cap B_1. \end{aligned}$$

Similarly $B_2 \cap [M_1 + (Rm \cap M_2)] \leq Rm \cap B_2$. It follows from $Rm \cap (B_1 + B_2) \leq (Rm \cap B_1) + (Rm \cap B_2)$. So, we have that $Rm \cap (B_1 + B_2) = (Rm \cap B_1) + (Rm \cap B_2)$. Finally, if we use [11, Proposition 4.9], then we can say that $(Rm \cap B_1) + (Rm \cap B_2) \subseteq Soc_\delta (B_1 + B_2)$ because $A_1 \cap B_1 \subseteq Soc_\delta (B_1)$ and $A_2 \cap B_2 \subseteq Soc_\delta (B_2)$. Thus, M is principally $\delta_{ss}-$supplemented. □

Recall that, a module M is said to be *principally semisimple* if each cyclic submodule is a direct summand of M. It is clear that every semisimple module is principally semisimple.

Lemma 3.5. *Let M be a principally $\delta_{ss}-$supplemented distributive module. Then $\frac{M}{Soc_\delta(M)}$ is a principally semisimple module.*

Proof. By the hypothesis, there is a submodule N of M provided that $M = Rm + N$ and $Rm \cap N \subseteq Soc_\delta (N) \subseteq Soc_\delta (M)$. Considering that the module M is distributive module, we have $Soc_\delta (M) = (Rm \cap N) + Soc_\delta (M) = (Rm + Soc_\delta (M)) \cap (N + Soc_\delta (M))$. In addition,

$$\frac{M}{Soc_\delta (M)} = \frac{Rm + Soc_\delta (M)}{Soc_\delta (M)} + \frac{N + Soc_\delta (M)}{Soc_\delta (M)}.$$

Therefore, $\frac{M}{Soc_\delta(M)} = \frac{Rm+Soc_\delta(M)}{Soc_\delta(M)} \oplus \frac{N+Soc_\delta(M)}{Soc_\delta(M)}$ and so $\frac{M}{Soc_\delta(M)}$ is principally semisimple. □

We investigate under which conditions their direct summand of principally δ_{ss}-supplemented modules are principally $\delta_{ss}-$supplemented modules.

Lemma 3.6. *Let $M = M_1 \oplus M_2$ be a module. Then M_2 is principally $\delta_{ss}-$ supplemented if and only if for each cyclic submodule $\frac{L}{M_1}$ of $\frac{M}{M_1}$, there exists a submodule N of M_2 such that $M = N + L$ and $N \cap L \subseteq Soc_\delta (N)$.*

Proof. Assume that M_2 be principally $\delta_{ss}-$supplemented and $\frac{L}{M_1}$ be cyclic of $\frac{M}{M_1}$. Now, we take $m = m_1 + m_2$ where $m_1 \in M_1$, $m_2 \in M_2$ and $\frac{L}{M_1} = \frac{Rm_1+M_1}{M_1}$. Then $\frac{L}{M_1} = \frac{Rm_2+M_1}{M_1}$. By the hypothesis, there exists a submodule N of M_2 such that $M_2 = Rm_2 + N$ with $Rm_2 \cap N \subseteq Soc_\delta (N)$. From here $L = Rm_2 + M_1$ and $M = N + L$. Otherwise

$$\begin{aligned} L \cap N &= (Rm_2 + M_1) \cap N \leq Rm_2 \cap (M_1 + N) + M_1 \cap (N + Rm_2) \\ &\leq N \cap (M_1 + Rm_2) + M_1 \cap (N + Rm_2). \end{aligned}$$

Since $M_1 \cap (N + Rm_2) = 0$, $M_1 \cap (N + Rm_2) = Rm_2 \cap (Rm_1 + K)$. Therefore, $N \cap L \leq Rm_2$ and $N \cap L \subseteq Soc_\delta (N)$. Conversely, suppose that $\frac{L+M_1}{M_1}$ be a cyclic submodule of $\frac{M}{M_1}$ for a cyclic submodule L of M_2. By the assumption, there is a submodule N of M_2 provided that $M = (L + M_1) + N$ and $N \cap (L + M_1) \subseteq Soc_\delta (N)$. It is clear that $M_2 = N + L$. If we show that $N \cap (M_1 + L) = L \cap (M_1 + N) = L + N$, the proof is completed. Here $L \cap (M_1 + N) \leq M_1 \cap (N + L) + N \cap (L + M_1) \leq L \cap (M_1 + N) + M_1 \cap (N + L) = L \cap (M_1 + N)$ because $M_1 \cap (N + L) = 0$. Hence $L \cap (M_1 + N) = N \cap (L + M_1)$. It is obvious that $(M_1 + N) \cap L = N \cap (M_1 + L) = L \cap N$. Finally, M_2 is principally $\delta_{ss}-$supplemented. □

Proposition 3.7. *Let $M = M_1 + M_2$ be a module with M_1 and M_2 are principally $\delta_{ss}-$supplemented modules. Then M is principally $\delta_{ss}-$supplemented if and only if for each cyclic submodule L of M with $M = N + L$ for any proper submodule N of M has a $\delta_{ss}-$supplement in M.*

Proof. ($\Rightarrow$) The proof is clear.

($\Leftarrow$)By the hypothesis, for every cyclic submodule L of M with $M = N + L$ for any proper submodule N of M has a $\delta_{ss}-$supplement in M. Suppose that $L = Rm$ be a cyclic submodule of M. If $M = L + M_i$ or $L \leq M_i$, the proof is completed. Other than we can accept $m = m_1 + m_2$ and m_1, m_2 are non-zero. By the assumption, there are submodules T_1,T_2 of M such that $M_1 = Rm_1 + T_1$, $Rm_1 \cap T_1 \subseteq Soc_\delta(T_1)$ and $M_2 = Rm_2 + T_2$, $Rm_2 \cap T_2 \subseteq Soc_\delta(T_2)$. $Rm_1 + Rm_2 = L + Rm_2 = L + Rm_1$ and $M = L + Rm_1 + T_1 + T_2 = L + M_1 + T_2$. In the same way, we have $M = L + M_2 + T_1$. Suppose that $M = M_1 + T_2$. It follows that $M = T_2$ and so $m_2 = 0$ and $L \leq M_1$. This is contradiction. Therefore $M_1 + T_2$ is a proper submodule of M. In the same way, $M_2 + T_1$ is proper. Finally, L has a $\delta_{ss}-$supplement in M. □

Recall from [4] that, a non-zero module M is said to be *principally $\delta-$hollow* if every cyclic module is $\delta-$small in M. It is clear that every principally $\delta_{ss}-$lifting module is principally $\delta_{ss}-$supplemented.

It is clear that principally $\delta-$hollow module is principally $\delta_{ss}-$supplemented. The following lemma shows that notions of principally $\delta_{ss}-$lifting modules and principally $\delta-$hollow modules are the same.

Lemma 3.8. *Let M be indecomposable module. If M is principally $\delta_{ss}-$lifting module, then M is principally $\delta-$hollow module.*

Proof. Suppose that $m \in M$. By the hypothesis, there are submodules N and L of M such that $N \leq Rm$, $Rm \cap L \subseteq Soc_\delta(L)$ and $M = N \oplus L$. Since M is indecomposable, $L = M$. Hence $Rm \cap L = Rm$ is $\delta-$small in M. Thus, M is a principally $\delta-$hollow module. □

Theorem 3.9. *Let M be a module with non-zero $\delta(M)$. Then the following conditions are equivalent: (i) M is principally $\delta_{ss}-$supplemented.*
(ii) M is principally $\delta-$supplemented.
(iii) M is projective semisimple.

Proof. The proof is clear by [11, Theorem 3.14]. □

Corollary 3.10. *Let M be a module with non-zero $\delta(M)$. Then the following statements are equivalent:*
(i) M is principally $\delta_{ss}-$supplemented.
(ii) M is principally $\delta-$supplemented.
(iii) M is regular.

Recall from [9] that, M is said to have the *summand intersection property* (SIP) if the intersection of any direct summands of M is a direct summand.

Theorem 3.11. *Let M be a refinable module. Then (i)⇒(ii)⇒(iii).*
(i) M is principally $\delta_{ss}-$lifting.
(ii) M is principally $\delta_{ss}-$supplemented.
(iii) M is weakly principally $\delta_{ss}-$supplemented.
If M has the (SIP), then (iii)⇒(i).

Proof. (i)⇒(ii)⇒ (iii) proofs are clear from definitions.

(iii)⇒(i) Suppose that M has the (SIP) and let $m \in M$. Since M is weakly principally $\delta_{ss}-$supplemented, there is a submodule N of M with $M = Rm + N$ and $Rm \cap B \subseteq Soc_s(M)$. By the assumption, there is a direct summand X_1 of M provided that $X_1 \leq N$ and $M = Rm + X_1 = X_1' \oplus X_1$. $Rm \cap X_1 \ll_\delta X_1$, since $Rm \cap N \ll_\delta M$ and X_1 is a direct summand of M. As M is a refinable module, there is a direct summand X_2 of M such that $X_2 \leq Rm$ and $M = X_2 + X_1 = X_2 \oplus X_2'$. We obtain that $X_2 \cap X_1$ is a direct summand of and so we can write $M = (X_2 \cap X_1) \oplus T$ for some submodule T of M because M has the (SIP) property. Then $X_1 = (X_2 \cap X_1) \oplus (T \cap X_1)$ and $M = X_2 \oplus (T \cap X_1)$. Since $Rm \cap (T \cap X_1) \leq Rm \cap X_1 \leq X_1$, $Rm \cap X_1 \ll_\delta X_1$ and $T \cap X_1$ is a direct summand of M, we have $Rm \cap (T \cap X_1) \ll_\delta T \cap X_1$. Also, $Rm \cap (T \cap X_1)$ is semisimple because $Rm \cap (T \cap X_1) \subseteq Rm \cap (T \cap N) \subseteq Rm \cap N$ is semisimple. Therefore, M is principally $\delta_{ss}-$lifting. □

Theorem 3.12. *Let M be a principally $\delta-$supplemented module with $\delta(M) \subseteq Soc(M)$. Then M is principally $\delta_{ss}-$supplemented.*

Proof. The proof is clear from [11, Proposition 4.6]. □

Proposition 3.13. *Let M be a principally $\delta-$lifting module with $\delta(M) \subseteq Soc_\delta(M)$. Then M is principally $\delta_{ss}-$lifting.*

Proof. Since M is a principally $\delta-$lifting module, we get that $M = N \oplus K$ with $N \subseteq Rm$ and $Rm \cap K \ll_{\delta} K$ for each cyclic submodule Rm of M. Because of $\delta(M) \subseteq Soc_{\delta}(M)$ it is obvious that $Rm \cap K \subseteq Soc_{\delta}(M)$. So M is principally $\delta_{ss}-$lifting. □

Proposition 3.14. *Let M be a projective module. If M is semilocal and principally $\delta_{ss}-$supplemented, then M is principally $ss-$supplemented.*

Proof. The proof can be obtained by using the similar method in [11, Proposition 5.9]. □

Theorem 3.15. *Let M be a module with non-zero $\delta(M)$. Then the following statements are equivalent:*

(1) M is principally $\delta_{ss}-$supplemented.
(2) M is principally $\delta-$supplemented.
(3) M is projective semisimple.

Proof. The proof can be done in a similar way in [11, Theorem 3.14]. □

Recall from [11] that, a module M is said to be *strongly $\delta-$local* if it is $\delta-$local and $\delta(M) \subseteq Soc(M)$. Here, M is said to be $\delta-$local if $\delta(M) \ll_{\delta} M$ and $\delta(M)$ is a maximal submodule of M.

Before completing this section, let's see that notions of a strongly $\delta-$local module and principally $\delta_{ss}-$supplemented module coincide in $\delta-$local module.

Lemma 3.16. *Let M be strongly $\delta-$local module. Then it is principally $\delta_{ss}-$ supplemented.*

Proof. Suppose that $A = Rm$ be a cyclic submodule of M and $A \subseteq \delta(M)$. A is semisimple because $\delta(M)$ is semisimple. If we consider [11, Lemma 2.2], we get that $A \ll_{\delta} M$ and so M is the $\delta_{ss}-$supplement of A in M. Assume that $A \nsubseteq \delta(M)$. We have the equality $M = A + \delta(M)$ because $\delta(M)$ is maximal. Since $\delta(M) \ll_{\delta} M$, there exists a projective semisimple submodule B of $\delta(M)$ such that $M = A \oplus B$. Finally, M is principally $\delta_{ss}-$supplemented. □

Corollary 3.17. *Let M be $\delta-$local module. Then M is principally $\delta_{ss}-$supplemented if and only if M is strongly $\delta-$local.*

In Example 2.1, since the $\mathbb{Z}-$module M is $\delta-$local and strongly $\delta-$local, M may not be principally $\delta_{ss}-$supplemented by Corollary 3.17.

Example 3.18. ([11, Example 4.4.(1)]) Let S be the non-noetherian commutative ring S which is the direct product $\prod_{i\geq 1}^{\infty} F_i$, where $F_i = \mathbb{Z}_2$ and R be a subring of S generated by $\bigoplus_{i\geq 1}^{\infty} F_i$ and 1_S. Consider the module $M =_R R$. M is a $\delta-$local module and strongly $\delta-$local but not $ss-$supplemented. By Corollary 3.17, M is principally $\delta_{ss}-$supplemented. Otherwise, M is not principally $ss-$supplemented.

Competing Interests and Funding

The authors have no direct or indirect financial or non-financial interest in the work submitted for publication.

4 Conclusion remarks

In this article, we have focused on principally $ss-$supplemented modules and in particular we have related the notions of an $ss-$supplement submodule, a $\delta_{ss}-$supplement submodule via cyclic and semisimple submodules. This study is an important study that contains many characterizations as a possession of a strong notion of the principally supplemented module class. In addition it is aimed to define principally $\delta-$lifting (principally δ_{ss}-supplemented) modules as its proper generalization of the principally ss-lifting (principally ss-supplemented) modules on the occasion of this study. Relevant properties of these modules classes have been obtained. Especially, the class of principally $ss-$supplemented modules is closed under extensions under certain conditions. We show that the notion of principally $\delta_{ss}-$supplemented modules is conserved in a finite sum in a distributive module. We prove that the notion of principally $\delta_{ss}-$lifting and the notion of principally $\delta_{ss}-$supplemented coincide with a refinable module. We believe that this study may open up new research areas on the subject.

References

[1] U. Acar and A. Harmancı, *Principally supplemented modules*, Albanian J. Math., **4(3)**, 79-88, (2010).

[2] S. K. Choubey, L.K. Das and M. K. Patel , *Closed weak Rad-supplemented modules*, Palestine Journal of Mathematics, **11(4)**, 325-330 , (2022).

[3] F. Eryılmaz, $SS-$ *lifting modules and rings*, Miskolc Math. Notes, **22(2)**, 655-662, (2021).

[4] H. İnankıl, S. Halıcıoğlu and A. Harmancı, *On a class of lifting modules*, Vietnam J. Math., **38(2)**, 189-201, 2010.

[5] H. İnankıl, S. Halıcıoğlu and A. Harmancı, *A generalization of supplemented modules*, Algebra and Discrete Mathematics, **11(1)**, 59-74, (2011).

[6] M.A. Kamal and A. Yousef, *On principally lifting modules*, Int. Electron J. Algebra, **2**, 127-137, (2007).

[7] F. Kasch, *Modules and Rings*, London Mathematical Society Monographs, London New York, (1982).

[8] E. Kaynar, H. Çalışıcı and E. Türkmen, $SS-$*supplemented modules*, Comm. Fac. Sci. Univ. Ank. Ser. A1 Math. Sci. Stat., **69(1)**, 473-485, (2020).

[9] S.H. Mohamed and B.J. Müller, *Continuous and Discrete Modules*, London Mathematicial Society Lecture Note Series, Cambridge University Press, Cambridge, (1990).

[10] W.K. Nicholson and M.F. Yousif, *Quasi-Frobenius rings*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, (1990).

[11] B. Nişancı Türkmen and E. Türkmen, $\delta_{ss}-$*supplemented modules and rings*, An. Şt. Univ. Ovidius Constanta Ser. Mat., **28(3)**, 193-216, 2020.

[12] A.Ç. Özcan and M. Alkan, *Duo modules*, Glasgov Mathematical Journal, **48(3)**, 533-545, (2006).

[13] Y. Talebi, M. Hosseinpour and S. Khajvand Sany, *Strongly FI-δ-lifting modules*, Palestine Journal of Mathematics, **4(2)**, 380-385, 2015.

[14] A. A. Tuganbaev, *Semiregular, weakly regular and $\pi-$regular rings*, J. Math. Sci., **109(3)**, 1509-1588, (2002).

[15] B. Üngör, S. Halıcıoğlu and A. Harmancı, *On a class of $\oplus-$supplemented modules*, Ring Theory and Its Applications, Contemp. Math., **609**, 123-136, (2014).

[16] R.Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, Philadelphia, (1991).

[17] R. Wisbauer, *Modules and Algebras: Bimodule Structure on Group Actions and Algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman, Harlow, (1996).

[18] Y. Zhou, *Generalizations of perfect, semiperfect, and semiregular rings*, Algebra Colloq. , **7(3)**, 305-318, (2000).

[19] D.X. Zhou and X.R. Zhang , *Small-essential submodules and Morita duality*, Southeast Asian Bull. Math., **35(6)**, 1051–1062, (2011).

Author information

Figen ERYILMAZ, Ondokuz Mayıs University, Department of Mathematics Education, Samsun, Turkey.
E-mail: fyuzbasi@omu.edu.tr

Burcu NİŞANCI TÜRKMEN, Amasya University, Faculty of Art and Science, Department of Mathematics, Amasya, Turkey.
E-mail: burcu.turkmen@amasya.edu.tr

Received: 2022-10-09
Accepted: 2024-05-03