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Graph Theoretical Characterizations of Non integrally closed ring extensions with few non-Artinian intermediate rings

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Abstract The main purpose of this paper is to provide a complete characterization of non integrally closed ring extensions with exactly three non-Artinian intermediate rings.

1 Introduction

In this work, all rings and algebras are commutative and unital; all inclusions of rings, ring extensions and algebra/ring homomorphisms are unital. We will adopt the same notation and conventions as in [1, 2]. Fix, a ring extension $R \subset S$. The symbol $[R, S]$ designates the set of intermediate rings; that is, rings T such that $R \subseteq T \subseteq S$. In particular, if $S = \text{tq}(R)$ is the total quotient ring of R, then any ring in $[R, \text{tq}(R)]$ is called an *overring* of R. Given a ring theoretic property $\mathcal{P}$, we let $[R, S]_{\text{non}-\mathcal{P}} := \{T \in [R, S] \mid T \text{ does not satisfy } \mathcal{P}\}$ and $[R, S]_{\mathcal{P}}$ its complement in $[R, S]$. The second named author and Al Subaiei in [2] raised the following problem, which we label it as **SJ-Probem**: “Let $\mathcal{P}$ be a ring-theoretic property and let n be a nonnegative integer. Provide necessary and sufficient conditions in order that a ring extension $R \subset S$ satisfies $|[R, S]_{\text{non}-\mathcal{P}}| = n$.” As usual, $|\Omega|$ denotes the cardinal number of a set Ω. It is worth noticing that SJ-Probem generalizes those related to pairs of rings and maximal non-$\mathcal{P}$ subrings. In fact, (R, S) is a $\mathcal{P}$-pair if $|[R, S]_{\text{non}-\mathcal{P}}| = 0$ and R is a maximal non-$\mathcal{P}$ subring of S if $[R, S]_{\text{non}-\mathcal{P}} = \{R\}$. $\mathcal{P}$-pairs and maximal non-$\mathcal{P}$ subrings have been studied for many properties $\mathcal{P}$ such as Noetherian [5, 34], Jaffard [9, 10, 11, 23], S-domain [30], treed [6, 7], valuation [13], Prüfer [31], universally catenarian [3, 8, 12], Artinian [26, 33], integrally closed [32, 35], pseudo-valuation [36]. Solutions to SJ-Probem were given in [33] (resp., [2]) in case $0 \leq n \leq 2$ and $\mathcal{P}$:= Artinian (resp., Prüfer). For $n = 3$ and $\mathcal{P}$:= Artinian, a complete answer was provided in case R is integrally closed in S (cf. [1]). To complete this circle of ideas, we aim to answer SJ-problem in case $\mathcal{P}$:= Artinian, $n = 3$ and R is not integrally closed in S. Our work is motivated both by the importance of Artinian rings in commutative algebra and several related fields, and the recent growing interest in the study of extensions with few non-$\mathcal{P}$ intermediate rings as described above. In Theorem 2.4, we give a complete characterization of such pairs of rings by means of the shape of the ordered set $[R, S]$ as a directed graph. Corollary 2.5 takes care of the case of integral domains.

We let $\overline{R}_S$ denote the integral closure of R in S and R' denote the integral closure of R (in its total quotient ring). The symbol “$\subset$” denotes proper containment. $\text{qf}(R)$ is the quotient field of the integral domain R. For a ring extension $R \subset S$, we let $]R, S] := [R, S] \setminus \{R\}$. Most of our notation is standard and can for instance be found in [24] and [37].

2 Main results

We call a ring extension $R \subset S$ *minimal* if $|[R, S]| = 2$ (cf. [15, 22]). If moreover, $\overline{R}_S = R$ (resp., $\overline{R}_S = S$), then $R \subset S$ is called *closed* (resp., *integral*). We start our investigation with the following two lemmata.

Lemma 2.1. *If $R \subset S$ is an integral extension with $|[R,S]_{\text{non}-\mathcal{P}}| = n \geq 1$, then $[R,S] = [R,S]_{\text{non}-\mathcal{P}}$ and so $n \geq 2$.*

Proof. Since $|[R,S]_{\text{non}-\mathcal{P}}| = n \geq 1$, then [33, Proposition 1] ensures that R is not Artinian. Now, let $T \in]R,S]$ and assume, by way of contradiction, that T is Artinian. As $[R,T]_{\text{non}-\mathcal{P}}$ is finite, we can choose C in $[R,T]$ maximal with respect to being non-Artinian. Thus, C would be a maximal non-Artinian subring of T. Hence, $C \subset T$ would be a closed minimal extension according to [33, Theorem 1]. This contradicts the integrality of $C \subset T$. Hence, we have proved that $[R,S] = [R,S]_{\text{non}-\mathcal{P}}$ and so $n = |[R,S]| \geq 2$. The proof is complete. □

The following lemma generalizes [1, Lemma 3.1], which was proved only for integrally closed extensions. But, first recall that a ring extension $R \subseteq S$ is said to be an *FIP extension* (for the "finitely many intermediate algebras property") if $[R,S]$ is finite.

Lemma 2.2. *Let $R \subset S$ be a ring extension such that $|[R,S]_{\text{non}-\mathcal{P}}| = n \geq 1$. Then $R \subset S$ is an FIP extension. Moreover, (A,S) is an Artinian pair for any $A \in [R,S]_{\mathcal{P}}$. In particular, $A \subseteq S$ is an integral extension.*

Proof. Firstly, we demonstrate that $R \subset S$ is an FIP extension. The case where $R \subset S$ is integrally closed was already treated in [1, Lemma 3.1]. Thus, we will assume that $R \subset S$ is not integrally closed. Since $|[R,S]_{\text{non}-\mathcal{P}}| = n \geq 1$, then [33, Proposition 1] ensures that R is not Artinian. Hence, $1 \leq |[R,\overline{R}_S]_{\text{non}-\mathcal{P}}| < \infty$. It follows from Lemma 2.1 that $[R,\overline{R}_S]$ is finite (that is, $R \subset \overline{R}_S$ is an FIP extension) and $\overline{R}_S$ is not Artinian. Thus, $1 \leq |[\overline{R}_S,S]_{\text{non}-\mathcal{P}}| < \infty$. An application of [1, Lemma 3.1] ensures that $\overline{R}_S \subset S$ is an FIP extension. This yields that $R \subset S$ is also an FIP extension accordingly to [18, Theorem 3.13]. For the "Moreover" statement, let A be an Artinian ring in $[R,S]$ and let $B \in [A,S]$. We need to show that B is Artinian. Since $R \subset S$ is an FIP extension, then so is $A \subseteq B$. Therefore, there exists a (finite) chain of rings $A = A_0 \subset A_1 \subset ... \subset A_l = B$ going from A to B. As $A \subset A_1$ is a minimal extension and A is Artinian, then so is A_1 accordingly to [26, Theorem 2]. Again, as $A_1 \subset A_2$ is minimal and A_1 is Artinian, then so is A_2. Proceed along the same lines, one can derive easily that B is Artinian. Hence, (A,S) is an Artinian pair. It follows from [27, Corollary 4.2] that $A \subseteq S$ is an integral extension. □

In order to state our next results, we introduce the following definition.

Definition 2.3. A graph that can be drawn in the shape of

(i) a kite, as in Fig. 1, is called a *kite-graph* (cf. [29, Definition 1]). A kite-graph is said to be of dimension d if it contains a chain with exactly d edges and the number of edges in any other chain of this graph is $\leq d$.

(ii) a rectangle, as in Fig. 2, is called a *rectangle-graph*.

(iii) a pentagon, as in Fig. 3, is called a *pentagon-graph*.

(iv) the Greek letter θ, as in Fig. 4, is called a *theta-graph*.

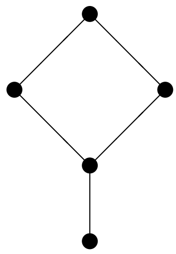


Fig.1 (Kite-graph of dimension 3)

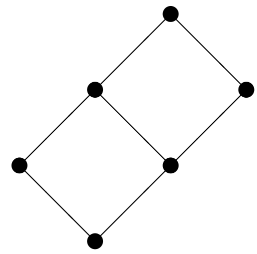


Fig. 2 (Rectangle-graph)

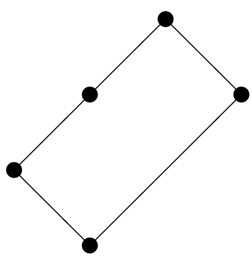


Fig. 3 (pentagon-graph)

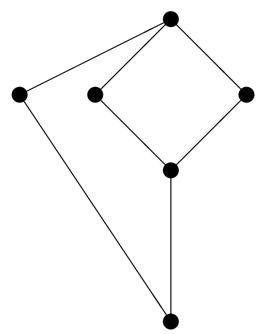


Fig. 4 (theta-graph)

In what follows we assume that R is not integrally closed in S. Then we establish necessary and sufficient conditions for the ring extension $R \subset S$ to have exactly three non-Artinian intermediate rings. We recall some background. We call a ring extension $R \subseteq S$ *Prüfer* if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$ (cf. [38]). Any closed minimal extension is a Prüfer extension and any ring extension $R \subseteq S$ has a greatest Prüfer subextension $R \subseteq \widetilde{R}^S$, called the *Prüfer hull* of R in S (cf. [38]). If $R = \widetilde{R}^S$, then $R \subseteq S$ is called *Prüfer-closed* (see also [19]). For instance, any integral extension is Prüfer-closed. For any ring extension $R \subseteq S$, we say that (R, S) is a *normal pair* if $T \subseteq S$ is an integrally closed extension; that is, T is integrally closed in S, for each ring $T \in [R, S]$. The concept of normal pairs (R, S) was introduced and investigated, in case S is an (integral) domain, by Davis [17]. The typical example of a normal pair (R, S) arises when R is a Prüfer domain and S is its quotient field (cf. [17, Theorem 1] or [24, Theorems 23.4(1) and 26.1(1)]). Many interesting characterizations of these pairs have been established in [4]. Normal pairs of rings with zero divisors have attracted several researchers, so many results have been generalized from the domain-theoretic case to arbitrary rings (see for instance, [14], [21] and [38]). It was proved in [38, Theorem 5.2] that $R \subseteq S$ is a Prüfer extension if and only if (R, S) is a normal pair. We recall from [28] that if $R \subseteq S$ is a ring extension, then an element s of S is said to be *primitive* over R if s is a root of a polynomial $f(X) \in R[X]$ with unit content. We say that $R \subseteq S$ is a *P-extension* if any element of S is primitive over R. The relationship between P-extensions and normal pairs was established in [14, Theorem 1]. More precisely, it was shown that (R, S) is a normal pair if and only if $R \subseteq S$ is an integrally closed

P-extension.

Recall also that $R \subseteq S$ is said to be an *FCP extension* if each chain in $[R, S]$ is finite. Clearly, each FIP extension is an FCP extension. In [18], the authors have characterized FCP and FIP extensions. Remark that if $R \subset S$ has FCP, then any maximal (necessarily finite) chain $\mathcal{C}$ of R-subalgebras of R, $R = R_0 \subset R_1 \subset ...R_{m-1} \subset R_m = S$, with *length* $\ell(\mathcal{C}) := m < \infty$, results from juxtaposing m minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq m-1$. For an FCP extension $R \subset S$, the length $\ell[R, S]$ of $[R, S]$ is the supremum of the lengths of chains of R-subalgebras of S. Observe that this length is finite and there does exist some maximal chain of R-subalgebras of S with length $\ell[R, S]$ [20, Theorem 4.11].

Theorem 2.4. *Let $R \subset S$ be a ring extension such that R is not integrally closed in S. Then the following statements are equivalent:*

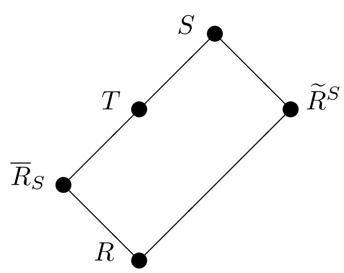
(i) $|[R, S]_{\text{non}-\mathcal{P}}| = 3$.

(ii) (*Exactly*) *one of the following conditions holds true:*

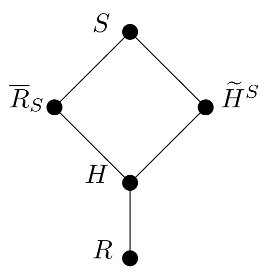
- *a.* $[R, S]$ *ordered by the usual set inclusion is a chain of length* 2 *and S is not Artinian.*
- *b.* $[R, S]$ *ordered by the usual set inclusion is a chain of length* 3 *such that S is Artinian and R is not Artinian.*
- *c.* $[R, S]$ *ordered by the usual set inclusion is order isomorphic to a kite-graph of dimension* 3 *such that S is Artinian, R is not Artinian and $[R, \overline{R}_S]$ is a chain of length* 2.
- *d.* $[R, S]$ *ordered by the usual set inclusion is order isomorphic to a pentagon-graph such that S is Artinian and R is not Artinian.*
- *e.* $[R, S]$ *ordered by the usual set inclusion is either order isomorphic to a rectangle-graph or a theta-graph such that S is Artinian, R is not Artinian, $[R, \overline{R}_S]$ is a chain of length* 2 *and $R \subset S$ is not a Prüfer-closed extension.*

Proof. (i)⇒(ii) We distinguish the following two cases.
case 1. $\overline{R}_S = S$.
It follows, from Lemma 2.1, that $[R, S] = [R, S]_{\text{non}-\mathcal{P}}$. Hence, $[R, S]$ ordered by the usual set inclusion is a chain of length 2 and S is not Artinian.
case 2. $\overline{R}_S \neq S$.
As $R \subset \overline{R}_S \subset S$, then two subcases may occur.
subcase 2.1. $R \subset \overline{R}_S$ is a minimal extension.
Since R and $\overline{R}_S$ are not Artinian (Lemma 2.1) and $|[R, S]_{\text{non}-\mathcal{P}}| = 3$, then necessarily there exists a non-Artinian ring $T \in [R, S] \setminus \{R, \overline{R}_S\}$. If $T = S$, then $[R, S]_{\text{non}-\mathcal{P}} = \{R, \overline{R}_S, S\}$. Thus, $|[\overline{R}_S, S]_{\text{non}-\mathcal{P}}| = 2$. But, as $\overline{R}_S$ is integrally closed in S, then [1, Lemma 3.2] ensures that $\overline{R}_S \subset S$ is a minimal extension. Now, we claim that $[R, S] = \{R \subset \overline{R}_S \subset S\}$ is a chain of length 2. For, let $A \in [R, S] \setminus \{R, \overline{R}_S, S\}$. Then A is Artinian. Hence, S is Artinian by virtue of Lemma 2.2, which is a contradiction. So in this case $[R, S]$ is a chain of length 2, where S is not Artinian. Assume now that $T \neq S$. Then, a fortiori S is Artinian. We claim that T and $\overline{R}_S$ are comparable under inclusion. Indeed, assume the contrary. Then $R \subset T$ is a minimal extension by using [26, Theorem 2] and it is closed since $T \nsubseteq \overline{R}_S$. Since $\overline{R}_S \notin [T, S]$, then T would be a maximal non-Artinian subring of S. Thus, $T \subset S$ would be a (closed) minimal extension according to [33, Theorem 1]. It follows that $R \subset S$ is an integrally closed extension, which contradicts our assumption. We conclude that T and $\overline{R}_S$ are comparable under inclusion as claimed. It follows that $R \subset \overline{R}_S \subset T \subset S$. As $|[R, S]_{\text{non}-\mathcal{P}}| = 3$, then $|[\overline{R}_S, S]_{\text{non}-\mathcal{P}}| = 2$. It follows from [1, Lemma 3.2] that $|[\overline{R}_S, S]| = 3$. In particular, $\overline{R}_S \subset T$ and $T \subset S$ are (closed) minimal extensions. Next, we will distinguish the following two subsubcases.
subsubcase 2.1.1. $R \subset S$ is Prüfer-closed.
We claim that $[R, S] = \{R \subset \overline{R}_S \subset T \subset S\}$ is a chain of length 3. Indeed, suppose that there exists a ring $A \in [R, S] \setminus \{R, \overline{R}_S, T, S\}$. Then A is Artinian. Note that on one hand, $\overline{R}_S \nsubseteq A$ since $[\overline{R}_S, S] = \{\overline{R}_S, T, S\}$. On the other hand, $T \nsubseteq A$ because $T \subset S$ is a minimal extension. Therefore, R is a maximal non-Artinian subring of A. Hence $R \subset A$ should be a closed minimal extension (see [33, Theorem 1]) and so $A \subseteq \widetilde{R}^S$, the desired contradiction since $R \subset S$ is assumed to be Prüfer-closed. This proves our claim.

subsubcase 2.1.2. $R \subset S$ is not Prüfer-closed.
In this case $\widetilde{R}^S \neq R$. Observe that $\overline{R}_S, T \not\in]R, \widetilde{R}^S]$ since $R \subset \widetilde{R}^S$ is a Prüfer extension. It follows that each ring in $]R, \widetilde{R}^S]$ is Artinian. Therefore, [33, Theorem 1] ensures that $R \subset \widetilde{R}^S$ is a (closed) minimal extension. Since $\widetilde{R}^S$ is Artinian, then Lemma 2.2 guarantees that $\widetilde{R}^S \subset S$ is an integral extension. We claim that $\widetilde{R}^S \subset S$ is a minimal extension. Indeed, let $B \in [\widetilde{R}^S, S]$. Then either $B \cap \overline{R}_S = \overline{R}_S$ or $B \cap \overline{R}_S = R$ because $R \subset \overline{R}_S$ is a minimal extension. If $B \cap \overline{R}_S = R$, then $R \subset B$ is a Prüfer extension. Hence, $B \subseteq \widetilde{R}^S$, which implies that $B = \widetilde{R}^S$. If $B \cap \overline{R}_S = \overline{R}_S$, then $B \supseteq \overline{R}_S$. So $B \subseteq S$ is integrally closed. But $B \subset S$ is an integral extension, which yields that $B = S$. Thus, $\widetilde{R}^S \subset S$ is a minimal extension, as claimed. Therefore, we get the following two maximal chains of rings: $R \subset \overline{R}_S \subset T \subset S$, and $R \subset \widetilde{R}^S \subset S$. We claim that $[R, S] = \{R, \widetilde{R}^S, \overline{R}_S, T, S\}$ is a pentagon-graph as drawn below.



To this end, let $A \in [R, S] \setminus \{R, \widetilde{R}^S, \overline{R}_S, T, S\}$. Then A is Artinian. Moreover, $\overline{R}_S \not\in]R, A]$ since $A \not\in [\overline{R}_S, S]$. On the other hand, $T \not\in]R, A]$ since $\overline{R}_S \not\in]R, A]$. Thus, each ring in $]R, A]$ is Artinian. Hence, $R \subset A$ is a (closed) minimal extension by virtue of [33, Theorem 1]. It follows that $A \subseteq \widetilde{R}^S$, which is impossible since $R \subset \widetilde{R}^S$ is a minimal extension.
subcase 2.2. $R \subset \overline{R}_S$ is not a minimal extension.
In this case $[R, \overline{R}_S] = \{R \subset H \subset \overline{R}_S\}$ is a chain of length 2. Lemma 2.1 shows that each ring in $[R, \overline{R}_S]$ is not Artinian. Thus, each ring in $]\overline{R}_S, S]$ is Artinian. Therefore, $\overline{R}_S \subset S$ is a closed minimal extension (cf. [33, Theorem 1]). Moreover, we have $|[H, S]_{\text{non}-\mathcal{P}}| = 2$. As H is not integrally closed in S, then $[H, S]$ is either a chain $H \subset \overline{R}_S \subset S$ of length 2; or consists of two chains of length 2, namely, $H \subset \overline{H}_S = \overline{R}_S \subset S$ and $H \subset \widetilde{H}^S \subset S$ (for more details, see the proof of [33, Theorem 2]).
subsubcase 2.2.1. $R \subset S$ is Prüfer-closed.
If $[H, S] = \{H \subset \overline{R}_S \subset S\}$, then we show that $[R, S] = \{R \subset H \subset \overline{R}_S \subset S\}$ is a chain of length 3. Indeed, let $A \in [R, S] \setminus \{R, H, \overline{R}_S, S\}$, then obviously $\overline{R}_S \not\subseteq A$ since $\overline{R}_S \subset S$ is a minimal extension. Moreover, $H \not\subseteq A$ since $A \not\in [H, S]$. Hence, each ring in $]R, A]$ is Artinian. Thus, $R \subset A$ is a closed minimal extension. Therefore, $A \subseteq \widetilde{R}^S$, which is impossible since $\widetilde{R}^S = R$. Thus, $[R, S] = \{R \subset \overline{R}_S \subset H \subset S\}$ is a chain of length 3 as desired. Now, if $[H, S]$ consists of the two chains $H \subset \overline{H}_S = \overline{R}_S \subset S$ and $H \subset \widetilde{H}^S \subset S$, we demonstrate that $[R, S] = \{R, H, \overline{R}_S, \widetilde{H}^S, S\}$ is a kite-graph of dimension 3 as drawn below.

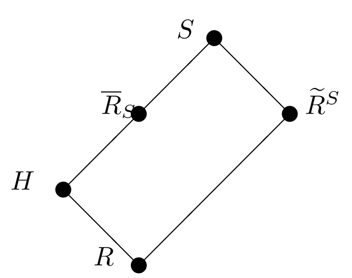


Indeed, let $A \in [R,S] \setminus \{R, H, \overline{R}_S, \widetilde{H}^S, S\}$. Then $A \not\supseteq \overline{R}_S$ since otherwise, $A \in [\overline{R}_S, S] = \{\overline{R}_S, S\}$, which is a contradiction. If moreover, $H \not\subseteq A$, then each ring in $]R, A]$ would be Artinian. Thus, $R \subset A$ would be a closed minimal extension. Hence, $A \subseteq \widetilde{R}^S$, which is impossible since $\widetilde{R}^S = R$. We conclude that $H \subset A$. Hence $A \in \{\overline{R}_S, \widetilde{H}^S, S\}$, which is a contradiction.

subsubcase 2.2.2. $R \subset S$ is not Prüfer-closed.

Firstly, note that as $R \subset \overline{R}_S$ and $R \subset H$ are integral extensions, then $\overline{R}_S, H \notin]R, \widetilde{R}^S]$. Thus, each ring in $]R, \widetilde{R}^S]$ is Artinian and so $R \subset \widetilde{R}^S$ is a closed minimal extension. Moreover, since $\widetilde{R}^S$ is Artinian, then $\widetilde{R}^S \subset S$ is an integral extension accordingly to Lemma 2.2.

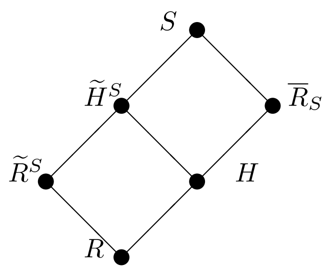
Suppose that $[H, S] = \{H \subset \overline{R}_S \subset S\}$, then we claim that $[R,S] = \{R, H, \overline{R}_S, \widetilde{R}^S, S\}$ is a pentagon-graph as drawn below.



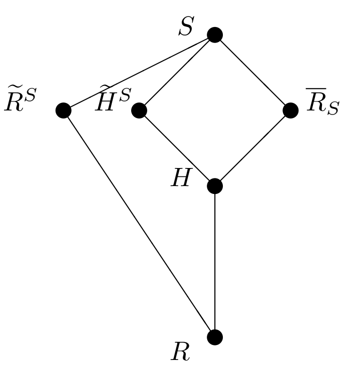
To this end, notice that $\widetilde{R}^S \subset S$ is a minimal extension. For, let $B \in [\widetilde{R}^S, S]$. Then $B \cap \overline{R}_S \in \{R, H, \overline{R}_S\}$. If $B \cap \overline{R}_S = R$, then $R \subset B$ is a Prüfer extension. So $B \subseteq \widetilde{R}^S$, which yields that $B = \widetilde{R}^S$. If $B \cap \overline{R}_S = \overline{R}_S$, then $B \in [\overline{R}_S, S] = \{\overline{R}_S, S\}$, which implies $B = S$. Finally, if $B \cap \overline{R}_S = H$, then $B \in [H, S]$. Hence, $B = S$. It follows that $\widetilde{R}^S \subset S$ is a minimal integral extension, as desired.

Next, we prove the above claim. Let $A \in [R,S] \setminus \{R, H, \overline{R}_S, \widetilde{R}^S, S\}$. If $H \subset A$, then $A \in]H, S] = \{\overline{R}_S, S\}$, which is impossible. If $\overline{R}_S \subset A$, then $A \in]\overline{R}_S, S] = \{S\}$, which is impossible. Thus each ring in $]R, A]$ is Artinian and so $R \subset A$ would be a closed minimal extension. This implies that $A \subseteq \widetilde{R}^S$ and so $A \in]R, \widetilde{R}^S] = \{\widetilde{R}^S\}$, which is a contradiction completing

the proof of our claim.
Assume now that $[H,S]$ consists of two chains of length 2, namely, $H\subset\overline{H}_S=\overline{R}_S\subset S$ and $H\subset\widetilde{H}^S\subset S$. Note that as $R\subset H$ is an integral extension and $R\subset\widetilde{R}^S$ is a Prüfer extension, then $\widetilde{R}^S\neq\widetilde{H}^S$. Now, we claim that $[R,S]=\{R,H,\overline{R}_S,\widetilde{R}^S,\widetilde{H}^S,S\}$ is either a rectangle-graph as drawn below:

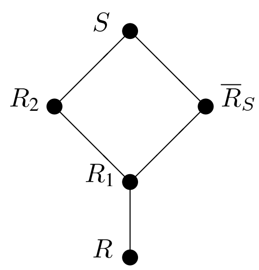


or a theta-graph as follows:

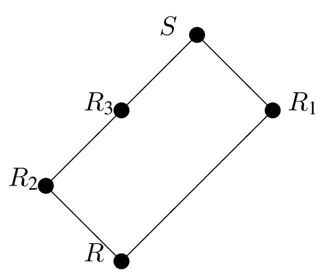


Indeed, assume that $\widetilde{R}^S$ and $\widetilde{H}^S$ are comparable under inclusion, then a fortiori we have $\widetilde{R}^S\subset\widetilde{H}^S$. Let $A\in[R,S]\setminus\{R,H,\overline{R}_S,\widetilde{R}^S,\widetilde{H}^S,S\}$. If $H\in]R,A]$, then $A\in[H,S]=\{H,\overline{R}_S,\widetilde{H}^S,S\}$, which is impossible. If $\overline{R}_S\in]R,A]$, then $A\in[\overline{R}_S,S]=\{\overline{R}_S,S\}$, which is impossible. Thus, each ring in $]R,A]$ is Artinian and hence $R\subset A$ is a closed minimal extension. It follows that $A\in]R,\widetilde{R}^S]$, which implies that $A=\widetilde{R}^S$, which is another contradiction. Therefore, $[R,S]=\{R,H,\overline{R}_S,\widetilde{R}^S,\widetilde{T}^S,S\}$ is a rectangle-graph as claimed.
Assume now that $\widetilde{R}^S$ and $\widetilde{H}^S$ are incomparable under inclusion. Then $R\subset\widetilde{R}^S\subset S$ is a chain of length 2. Thus, $[R,S]$ ordered by the usual set inclusion is a theta-graph.
(ii)⇒(i) Assume condition (a) satisfied. If $\overline{R}_S=S$, then by assumption $[R,S]=\{R\subset T\subset\overline{R}_S=S\}$ is a chain of length 2 such that S is not Artinian. An application of Lemma 2.1 shows that $[R,S]_{\text{non}-\mathcal{P}}=[R,S]$. If $\overline{R}_S\neq S$, then by assumption $[R,S]=\{R\subset\overline{R}_S\subset S\}$ is a chain of length 2 and S is not Artinian. It follows from [26, Theorem 2] that $\overline{R}_S$ is not Artinian. Another appeal to [26, Theorem 2] shows that R is not Artinian. Hence, $|[R,S]_{\text{non}-\mathcal{P}}|=3$. Assume now that condition (b) is satisfied. Then either $[R,S]=\{R\subset\overline{R}_S\subset T\subset S\}$ or

$[R, S] = \{R \subset T \subset \overline{R}_S \subset S\}$. In both cases, one can easily check that $\overline{R}_S \subset S$ is a Prüfer extension because $\overline{R}_S$ is integrally closed in S and $\overline{R}_S \subset S$ is a P-extension since it satisfies FIP. Thus, in the former case, it follows that $T \subset S$ is a closed minimal extension. As S is Artinian, then T cannot be Artinian. Thus, [26, Theorem 2] ensures that $\overline{R}_S$ and R are not Artinian. In the later case, $\overline{R}_S \subset S$ is a closed minimal extension. Thus, $\overline{R}_S$ cannot be Artinian since S is Artinian. Now, Lemma 2.1 infers that R and T are non-Artinian too. Hence, $|[R, S]_{\text{non}-\mathcal{P}}| = 3$. Assume condition (c) satisfied, then $[R, S]$ is as follows:



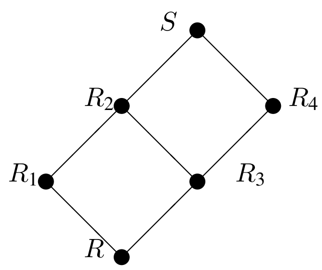
As R is not Artinian, it follows from Lemma 2.1 that R_1 and $\overline{R}_S$ are also non-Artinian. We need to show that R_2 is Artinian. Assume the contrary. Then $R_2 \subset S$ would be a closed minimal extension. Thus $R_1 \subset R_2$ would be integral. Hence $R_2 \subset \overline{R}_S$, which is a contradiction. It follows that $[R, S]_{\text{non}-\mathcal{P}} = \{R, R_1, \overline{R}_S\}$, as desired. Assume now condition (d) satisfied. Then $[R, S]$ looks like a pentagon-graph as follows:



It follows from Lemma 2.1 that $\overline{R}_S$ cannot be Artinian. Thus $\overline{R}_S \neq S$. Next, we claim that $R_1 \neq \overline{R}_S$. Indeed, assume the contrary. As R_3 and $\overline{R}_S$ are incomparable under inclusion, then $R_3 \cap \overline{R}_S = R$. Thus $R \subset R_3$ is an integrally closed extension. Hence, $R_3 \subseteq \widetilde{R}^S$. But $S \neq \widetilde{R}^S$, thus $R_3 = \widetilde{R}^S$. In this case, $R_3 \subset S$ is an integral extension. Hence, R_3 is an Artinian ring. As $R_2 \subset \widetilde{R}^S$ is a closed minimal extension and $\widetilde{R}^S$ is Artinian, then R_2 is not Artinian. It follows that R_2 is a maximal non-Artinian subring of S. Thus $R_2 \subset S$ should be a closed minimal extension according to [33, Theorem 1], which is a contradiction completing the proof of our claim. Next, we handle the following two cases.

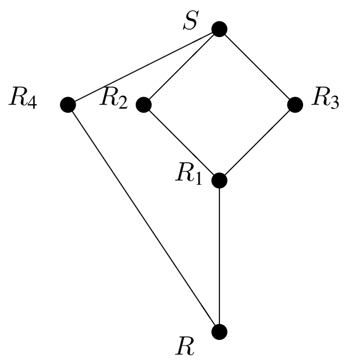
case 1. $R_2 = \overline{R}_S$.
In this case, $R_2 \subset S$ is a Prüfer extension. So $R_3 \subset S$ is a closed minimal extension. It follows that R_3 is not Artinian, since otherwise $R_3 \subset S$ would be integral. Now, as $R_1 \nsubseteq \overline{R}_S$, then $R \subset R_1$ is a closed minimal extension. It follows that $R_1 \subset S$ should be integral. Hence, R_1 is Artinian. We get $[R,S]_{\text{non}-\mathcal{P}} = \{R, \overline{R}_S, R_3\}$, as desired.
case 2. $R_3 = \overline{R}_S$.
As $R_1 \nsubseteq \overline{R}_S$, then $R \subset R_1$ is a closed minimal extension. Thus, $R_1 \subset S$ should be integral, which yields that R_1 is Artinian. Hence, $[R,S]_{\text{non}-\mathcal{P}} = \{R, R_2, \overline{R}_S\}$, as desired.

Assume now condition (e) satisfied. We start with the case where $[R,S]$ is as follows:



As $[R, \overline{R}_S]$ is a chain of length 2, then either $\overline{R}_S = R_2$ or $\overline{R}_S = R_4$. We claim that $\overline{R}_S = R_4$. Indeed, assume that $\overline{R}_S = R_2$. Then $R \subset R_1$ and $R \subset R_3$ would be integral extensions and this contradicts the fact that $R \neq \widetilde{R}^S$. It follows that $\overline{R}_S = R_4$. Thus, Lemma 2.1 ensures that R, R_3 and $\overline{R}_S$ are not Artinian. On the other hand, as $R \neq \widetilde{R}^S$, then either $\widetilde{R}^S = R_1$ or $\widetilde{R}^S = R_2$. If $\widetilde{R}^S = R_1$, then $R_1 \subset S$ would be an integral extension, and a fortiori R_1 and R_2 would be Artinian. Hence, $[R,S]_{\text{non}-\mathcal{P}} = \{R, R_3, \overline{R}_S\}$. If $\widetilde{R}^S = R_2$, then $R_2 \subset S$ is integral, so R_2 is Artinian. It follows that R_1 is also Artinian, since otherwise R_1 would be a maximal non-Artinian subring of S. So $R_1 \subset S$ would be a closed minimal extension by [33, Theorem 1], which is a contradiction. It follows that $[R,S]_{\text{non}-\mathcal{P}} = \{R, R_3, \overline{R}_S\}$, as desired.

Now, assume that $[R,S]$ looks like the following graph:



As $R \subset S$ is not Prüfer-closed, then a fortiori $\overline{R}_S \neq S$. Now, since $[R, \overline{R}_S]$ is a chain of length

2, then either $\overline{R}_S = R_2$ or $\overline{R}_S = R_3$. Without loss of generality, we can assume that $\overline{R}_S = R_2$. As $R \neq \widetilde{R}^S$ and $R \subset R_1$ is integral, then a fortiori $R_4 = \widetilde{R}^S$. Hence, $R_4 \subset S$ is integral. So R_4 is Artinian. We need to show that R_3 is also Artinian. The ring extension $R_1 \subset R_3$ is not integral since otherwise $R \subset R_3$ would be integral and so $R_3 \subset \overline{R}_S$, which is absurd. Thus, $R_1 \subset R_3$ is a closed minimal extension. Thus, $R_3 \subset S$ is integral. Hence, R_3 is Artinian. It follows that $[R, S]_{\text{non}-\mathcal{P}} = \{R, R_1, \overline{R}_S\}$, as desired. The proof of this theorem is complete. □

We close the paper with the following result.

Corollary 2.5. *Let $R \subset S$ be an extension of integral domains such that R is not integrally closed in S. Then the following statements are equivalent:*

(i) $|[R, S]_{\text{non}-\mathcal{P}}| = 3$.

(ii) (*Exactly*) *one of the following conditions holds true:*

a. $[R, S]$ *ordered by the usual set inclusion is a chain of length* 2 *and* S *is not a field.*

b. $S = \text{qf}(R)$ *and either* R' *is a rank 2 valuation domain and* $R \subset R'$ *is a minimal extension; or* R' *is a rank 1 valuation domain and* $[R, R']$ *is a chain of length 2.*

Proof. (i)⇒(ii) If S is not a field, it follows from Theorem 2.4 that $[R, S]$ ordered by the usual set inclusion is a chain of length 2. Assume now that S is a field. Then conditions (c), (d) and (e) in Theorem 2.4 cannot hold since an integral domain is Artinian if and only if it is a field. It remains only to discuss condition (b) of this theorem. According to Theorem 2.4, $[R, S]$ ordered by the usual set inclusion is a chain of length 3 such that S is a field and R is not a field. We have either $[R, S] = \{R \subset R' \subset T \subset S\}$ or $[R, S] = \{R \subset T \subset R' \subset S\}$. In the former case R' is a rank 2 valuation domain with quotient field S and in the latter case R' is a rank 1 valuation domain with quotient field S.
(ii)⇒(i) If $[R, S] = \{R \subset T \subset S\}$ is a chain of length 2 and S is not a field, then T cannot be a field by the Ferrand-Olivier classification of minimal extensions of a field (see [22, Théorème 2.2]). Again, R cannot be a field for the same reasons. Hence, $|[R, S]_{\text{non}-\mathcal{P}}| = 3$. If R' is a rank two valuation domain with quotient field S, then $[R', S] = \{R', V, S\}$, where V is the rank 1 valuation overring of R'. As $R \subset R'$ is a minimal extension, then [25, Theorem 2.4] ensures that $[R, S] = \{R, R', V, S\}$. Thus, $|[R, S]_{\text{non}-\mathcal{P}}| = 3$. Assume now that R' is a rank 1 valuation domain and $[R, R']$ is a chain of length 2. As $R \subset S$ satisfies FCP and each ring in $[R, R']$ is local since R' is local, then [16, Proposition 2.2] guarantees that $[R, S] = [R, R'] \cup [R', S]$. Hence, $|[R, S]_{\text{non}-\mathcal{P}}| = 3$. The proof is complete. □

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References

[1] N. Al-Kuleab, Integrally closed extensions with three non-Artinian intermediate rings, *Palest. J. Math.* **12** (2) (2023), 499-503.

[2] B. Al Subaiei, N. Jarboui, On the commutative ring extensions with at most two non-Prüfer intermediate rings, *Mediterr. J. Math.* **18** (4) (2021), 132, 19 pp.

[3] A. Ayache, M. Ben Nasr, O. Echi, N. Jarboui, Universally catenarian and going-down pairs of rings. *Math. Z.* **238** (4) (2001), 695-731.

[4] A. Ayache, A. Jaballah, Residually algebraic pairs of rings. *Math. Z.* **225** (1) (1997), 49–65.

[5] A. Ayache, N. Jarboui, Maximal non-Noetherian subring of a domain, *J. Algebra* **248** (2) (2002), 806–823.

[6] A. Ayache, N. Jarboui, An answer to a Dobbs conjecture about treed domains, *J. Algebra* **320** (10) (2008), 3720-3725.

[7] A. Ayache, N. Jarboui, E. Massaoud, Pairs of domains where all intermediate domains are treed, *Arab. J. Sci. Eng.* **36** (6) (2011), 933–946.

[8] M. J. Ben Abdallah, N. Jarboui, On universally catenarian pairs, *J. Pure Appl. Algebra* **212** (10) (2008), 2170-2175.

[9] M. Ben Nasr, O. Echi, L. Izelgue, N. Jarboui, Pairs of domains where all intermediate domains are Jaffard, *J. Pure Appl. Algebra* **145** (1) (2000), 1-18.

[10] M. Ben Nasr, N. Jarboui, Maximal non-Jaffard subrings of a field, *Publ. Math.* **44** (1) (2000), 157-175.

[11] M. Ben Nasr, N. Jarboui, Intermediate domains between a domain and some intersection of its localizations, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* **(8) 5** (3) (2002), 701-713.

[12] M. Ben Nasr, N. Jarboui, On maximal non-universally catenarian subrings, *J. Algebra Appl.* **7** (5) (2008), 553-556.

[13] M. Ben Nasr, N. Jarboui, On maximal non-valuation subrings. *Houston J. Math.*, **37** (1) (2011), 47-59.

[14] M. Ben Nasr, N. Jarboui, New results about normal pairs of rings with zero divisors, *Ric. Mat.* **63** (1) (2014), 149–155.

[15] M. Ben Nasr, N. Jarboui, A simple proof of the Ferrand-Olivier classification of the minimal ring extensions of a field, *Palest. J. Math.* **11** (4) (2022), 31-32.

[16] M. Ben Nasr, N. Zeidi, When is the integral closure comparable to all intermediate rings, *Bull. Aust. Math. Soc.* **95** (1) (2017), 14–21.

[17] E. D. Davis, Overrings of commutative rings, III: Normal pairs, *Trans. Amer. Math. Soc.* **182** (1973), 175-185.

[18] D. E. Dobbs, G. Picavet, M. Picavet-L'Hermitte, Characterizing the ring extensions that satisfy FIP or FCP, *J. Algebra* **371** (2012), 391-429.

[19] D. E. Dobbs, N. Jarboui, Prüfer-closed extensions and FCP λ-extensions of commutative rings, *Palest. J. Math.* **11** (3) (2022), 362-378.

[20] D. E. Dobbs, G. Picavet, M. Picavet-L'Hermitte, Transfer results for the FIP and FCP properties of ring extensions, *Comm. Algebra* **43** (2015), 1279-1316.

[21] D. E. Dobbs, J. Shapiro, Normal pairs with zero-divisors, *J. Algebra Appl.* **10** (2) (2011), 335-356.

[22] D. Ferrand, J.-P. Olivier, Homomorphismes minimaux d'anneaux, *J. Algebra* **16** (1970), 461-471.

[23] B. Gasmi, N. Jarboui, A note on intermediate rings between $D + I$ and $K[y_1]] \cdots [y_t]]$, *Palest. J. Math.* **7** (2) (2018), 414–416.

[24] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, (1972).

[25] R. Gilmer, W. Heinzer, Intersections of quotient rings of an integral domain, *J. Math. Kyoto Univ.* **7** (1967), 133-150.

[26] R. Gilmer, W. Heinzer, An application of Jonsson modules to some questions concerning proper subrings, *Math. Scand.* **70**(1) (1992), 34–42.

[27] R. Gilmer, W. Heinzer, Products of commutative rings and zero-dimensionality, *Trans. Amer. Math. Soc.* **331**(2) (1992), 663–680.

[28] R. Gilmer, J. Hoffmann, A characterization of Prüfer domains in terms of polynomials, *Pacific J. Math.* **60** (1) (1975), 81–85.

[29] A. Jaballah, Graph theoretic characterizations of maximal non-valuation subrings of a field, *Beitr. Algebra Geom.* **54** (1) (2013), 111–120.

[30] N. Jarboui, When is each proper overring of R an S(eidenberg)-domain? *Publ. Mat.* **46** (2) (2002), 435-440.

[31] N. Jarboui, Pairs of domains where most of the intermediate domains are Prüfer, *J. Algebra Appl.*, **20**(6) (2021), 2150101, 13 pp.

[32] N. Jarboui, S. Aljubran, Maximal non-integrally closed subrings of an integral domain, *Ric. Mat.* **71** (2) (2022), 325-332.

[33] N. Jarboui, N. Al-Kuleab, O. Almallah, Ring extensions with finitely many non-Artinian intermediate rings, *J. Math.* (2020), Art. ID 7416893, 6 pp.

[34] N. Jarboui, A. Jerbi, A note on maximal non-Noetherian subrings of a domain, *Beitr. Algebra Geom.* **53** (1) (2012), 159-172.

[35] N. Jarboui, M. Toumi, S. Trabelsi, Some questions concerning proper subrings, *Ric. Mat.* **64** (1) (2015), 51–55.

[36] N. Jarboui, S. Trabelsi, Some results about proper overrings of pseudo-valuation domains, *J. Algebra Appl.* **15** (5) (2016), 1650099, 16 pp.

[37] I. Kaplansky, *Commutative Rings*, rev. ed., Univ. Chicago Press, Chicago, (1974).

[38] M. Knebusch, D. Zhang, *Manis Valuations and Prüfer Extensions I*, Lecture Notes Math. 1791, Springer-Verlag, Berlin-Heidelberg, (2002).

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