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TWO-PARAMETER POWER CALCULUS

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Abstract In this paper, we introduce a new type of two-parameter power-calculus. The differentiation and inverse of differentiation in this calculus are investigated and various properties of these concepts are given. The fundamental theorem and formulas of integration by part regarding this calculus are also presented.

1 Introduction

Classically, the derivative of a function $f(x)$ of a variable x is denoted as

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}. \tag{1.1}$$

In 2002, Kac et al. published a work that reviewed two types of quantum calculus, the q-calculus and h-calculus. The definitions of q-derivative and h-derivative of $f(x)$ come with, if we do not take the limit in the derivative of a function f, and taking $y = qx$ and $y = x + h$, respectively. In [18], one can see more details about these two types of quantum calculus. Also for further information about the history of q-calculus, we refer to readers to [8].

In recent years, researchers have shown an increasing interest in q-calculus. A considerable amount of literature has been published on this subject. These studies have examined interesting applications in various mathematical areas such as fractional calculus, discrete function theory, umbral calculus, difference equations, summability etc. See [1, 2, 3, 4, 9, 10, 11, 19, 22, 24]. Also, as a special form of q-calculus, (p, q)-calculus and its applications have been very attractive, see [5, 6, 7, 12, 13, 15, 16, 17, 23].

In a similar manner, assuming not taking limit and also setting $y = x^p$, where p is not equal to 1, that is, by dealing with

$$\frac{f(x^p) - f(x)}{x^p - x}, \tag{1.2}$$

we come across a new type of quantum calculus, power calculus denoted by p-calculus, and also (1.2), as the definition of the p-derivative, is given. The formula (1.2) and some properties of it are defined and investigated in [20]. In [21, 25], authors bring forward some new properties of functions in p-calculus. In [14], a necessary optimality condition and a sufficient optimality condition for a p-variational problem were given with the concept of p-calculus.

Accordingly, the objective of the present work paper is to develop notation and terminology for two-parameter power calculus, namely the ^{pq}-calculus. By with appropriate choices, all results in this paper can be reduced in [20].

The main findings of this paper can be summarized as follows. In Section 2, ^{pq}-derivative is described and given its attributes. In Section 3, we present the ^{pq}- antiderivative, ^{pq}-integral and investigated their convergence. In the following section, we express definite the ^{pq}-integral and improper ^{pq}-integral. In the last section, the fundamental theorem of ^{pq}-calculus is proved and formulas of ^{pq}-integration by parts is obtained.

2 ^{p,q}-derivative

In this section, we give the concepts ^{p,q}-differential, ^{p,q}-derivative and investigate some of their properties. Throughout this paper, it has been assumed that p and q are a fixed number not equal to 1 and function $f(x)$ is defined on $[0, \infty)$.

Definition 2.1. The ^{p,q}-differential of the function f is denoted by

$$d_{pq} f(x) = f(x^p) - f(x^q). \tag{2.1}$$

In particular, $d_{p,q} x = x^p - x^q$. By using (2.1), it can be given ^{p,q}-derivative of a function.

Definition 2.2. The ^{p,q}-derivative of the function f is denoted by

$$\mathrm{D}_{\mathrm{p,q}}\, f(x) = \frac{f(x^p) - f(x^q)}{x^p - x^q}, \quad x \neq 0, 1, \tag{2.2}$$

and

$$\mathrm{D}_{\mathrm{p,q}}\, f(0) = \lim_{x \to 0^+} \mathrm{D}_{\mathrm{p,q}}\, f(x), \mathrm{D}_{\mathrm{p,q}}\, f(1) = \lim_{x \to 1} \mathrm{D}_{\mathrm{p,q}}\, f(x). \tag{2.3}$$

Definition 2.3. The nth order ^{p,q}-derivative of function $f(x)$ can be denoted by for $n \in \mathbb{N}$

$$\left(\mathrm{D}_{\mathrm{p,q}}{}^0 f\right)(x) = f(x), \left(\mathrm{D}_{\mathrm{p,q}}{}^n f\right)(x) = \mathrm{D}_{\mathrm{p,q}} \left(\mathrm{D}_{\mathrm{p,q}}{}^{n-1} f\right)(x). \tag{2.4}$$

Example 2.4. Set $f(x) = c$ (c a constant), $g(x) = x^n$, $n \in \mathbb{N}$ and $h(x) = \ln(x)$. Then, it can be obtained as

- $\mathrm{D}_{\mathrm{p,q}}\ f(x) = 0,$
- $\mathrm{D}_{\mathrm{p,q}}\ g(x) = \dfrac{g(x^p) - g(x^q)}{x^p - x^q} = \dfrac{x^{pn} - x^{qn}}{x^p - x^q} = \dfrac{x^{(p-1)n} - x^{(q-1)n}}{x^{p-1} - x^{q-1}} x^{n-1},$
- $\mathrm{D}_{\mathrm{p,q}}\ h(x) = \dfrac{h(x^p) - h(x^q)}{x^p - x^q} = \dfrac{(p-q)\ln(x)}{x^p - x^q} = \dfrac{(p-q)\ln(x)}{x^{p-1} - x^{q-1}} \dfrac{1}{x}.$

Also, we note that $\mathrm{D}_{\mathrm{p,q}}$ has the linearity property, that is for arbitrary two constants α, β and any two functions $f(x)$ and $g(x)$, we get

$$\mathrm{D}_{\mathrm{p,q}}(\alpha f(x) + \beta g(x)) = \alpha\, \mathrm{D}_{\mathrm{p,q}}\, f(x) + \beta\, \mathrm{D}_{\mathrm{p,q}}\, g(x). \tag{2.5}$$

Proposition 2.5. *^{p,q}-derivative has the following product rules*

$$\mathrm{D}_{\mathrm{p,q}}\,(f(x)g(x)) = g(x^p)\, \mathrm{D}_{\mathrm{p,q}}\, f(x) + f(x^q)\, \mathrm{D}_{\mathrm{p,q}}\, g(x), \tag{2.6}$$

$$\mathrm{D}_{\mathrm{p,q}}\,(f(x)g(x)) = g(x^q)\, \mathrm{D}_{\mathrm{p,q}}\, f(x) + f(x^p)\, \mathrm{D}_{\mathrm{p,q}}\, g(x). \tag{2.7}$$

Proof. On account of ^{p,q}-derivative (2.2), it can be seen that

$$\begin{aligned}\mathrm{D}_{\mathrm{p,q}}\,(f(x)g(x)) &= \frac{f(x^p)g(x^p) - f(x^q)g(x^q)}{x^p - x^q} \\ &= \frac{g(x^p)[f(x^p) - f(x^q)] + f(x^q)[g(x^p) - g(x^q)]}{x^p - x^q} \\ &= g(x^p)\, \mathrm{D}_{\mathrm{p,q}}\, f(x) + f(x^q)\, \mathrm{D}_{\mathrm{p,q}}\, g(x).\end{aligned}$$

This proves (2.6). Similarly, by symmetry, (2.7) can be deduced. □

Proposition 2.6. *^{p,q}-derivative has the following quotient rules*

$$\mathrm{D}_{\mathrm{p,q}} \left(\frac{f(x)}{g(x)}\right) = \frac{g(x^q)\, \mathrm{D}_{\mathrm{p,q}}\, f(x) - f(x^q)\, \mathrm{D}_{\mathrm{p,q}}\, g(x)}{g(x^q)g(x^p)}, \tag{2.8}$$

$$\mathrm{D}_{\mathrm{p,q}} \left(\frac{f(x)}{g(x)}\right) = \frac{g(x^p)\, \mathrm{D}_{\mathrm{p,q}}\, f(x) - f(x^p)\, \mathrm{D}_{\mathrm{p,q}}\, g(x)}{g(x^q)g(x^p)}. \tag{2.9}$$

Proof. The correctness of these rules can be obtained by considering (2.6) and (2.7) with functions $\dfrac{f(x)}{g(x)}$ and $g(x)$. □

3 ^{p,q}-integral

In this section, we formalize the notion of ^{p,q}-integral. From now on, it can be assumed that $0 < q < p < 1$.

Definition 3.1. An ^{p,q}-antiderivative of a function $f(x)$ is a function whose ^{p,q}-derivative is equal to $f(x)$. That is, if $\mathrm{D}_{p,q} F(x) = f(x)$, then the function $F(x)$ is a ^{p,q}-antiderivative of $f(x)$. It is denoted as

$$\int f(x) d_{p,q} x. \tag{3.1}$$

The following procedure is the key to constructing ^{p,q}-integral for an arbitrary function $f(x)$. For this, we need two operators which are $\widetilde{M_p}$, $\widetilde{M_q}$ defined by $\widetilde{M_p} f(x) := f(x^p)$, $\widetilde{M_q} f(x) := f(x^q)$. From the definition of ^{p,q}-derivative, by using operators $\widetilde{M_p}$ and $\widetilde{M_q}$, we have

$$\mathrm{D}_{p,q} F(x) = \frac{1}{x^p - x^q} \left(\widetilde{M_p} - \widetilde{M_q} \right) F(x) = f(x), \tag{3.2}$$

then by using geometric series expansion, we have

$$\begin{aligned} F(x) &= \frac{1}{\left(\widetilde{M_p} - \widetilde{M_q} \right)} ((x^p - x^q) f(x)) \\ &= \frac{1}{\widetilde{M_p} \left(1 - \frac{\widetilde{M_q}}{\widetilde{M_p}} \right)} ((x^p - x^q) f(x)) \\ &= \sum_{j=0}^{\infty} \left(\widetilde{M_p}^{-j-1} \widetilde{M_q^j} \right) ((x^p - x^q) f(x)) \\ &= \sum_{j=0}^{\infty} \left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}} \right) f\left(x^{q^j p^{-j-1}} \right). \end{aligned} \tag{3.3}$$

Definition 3.2. The ^{p,q}-integral of $f(x)$ is denoted by with serial expansion

$$\int f(x) d_{p,q} x = \sum_{j=0}^{\infty} \left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}} \right) f\left(x^{q^j p^{-j-1}} \right). \tag{3.4}$$

From this definition, it can be obtained a more general formula:

$$\begin{aligned} \int f(x) \mathrm{D}_{p,q} g(x) d_{p,q} x &= \sum_{j=0}^{\infty} \left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}} \right) f\left(x^{q^j p^{-j-1}} \right) \mathrm{D}_{p,q} g\left(x^{q^j p^{-j-1}} \right) \\ &= \sum_{j=0}^{\infty} \left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}} \right) f\left(x^{q^j p^{-j-1}} \right) \\ &\qquad \frac{g\left(\left(x^{q^j p^{-j-1}} \right)^p \right) - g\left(\left(x^{q^j p^{-j-1}} \right)^q \right)}{\left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}} \right)} \\ &= \sum_{j=0}^{\infty} f\left(x^{q^j p^{-j-1}} \right) \left(g\left(x^{q^j p^{-j}} \right) - g\left(x^{q^{j+1} p^{-j-1}} \right) \right) \end{aligned}$$

or otherwise stated

$$\int f(x) d_{p,q} g(x) = \sum_{j=0}^{\infty} f\left(x^{q^j p^{-j-1}} \right) \left(g\left(x^{q^j p^{-j}} \right) - g\left(x^{q^{j+1} p^{-j-1}} \right) \right). \tag{3.5}$$

It is noted that (3.4) is formal because the series does not always converge. The following theorem gives a sufficient condition for converge to a ^{p,q}-antiderivative.

Theorem 3.3. *Assuming that* $|f(x)x^{\alpha}|$ *is bounded on the interval* $(0, A]$ *for some* $0 \leq \alpha < 1$. *In this case, the* ^{p,q}*-integral* (3.4) *converges to a function* $H(x)$ *on* $(0, A]$, *which is a* ^{p,q}*-antiderivative of* $f(x)$. *In addition,* $H(x)$ *is continuous at the point* $x = 1$ *with* $H(1) = 0$.

Proof. For the correctness of the theorem, we deal with the following two cases.
Case 1. Let $x \in (1, A]$ be. By assuming that $|f(x)x^{\alpha}| < M$ on $(1, A]$, it can be obtain for $0 \leq j$,

$$\left|f\left(x^{q^j p^{-j-1}}\right)\right| < M\left(x^{q^j p^{-j-1}}\right)^{-\alpha} < M.$$

Thus for any $1 < x \leq A$, we have

$$\left|\left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}\right) f\left(x^{q^j p^{-j-1}}\right)\right| \leq \left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}\right) M.$$

Since $1 - \alpha > 0$ and $0 < q < p < 1$,

$$\sum_{j=0}^{\infty}\left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}\right) M \;=\; M(x-1),$$

it can be seen that the series, by a comparison test, ^{p,q}-integral converges to a function $F(x)$. By (3.3), we can obtain that $F(1) = 0$.

For $1 < x \leq A$,

$$|F(x)| = \left|\sum_{j=0}^{\infty}\left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}\right) f\left(x^{q^j p^{-j-1}}\right)\right| \leq M(x-1),$$

which tends to 0 when $x \to 1^{+}$. Because of $F(1) = 0$, thereby F is right continuous at the points $x = 1$.

Verifying that $F(x)$ is a ^{p,q}-antiderivative of $f(x)$, we ^{p,q}-differentiate it:

$$\begin{aligned}
\mathrm{D}_{p,q} F(x) &= \frac{F(x^p) - F(x^q)}{x^p - x^q} \\
&= \frac{\sum_{j=0}^{\infty}\left(x^{q^j p^{1-j}} - x^{p^{-j} q^{j+1}}\right) f\left(x^{q^j p^{-j}}\right)}{x^p - x^q} \\
&\quad - \frac{\sum_{j=0}^{\infty}\left(x^{q^{j+1} p^{-j}} - x^{p^{-j-1} q^{j+2}}\right) f\left(x^{q^{j+1} p^{-j-1}}\right)}{x^p - x^q} \\
&= \frac{\sum_{j=0}^{\infty}\left(x^{q^j p^{1-j}} - x^{p^{-j} q^{j+1}}\right) f\left(x^{q^j p^{-j}}\right)}{x^p - x^q} \\
&\quad - \frac{\sum_{j=1}^{\infty}\left(x^{q^j p^{-j+1}} - x^{p^{-j} q^{j+1}}\right) f\left(x^{q^j p^{-j}}\right)}{x^p - x^q} \\
&= f(x).
\end{aligned}$$

Case 2. Let $x \in (0, 1)$ be. By assuming $|f(x)x^{\alpha}| < M$ on $(0, 1)$, it can be obtained for $0 \leq j$,

$$\left|f\left(x^{q^j p^{-j-1}}\right)\right| < M\left(x^{q^j p^{-j-1}}\right)^{-\alpha} \leq M x^{-\alpha}.$$

Hence for any $0 < x < 1$, we have

$$\left|\left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}\right) f\left(x^{q^j p^{-j-1}}\right)\right| \leq \left(x^{q^{j+1} p^{-j-1}} - x^{q^j p^{-j}}\right) M x^{-\alpha}.$$

Since $1-\alpha>0$ and $0<q<p<1$,

$$\sum_{j=0}^{\infty}\left(x^{q^{j+1}p^{-j-1}}-x^{q^jp^{-j}}\right)Mx^{-\alpha}=Mx^{-\alpha}(1-x),$$

hence, by comparison test ^{p,q}-integral converges to a function $G(x)$ and by (3.3) it shows directly that $G(1)=0$.

Similar to Case 1, it can be seen that G is left continuous at $x=1$ and as well as is ^{p,q}-antiderivative of $f(x)$.

Bringing together the above two cases, it is now possible to define a new function $H(x)$ by

$$H(x)=G(x)\chi_{(0,1)}(x)+F(x)\chi_{(1,A]}(x),$$

where χ_I is characteristic function on I.

As can be easily seen that, ^{p,q}-integral converges to $H(x)$ on $(0,A]$ and it is also worth noting that $H(x)$ is ^{p,q}-antiderivative of $f(x)$ on $(0,1)\cup(1,A]$ and is continuous at the point $x=1$ with $H(1)=0$.

Assuming that $f(x)$ continuous at the point $x=1$, we have $\mathrm{D}_{p,q}\,H(1)=\lim\limits_{x\to1}H(x)=f(1)$ and we conclude that $H(x)$ is ^{p,q}-antiderivative of $f(x)$ on $(0,A]$, is the desired result. □

Remark 3.4. It has been demonstrated that if it satisfies the assumptions of Theorem 3.3, then the ^{p,q}-integral has the unique ^{p,q}-antiderivative which is continuous at the point $x=1$, up to a constant. In contrast, assuming that $F(x)$ is a ^{p,q}-antiderivative of $f(x)$ and $F(x)$ is continuous at the points $x=1$, $F(x)$ can be given, up to a constant, by the aid of (3.4), since

$$\begin{aligned}
&\sum_{j=0}^{N}\left(x^{q^jp^{-j}}-x^{q^{j+1}p^{-j-1}}\right)f\left(x^{q^jp^{-j-1}}\right)\\
&\quad=\sum_{j=0}^{N}\left(x^{q^jp^{-j}}-x^{q^{j+1}p^{-j-1}}\right)\mathrm{D}_{p,q}\,F(t)\big|_{t=x^{q^jp^{-j-1}}}\\
&\quad=\sum_{j=0}^{N}\left(x^{q^jp^{-j}}-x^{q^{j+1}p^{-j-1}}\right)\left(\frac{F\left(x^{q^jp^{-j}}\right)-F\left(x^{q^{j+1}p^{-j-1}}\right)}{\left(x^{q^jp^{-j}}-x^{q^{j+1}p^{-j-1}}\right)}\right)\\
&\quad=\sum_{j=0}^{N}\left(F\left(x^{q^jp^{-j}}\right)-F\left(x^{q^{j+1}p^{-j-1}}\right)\right)\\
&\quad=F(x)-F\left(x^{q^{N+1}p^{-N-1}}\right),
\end{aligned}$$

which approaches to $F(x)-F(1)$ as N tends to ∞, by the property of continuity of $F(x)$ at the point $x=1$.

In this theorem, the boundedness of the function can not be removed. In fact, let $f(x)=\dfrac{1}{x^p-x^q}$ be. By using definition of ^{p,q}-integral (3.4), one can get

$$\begin{aligned}
\int f(x)d_{p,q}x&=\sum_{j=0}^{\infty}\left(x^{q^jp^{-j}}-x^{q^{j+1}p^{-j-1}}\right)f\left(x^{q^jp^{-j-1}}\right)\\
&=\sum_{j=0}^{\infty}\left(x^{q^jp^{-j}}-x^{q^{j+1}p^{-j-1}}\right)\frac{1}{\left(x^{q^jp^{-j}}-x^{q^{j+1}p^{-j-1}}\right)}=\infty.
\end{aligned}$$

Hence, we note that the theorem does not work because of the unboundedness of $f(x)x^{\alpha}$ for any $0\le\alpha<1$ on $(0,1)\cup(1,A]$.

4 The definite ^{p,q}-integral

In this section, we are defining the definite ^{p,q}-integral, by using formula (3.4). As in proving Theorem 3.3, we need to consider the following three cases, to gain our aim:

Case 1. Suppose that $1 < a < b$. Let function f exist on $(1, b]$.

Definition 4.1. The definite ^{p,q}-integral of $f(x)$ on $(1, b]$ is denoted by

$$\int_1^b f(x)d_{p,q}x = \lim_{N\to\infty}\sum_{j=0}^{N}\left(b^{q^j p^{-j}} - b^{q^{j+1}p^{-j-1}}\right) f\left(b^{q^j p^{-j-1}}\right) \tag{4.1}$$

and

$$\int_a^b f(x)d_{p,q}x = \int_1^b f(x)d_{p,q}x - \int_1^a f(x)d_{p,q}x. \tag{4.2}$$

Example 4.2. Set $b = 4$ and assume that $f(x) = c$ (c a constant), then we have

$$\begin{aligned}\int_1^4 c d_{p,q}x &= \lim_{N\to\infty}\sum_{j=0}^{N}\left(4^{q^j p^{-j}} - 4^{q^{j+1}p^{-j-1}}\right) f\left(4^{q^j p^{-j-1}}\right)\\ &= c\lim_{N\to\infty}\left[\left(4 - 4^{qp^{-1}}\right) + \left(4^{qp^{-1}} - 4^{q^2p^{-2}}\right) + \ldots + \left(4^{q^N p^{-N}} - 4^{q^{N+1}p^{-N-1}}\right)\right]\\ &= c\lim_{N\to\infty}\left[4 - 4^{q^{N+1}p^{-N-1}}\right] = c(4-1) = 3c,\end{aligned}$$

and if $a = 3$,

$$\int_3^4 c\, d_{p,q}x = \int_1^4 c\, d_{p,q}x - \int_1^3 c\, d_{p,q}x = 3c - 2c = c.$$

Example 4.3. Set $b = 3$ and assume that $f(x) = \dfrac{ln(x)}{x^p - x^q}$, then we have

$$\int_1^3 f(x)d_{p,q}x = \sum_{j=0}^{\infty}\left(3^{q^j p^{-j}} - 3^{q^{j+1}p^{-j-1}}\right)\frac{\ln(3^{q^j p^{-j}})}{3^{q^j p^{-j}} - 3^{q^{j+1}p^{-j-1}}} = \sum_{j=0}^{\infty} q^j p^{-j}\ln 3 = \frac{p\ln 3}{p-q}.$$

Case 2. Suppose that $0 < a < b < 1$. Let function f exist on $[b, 1)$.

Definition 4.4. The definite ^{p,q}-integral of $f(x)$ on $[b, 1)$ is given by

$$\begin{aligned}\int_b^1 f(x)d_{p,q}x &= \lim_{N\to\infty}\sum_{j=0}^{N}\left(b^{q^{j+1}p^{-j-1}} - b^{q^j p^{-j}}\right) f\left(b^{q^j p^{-j-1}}\right)\\ &= \sum_{j=0}^{\infty}\left(b^{q^{j+1}p^{-j-1}} - b^{q^j p^{-j}}\right) f\left(b^{q^j p^{-j-1}}\right).\end{aligned} \tag{4.3}$$

Example 4.5. Set $b = \frac{1}{3}$ and assume that $f(x) = c$ (c a constant), then we have

$$\int_{\frac{1}{3}}^{1} c d_{p,q} x = \lim_{N\to\infty} \sum_{j=0}^{N} \left(\left(\frac{1}{3}\right)^{q^{j+1}p^{-j-1}} - \left(\frac{1}{3}\right)^{q^j p^{-j}} \right) c$$

$$= c \lim_{N\to\infty} \Bigg[\left(\left(\frac{1}{3}\right)^{qp^{-1}} - \frac{1}{3} \right) + \left(\left(\frac{1}{3}\right)^{q^2p^{-2}} - \left(\frac{1}{3}\right)^{qp^{-1}} \right) + \dots$$

$$\dots + \left(\left(\frac{1}{3}\right)^{q^{N+1}p^{-N-1}} - \left(\frac{1}{3}\right)^{q^N p^{-N}} \right) \Bigg]$$

$$= c \lim_{N\to\infty} \left[-\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^{q^{N+1}p^{-N-1}} \right] = c\left(-\frac{1}{3} + 1\right)) = \frac{2c}{3}.$$

Remark 4.6. The definite ^{p,q}-integrals (4.1) and (4.3) are also given as

$$\int_{1}^{b} f(x) d_{p,q} x = \mathrm{I}^{+}_{\mathrm{p,q}} f(b), \tag{4.4}$$

$$\int_{b}^{1} f(x) d_{p,q} x = \mathrm{I}^{-}_{\mathrm{p,q}} f(b). \tag{4.5}$$

Case 3. Suppose that $0 < a < b < 1$. Let function f exist on $(0, b]$.

Definition 4.7. The definite ^{p,q}-integral of $f(x)$ on $(0, b]$ is given by

$$\mathrm{I}_{\mathrm{p,q}} f(b) = \int_{0}^{b} f(x) d_{p,q} x$$

$$= \lim_{N\to\infty} \sum_{j=0}^{N} \left(b^{q^{-j}p^{j}} - b^{q^{-j-1}p^{j+1}} \right) f\left(b^{q^{-j}p^{j+1}} \right)$$

$$= \sum_{j=0}^{\infty} \left(b^{q^{-j}p^{j}} - b^{q^{-j-1}p^{j+1}} \right) f\left(b^{q^{-j}p^{j+1}} \right) \tag{4.6}$$

and

$$\int_{a}^{b} f(x) d_{p,q} x = \int_{0}^{b} f(x) d_{p,q} x - \int_{0}^{a} f(x) d_{p,q} x. \tag{4.7}$$

Example 4.8. Set $a = \frac{1}{4}, b = \frac{1}{2}$ and assume that $f(x) = c$ (c a constant), then we have

$$\int_{0}^{\frac{1}{2}} c d_{p,q} x = \lim_{N\to\infty} \sum_{j=0}^{N} \left(\left(\frac{1}{2}\right)^{q^{-j}p^{j+1}} - \left(\frac{1}{2}\right)^{q^{-j}p^{j+1}} \right) c$$

$$= c \lim_{N\to\infty} \Bigg[\left(\frac{1}{2} - \left(\frac{1}{2}\right)^{q^{-1}p^{1}} \right) + \left(\left(\frac{1}{2}\right)^{q^{-1}p^{1}} - \left(\frac{1}{2}\right)^{q^{-2}p^{2}} \right) +$$

$$\dots + \left(\left(\frac{1}{2}\right)^{q^{-N}p^{N}} - \left(\frac{1}{2}\right)^{q^{-N-1}p^{N+1}} \right) \Bigg]$$

$$= c \lim_{N\to\infty} \left[\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{q^{-N-1}p^{N+1}} \right] = \frac{c}{2}.$$

In a similar way,

$$\int_0^{\frac{1}{4}} c\, d_{p,q}x = \frac{c}{4},$$

thus we have

$$\int_{\frac{1}{4}}^{\frac{1}{2}} c\, d_{p,q}x = \int_0^{\frac{1}{2}} c\, d_{p,q}x - \int_0^{\frac{1}{4}} c\, d_{p,q}x = \frac{c}{4}.$$

Definition 4.9. Assuming that $0 \le a < 1 < b$, the definite ^{p,q}-integral of $f(x)$ is given by

$$\int_a^b f(x)d_{p,q}x = \int_a^1 f(x)d_{p,q}x + \int_1^b f(x)d_{p,q}x. \tag{4.8}$$

Corollary 4.10. *Definitions of* ^{p,q}*-integrals and* (3.5) *imply a more general formula:*

i.) If $b > 1$*, then we note that*

$$\int_1^b f(x)d_{p,q}g(x) = \sum_{j=0}^{\infty} f\left(b^{q^j p^{-j-1}}\right)\left(g\left(b^{q^j p^{-j}}\right) - g\left(b^{q^{j+1}p^{-j-1}}\right)\right).$$

ii.) If $0 < b < 1$*, then we note that*

$$\int_0^b f(x)d_{p,q}g(x) = \sum_{j=0}^{\infty} f\left(b^{q^{-j}p^{j+1}}\right)\left(g\left(b^{q^{-j}p^{j}}\right) - g\left(b^{q^{-j-1}p^{j+1}}\right)\right).$$

Definition 4.11. The nth order ^{p,q}-integral of a function f can be denoted by

$$\left(\mathrm{I}^0_{\mathrm{p,q}} f\right)(x) = f(x), \quad \left(\mathrm{I}^{\mathrm{n}}_{\mathrm{p,q}} f\right)(x) = \mathrm{I}_{\mathrm{p,q}}\left(\mathrm{I}^{\mathrm{n\text{-}1}}_{\mathrm{p,q}} f\right)(x), \quad n \in \mathbb{N}.$$

5 Improper ^{p,q}-integral

In this section, we need to take a look at the improper ^{p,q}-integral of the function $f(x)$ and give a sufficient condition for its convergence.

Let $0 < q < p < 1$, thus $\left(\frac{q}{p}\right)^{-1} > 1$ and set $b = \left(\frac{q}{p}\right)^{-1}$. For any $j \in 0, \pm 1, \pm 2, \ldots$, we have $b^{\frac{q^j}{p^j}} > 1, \quad b^{\frac{q^{j+1}}{p^{j+1}}} < b^{\frac{q^j}{p^j}}$. Thus and so, on account of Equation (4.2), we first examine the integral

$$\begin{aligned}
\int_{b^{q^{j+1}p^{-j-1}}}^{b^{q^j p^{-j}}} f(x)d_{p,q}x &= \int_1^{b^{q^j p^{-j}}} f(x)d_{p,q}x - \int_1^{b^{q^{j+1}p^{-j-1}}} f(x)d_{p,q}x \\
&= \sum_{k=0}^{\infty}\left(\left(b^{q^j p^{-j}}\right)^{q^k p^{-k}} - \left(b^{q^j p^{-j}}\right)^{q^{k+1}p^{-k-1}}\right) f\left(\left(b^{q^j p^{-j}}\right)^{q^k p^{-k-1}}\right) \\
&\quad - \sum_{k=0}^{\infty}\left(\left(b^{q^{j+1}p^{-j-1}}\right)^{q^k p^{-k}} - \left(b^{q^{j+1}p^{-j-1}}\right)^{q^{k+1}p^{-k-1}}\right) f\left(\left(b^{q^{j+1}p^{-j-1}}\right)^{q^k p^{-k-1}}\right) \\
&= \sum_{k=0}^{\infty}\left(b^{q^{j+k}p^{-j-k}} - b^{q^{j+k+1}p^{-j-k-1}}\right) f\left(b^{q^{j+k}p^{-j-k-1}}\right) \\
&\quad - \sum_{k=0}^{\infty}\left(b^{q^{j+k+1}p^{-j-k-1}} - b^{q^{j+k+2}p^{-j-k-2}}\right) f\left(b^{q^{j+k+1}p^{-j-k-2}}\right),
\end{aligned}$$

and as follows, implies that

$$\int_{b^{q^{j+1}p^{-j-1}}}^{b^{q^{j}p^{-j}}} f(x)d_{p,q}x = \left(b^{q^{j}p^{-j}} - b^{q^{j+1}p^{-j-1}}\right) f\left(b^{q^{j}p^{-j-1}}\right).$$

Definition 5.1. Suppose that $0 < q < p < 1$ and set $b = \left(\frac{q}{p}\right)^{-1}$. The improper ^{p,q}-integral of $f(x)$ on $[1,\infty)$ is given by

$$\begin{aligned}\int_{1}^{\infty} f(x)d_{p,q}x &= \sum_{j=-\infty}^{\infty} \int_{b^{q^{j+1}p^{-j-1}}}^{b^{q^{j}p^{-j}}} f(x)d_{p,q}x \\ &= \sum_{j=-\infty}^{\infty} \left(b^{q^{j}p^{-j}} - b^{q^{j+1}p^{-j-1}}\right) f\left(b^{q^{j}p^{-j-1}}\right) \\ &= \sum_{j=0}^{\infty} \left(b^{q^{j}p^{-j}} - b^{q^{j+1}p^{-j-1}}\right) f\left(b^{q^{j}p^{-j-1}}\right) \\ &\quad + \sum_{j=1}^{\infty} \left(b^{q^{-j}p^{j}} - b^{q^{-j-1}p^{j+1}}\right) f\left(b^{q^{-j}p^{j+1}}\right). \end{aligned} \tag{5.1}$$

Definition 5.2. Assuming that $0 < q < p < 1$, for any $j \in 0, \pm 1, \pm 2, \ldots$, we observe that $p^{q^{j}p^{-j}} \in (0,1)$, $\quad p^{q^{j}p^{-j}} < p^{q^{j+1}p^{-j-1}}$ and

$$\int_{0}^{1} f(x)d_{p,q}x = \sum_{j=-\infty}^{\infty} \left(p^{q^{j+1}p^{-j-1}} - p^{q^{j}p^{-j}}\right) f\left(p^{q^{-j}p^{j+1}}\right).$$

Considering (4.7), we have

$$\begin{aligned}\int_{p^{q^{j}p^{-j}}}^{p^{q^{j+1}p^{-j-1}}} f(x)d_{p,q}x &= \int_{0}^{p^{q^{j+1}p^{-j-1}}} f(x)d_{p,q}x - \int_{0}^{p^{q^{j}p^{-j}}} f(x)d_{p,q}x \\ &= \left(p^{q^{j+1}p^{-j-1}} - p^{q^{j}p^{-j}}\right) f\left(p^{q^{-j}p^{j+1}}\right). \end{aligned}$$

It follows that,

$$\begin{aligned}\int_{0}^{1} f(x)d_{p,q}x &= \sum_{j=-\infty}^{\infty} \int_{p^{q^{j}p^{-j}}}^{p^{q^{j+1}p^{-j-1}}} f(x)d_{p,q}x \\ &= \sum_{j=-\infty}^{\infty} \left(p^{q^{j+1}p^{-j-1}} - p^{q^{j}p^{-j}}\right) f\left(p^{q^{-j}p^{j+1}}\right). \end{aligned}$$

Definition 5.3. Suppose that $0 < q < p < 1$. In this case, the improper ^{p,q}-integral of $f(x)$ on $[0,\infty)$ can be denoted by

$$\int_{0}^{\infty} f(x)d_{p,q}x = \int_{0}^{1} f(x)d_{p,q}x + \int_{1}^{\infty} f(x)d_{p,q}x.$$

Definition 5.4. Suppose that $0 < q < p < 1$. In this case, the improper ^{p,q}-integral of $f(x)$ on $[a,\infty)$ can be denoted by i.) Assuming that $0 < a < 1$, we have

$$\int_a^\infty f(x)d_{p,q}x = \int_a^1 f(x)d_{p,q}x + \int_1^\infty f(x)d_{p,q}x.$$

ii.) Assuming that $a > 1$, we have

$$\int_a^\infty f(x)d_{p,q}x = \lim_{N\to\infty}\sum_{j=1}^{N} \int_{a^{q^{-j+1}p^{j-1}}}^{a^{q^{-j}p^j}} f(x)d_{p,q}x.$$

Now, we wish to investigate about the convergence of the improper ^{p,q}-integral.

Proposition 5.5. *The improper ^{p,q}-integral of $f(x)$ defined above converges on $[1,\infty)$ if f satisfies that for $0 < r < \infty$*

$$|f(x)| < \min\left\{rx^{\alpha}, |x^{p^{-1}} - x^{q^{-1}}|^{-1}(\ln x)^{2\alpha}\right\},$$

in neighborhood of the point $x = 1$ with some $0 \le \alpha < 1$ and for sufficiently large x with some $-\varepsilon \le \alpha < 0$ where ε is a small positive number.

Proof. Consider $b = \left(\frac{q}{p}\right)^{-1}$. On account of Definition 5.1, the proof falls naturally into two parts:

$$\int_1^\infty f(x)d_{p,q}x = \sum_{j=0}^{\infty}\left(b^{q^jp^{-j}} - b^{q^{j+1}p^{-j-1}}\right) f\left(b^{q^jp^{-j-1}}\right) + \sum_{j=1}^{\infty}\left(b^{q^{-j}p^j} - b^{q^{-j-1}p^{j+1}}\right) f\left(b^{q^{-j}p^{j+1}}\right).$$

Under the conditions stated above and also by Theorem 3.3, the first sum converges. For the second sum, assume that for large x, we have $|f(x)| < |x^{p^{-1}} - x^{q^{-1}}|^{-1}(\ln x)^{2\alpha}$, where $-\varepsilon \le \alpha < 0$. Then, for all sufficiently large j, we get

$$\left|f\left(b^{q^{-j}p^{j+1}}\right)\right| < \left(b^{q^{-j}p^j} - b^{q^{-j-1}p^{j+1}}\right)^{-1}\left(\ln b^{q^{-j}p^{j+1}}\right)^{2\alpha}.$$

As follows,

$$\begin{aligned}\left|\left(b^{q^{-j}p^j} - b^{q^{-j-1}p^{j+1}}\right) f\left(b^{q^{-j}p^{j+1}}\right)\right| &\le \left(b^{q^{-j}p^j} - b^{q^{-j-1}p^{j+1}}\right)\left(b^{q^{-j}p^j} - b^{q^{-j-1}p^{j+1}}\right)^{-1}\left(\ln b^{q^{-j}p^{j+1}}\right)^{2\alpha}\\ &= \left(\ln b^{q^{-j}p^{j+1}}\right)^{2\alpha} = (\ln b)^{2\alpha}\left(p^{2\alpha}\right)^{j+1}\left(q^{-2\alpha}\right)^{j}.\end{aligned}$$

Therefore, the second sum is also majorized by a convergent geometric series, and thus it converges. □

6 Fundamental Theorem of ^{p,q}-Calculus

In this section, we give the fundamental theorem of ^{p,q}-calculus which explains the relation between ^{p,q}-derivative and ^{p,q}-integral. This relation will be proved once we prove some lemmas below.

Lemma 6.1. *Assume that $x > 1$. Then $\mathrm{D}_{p,q}\,\mathrm{I}^{+}_{p,q}\, f(x) = f(x)$ and also $\mathrm{I}^{+}_{p,q}\,\mathrm{D}_{p,q}\, f(x) = f(x) - f(1)$ holds if function f is continuous at $x = 1$.*

Proof. Since by concept of ^{p,q}-integral (3.4), one can get

$$\mathrm{I}^{+}_{p,q}\, f(x) = \int_1^x f(s)d_{p,q}s = \sum_{j=0}^{\infty}\left(x^{q^jp^{-j}} - x^{q^{j+1}p^{-j-1}}\right) f\left(x^{q^jp^{-j-1}}\right).$$

Hence, by concept of ^{p,q}-derivative (2.2), we have

$$\begin{aligned}
\mathrm{D}_{p,q}\, \mathrm{I}^{+}_{p,q} f(x) &= \frac{\mathrm{I}^{+}_{p,q} f(x^p) - \mathrm{I}^{+}_{p,q} f(x^q)}{x^p - x^q} \\
&= \frac{\sum_{j=0}^{\infty}\left(x^{q^j p^{-j+1}} - x^{q^{j+1} p^{-j}}\right) f\left(x^{q^j p^{-j}}\right) - \sum_{j=0}^{\infty}\left(x^{q^{j+1} p^{-j}} - x^{q^{j+2} p^{-j-1}}\right) f\left(x^{q^{j+1} p^{-j-1}}\right)}{x^p - x^q} \\
&= \frac{\left[(x^p - x^q) f(x) + (x^q - x^{q^2 p^{-1}}) f(x^{qp^{-1}}) + (x^{q^2 p^{-1}} - x^{q^3 p^{-2}}) + \dots\right]}{x^p - x^q} \\
&\quad - \frac{\left[(x^q - x^{q^2 p^{-1}}) f(x^{qp^{-1}}) + (x^{q^2 p^{-1}} - x^{q^3 p^{-2}}) + \dots\right]}{x^p - x^q} \\
&= \frac{(x^p - x^q) f(x)}{x^p - x^q} = f(x).
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
\mathrm{I}^{+}_{p,q}\, \mathrm{D}_{p,q} f(x) &= \lim_{N\to\infty} \sum_{j=0}^{N} \left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}\right) \mathrm{D}_{p,q} f\left(x^{q^j p^{-j-1}}\right) \\
&= \lim_{N\to\infty} \sum_{j=0}^{N} \left(x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}\right) \frac{f(x^{q^j p^{-j}}) - f(x^{q^{j+1} p^{-j-1}})}{x^{q^j p^{-j}} - x^{q^{j+1} p^{-j-1}}} \\
&= \lim_{N\to\infty} \sum_{j=0}^{N} \left[f(x^{q^j p^{-j}}) - f(x^{q^{j+1} p^{-j-1}})\right] \\
&= \lim_{N\to\infty} \left[f(x) - f(x^{qp^{-1}}) + f(x^{qp^{-1}}) - f(x^{q^2 p^{-2}}) + \dots + f(x^{q^N p^{-N}}) - f(x^{q^{N+1} p^{-N-1}})\right] \\
&= \lim_{N\to\infty} \left[f(x) - f(x^{q^{N+1} p^{-N-1}})\right] \\
&= f(x) - f(1).
\end{aligned}$$

We noted that the last equality holds because of the continuity of f at $x = 1$.

□

By a similar argument, the following lemmas can be easily obtained.

Lemma 6.2. *Assume that $x \in (0, 1)$. In this case,* $\mathrm{D}_{p,q}\, \mathrm{I}^{-}_{p,q} f(x) = -f(x)$*, and also* $\mathrm{I}^{-}_{p,q}\, \mathrm{D}_{p,q} f(x) = f(1) - f(x)$ *holds if function f is continuous at $x = 1$.*

Lemma 6.3. *Assume that $x \in (0, 1)$. In this case,* $\mathrm{I}_{p,q}$ *is defined by*

$$\mathrm{I}_{p,q} f(x) = \int_0^x f(s)\, d_{p,q} s,$$

then $\mathrm{D}_{p,q}\, \mathrm{I}_{p,q} f(x) = f(x)$*. Also* $\mathrm{I}_{p,q}\, \mathrm{D}_{p,q} f(x) = f(x) - f(0)$ *holds if function f is continuous at $x = 0$.*

The following theorem is a reformulation of the fundamental theorem of ordinary calculus, for ^{p,q}-calculus.

Theorem 6.4. *Assume that function $F(x)$ is an antiderivative of $f(x)$. If $F(x)$ is continuous at points $x = 0$ and $x = 1$, then one can obtain for $0 \le a < b \le \infty$*

$$\int_a^b f(x)\, d_{p,q} x = F(b) - F(a). \tag{6.1}$$

Proof. The proof falls naturally into three cases:

Case 1. Assume that a, b are finite and $1 < a < b$. By assumptions, we have $\mathrm{D}_{\mathrm{p,q}} F(x) = f(x)$. On account of Lemma 6.1, we obtain

$$F(x) - F(1) = \mathrm{I}^{+}_{\mathrm{pq}} f(x) = \int_1^x f(s) d_{p,q} s,$$

which yields

$$\int_1^a f(s) d_{p,q} s = F(a) - F(1) \text{ and } \int_1^b f(s) d_{p,q} s = F(b) - F(1).$$

By (4.2), thus we deduce that

$$\int_a^b f(x) d_{p,q} x = F(b) - F(a).$$

Case 2. Assume that $0 < a < b < 1$. On account of assumptions and Lemma 6.3, we have

$$F(x) - F(0) = \mathrm{I}_{\mathrm{p,q}} f(x) = \int_0^x f(s) d_{p,q} s,$$

which yields

$$\int_0^a f(s) d_{p,q} s = F(a) - F(0), \quad \int_0^b f(s) d_{p,q} s = F(b) - F(0).$$

By (4.7), thus we deduce that

$$\int_a^b f(x) d_{p,q} x = F(b) - F(a).$$

Case 3. Assume that b is finite and $0 < a < 1 < b$. On account of Definition 4.9 and also by Lemma 6.2, it can be deduced that

$$\int_a^1 f(x) d_{p,q} x = \mathrm{I}^{-}_{\mathrm{p,q}} f(a) = \mathrm{I}^{-}_{\mathrm{p,q}} \mathrm{D}_{\mathrm{p,q}} F(a) = F(1) - F(a).$$

Similarly,

$$\int_1^b f(x) d_{p,q} x = \mathrm{I}^{+}_{\mathrm{p,q}} f(b) = \mathrm{I}^{+}_{\mathrm{p,q}} \mathrm{D}_{\mathrm{p,q}} F(b) = F(b) - F(1).$$

Thus, we conclude finally that

$$\int_a^b f(x) d_{p,q} x = F(b) - F(a).$$

□

Remark 6.5. Assuming that $a > 1$ and setting $b = +\infty$, by the Definition 5.4, we have

$$\int_a^\infty f(x)d_{p,q}x = \lim_{N\to\infty}\sum_{j=1}^{N}\int_{a^{q^{-j+1}p^{j-1}}}^{a^{q^{-j}p^j}} f(x)d_{p,q}x$$
$$= \lim_{N\to\infty}\sum_{j=1}^{N}\left(F(a^{q^{-j}p^j}) - F(a^{q^{-j+1}p^{j-1}})\right)$$
$$= \lim_{N\to\infty}\left(F(a^{q^{-N}p^N}) - F(a)\right),$$

and if $\lim_{x\to\infty} F(x)$ exists, then for $b = +\infty$ (6.1) holds, too.

Corollary 6.6. *Assume that $f(x)$ is continuous at points $x = 0$ and $x = 1$, then we obtain*

$$\int_a^b \mathrm{D}_{p,q} f(x)d_{p,q}x = f(b) - f(a).$$

Corollary 6.7. *If $f(x)$ and $g(x)$ are continuous at $x = 0$ and $x = 1$, then we deduce the following formulas, which are the formulas of ^{p,q}-integration by parts.*

$$\int_a^b f(x^q)d_{p,q}g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x^p)d_{p,q}f(x), \tag{6.2}$$

$$\int_a^b f(x^p)d_{p,q}g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x^q)d_{p,q}f(x). \tag{6.3}$$

Proof. By using the product rule (2.6), we see that

$$\int_a^b \mathrm{D}_{p,q}(fg)(x)d_{p,q}x = \int_a^b \left(g(x^p)\,\mathrm{D}_{p,q} f(x) + f(x^q)\,\mathrm{D}_{p,q}\, g(x)\right) d_{p,q}x.$$

By the previous corollary, it yields

$$f(b)g(b) - f(a)g(a) = \int_a^b f(x^q)\,\mathrm{D}_{p,q}\, g(x)d_{p,q}x + \int_a^b g(x^p)\,\mathrm{D}_{p,q} f(x)d_{p,q}x.$$

Since $\mathrm{D}_{p,q}\, g(x)d_{p,q}x = d_{p,q}g(x)$, it gives (6.2). Similarly, by using the product rule (2.7), one can get (6.3).

□

Conclusion remarks

In this study, we presented a novel form of two-parameter power calculus. Within this calculus framework, we examined both differentiation and its inverse, offering insights into various aspects of these concepts. Additionally, we gave the fundamental theorem and integration by parts formulas that apply to this calculus. The results of this research have the potential to expand upon and enrich the existing literature.

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