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PARTIALLY NULL AND PSEUDO NULL SLANT HELICES OF (K,M)-TYPE IN SEMI EUCLIDEAN SPACE R_2^4

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Abstract This paper studies partially null and pseudo-null curves in semi-Euclidean Space R_2^4. The paper also gives characterizations for such curves to be certain (k, m) type slant helices. Non-existence results have also been proved for these curves to be (k, m) type helix.

1 Introduction

In 2004, Izumiya and Takeuchi [4] introduced the notion of a slant helix which is defined as a curve ξ in R^3 where the principal normal vector makes a constant angle with a fixed vector in R^3. Several geometers have studied slant helices [1, 5, 6, 9] and gave characterizations for being such curves. Because of having many rich properties and applications to the different branches of the sciences, k-type slant helices have been one of the most attractive cases which were studied in [2, 3, 7]. Different varieties of k-type slant helices, k-type partially null and pseudo-null helices etc. were further studied by Ergiiut et al [3] Ahmad T et.al.[2] and E. Nesovic et.al [7]. On the other hand, in 2006, the Frenet equations of partially-null and pseudo-null curves lying fully in the semi-Euclidean space R_2^4 are given in [10].

Later in 2020, the class of (k,m) type slant helices was considered in [8], which presented a study of (k, m) type slant helices for partially null and pseudo null curves in Minkowski space R_1^4. The aim of this paper is to discuss partially-null and pseudo-null curves to be (k, m) type slant helices in semi-Euclidean Space R_2^4 and give the characterization for a partially null and pseudo-null curve to be a certain (k,m) slant helix using a curvature function.

2 Preliminaries

Let us assume that R_2^4 is a 4-dimensional semi-Euclidean space with index 2. It is clear that, if the standard co-ordinate system of R_2^4 is $\{x_1, x_2, x_3, x_4\}$, then the metric can be written as [10].

$$ds^2 = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2. \tag{2.1}$$

The inner product on R_2^4 is denoted by $<,>$. We know that any non-zero vector $X \in R_2^4 - \{0\}$ is called timelike if< 0, spacelike if> 0 and null (lightlike) if$= 0$. If $X = 0$,then it will fall in the category of spacelike vectors. Also, we have

$||X|| = \sqrt{(||)}$.

Here $||X||$ denotes the norm of a vector X.

Two vectors X and Y are said to be orthogonal if $= 0$.

Definition.1 [11] *Any curve $\alpha : I \to R_2^4$ can locally be timelike, spacelike or null, if respectively, all of its velocity vectors $\alpha'(s)$ are timelike, spacelike or null .If $g(\alpha'(s), \alpha'(s)) = 1$ then α a unit speed curve.*

Definition.2 [12, 13] *A partially null curve α is a space like curve if N is space like and B_1 is lightlike. For such curves, the second bi-normal, B_2, is the only lightlike vector orthogonal to T*

and N such that $g(B_1, B_2) = 1$.
The Frenet equations for partially null curve is as follows:[10]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & -k_2 & 0 & -k_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2.2}$$

Here k_1,k_2 and k_3 are first , second and third curvatures of the curve α. We can also say that after a null rotation of the ambient space, the curvature k_3 can be chosen to be 0 and k_1 is determined up to a constant, which means that any partially null curve lies in a three dimensional lightlike subspace orthogonal to B_1.

Definition.3[12, 13] *A spacelike curve α is called a pseudo null curve if $\alpha''(s)$ is a lightlike vector for any s, where the normal vector N= T'. In case N' is lightlike, the curve α lies in the lightlike plane which we omit this trivial case. For the other cases, B_1 is a unit spacelike vector orthogonal to {T, N} and B_2 is the only lightlike vector orthogonal to T and B_1 such that g(N, B_2) =1.*

The Frenet equations are given as follows:[10]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & -k_2 \\ -k_1 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2.3}$$

Here k_1,k_2 and k_3 are first, second and third curvatures of the curve α. In this case the first curvature k_1, can take only 0 and 1 values. As is well known, the curve is a straight line if curvature vanishes. We will focus on the cases $k_1 = 1$ and $k_2, k_3 \neq 0$ [2]

3 (k,m)-type slant helices for partially null curves in semi Euclidean space

First we mention the definition of (k,m) type slant helix.It is clear from 2.2 that $k_3 = 0$. We also suppose that $k_2, k_3 \neq 0$.

Definition.4 [8] *Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be the frame fora curve ξ in R^4_2. Then ξ is known as a slant helix of (k, m) type, if we are able to find a fixed vector $U \neq 0 \in R^4_2$ such that $< \Gamma_k, U >= \alpha$, and $< \Gamma_m, U >= \beta$, where α, β are constants for $1 \leq k \leq 4$ and $1 \leq m \leq 4$.*

We can express U as U = $u_1T + u_2N + u_3B_1 + u_4B_2$, where u_i's are differentiable functions of 't'. For our case of study, we take $\Gamma_1 = T, \Gamma_2 = N, \Gamma_3 = B_1, \Gamma_4 = B_2$.

Theorem 3.1. There does not exist (k, m) type partially null slant helix in R^4_2 except the (2, 4) type which exists if the first and second curvatures satisfy the following differential equation.

$$\beta\frac{k_2}{k_1}k_1' - \alpha k_1^2 - \beta k_2' = 0$$

Proof. We first prove the non existence of (1, 2) type partially null slant helix in R^4_2.
Let ξ be a partially null slant helix of (1, 2) type in R^4_2.
Then, by definition, for any fixed vector U, we have

$$< T, U >= \alpha \quad and \quad < N, U >= \beta \tag{3.1}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.
Differentiating with respect to t, we get

$$< T', U >= 0 \quad and \quad < N', U >= 0 \tag{3.2}$$

Using equation 2.2, we get

$$< N, U >= 0 \quad and \quad < -k_1T + k_2B_1, U >= 0 \tag{3.3}$$

which contradicts our supposition . Hence, there does not exist a partially null slant helix of (1, 2) type in R^4_2.

Non existence of (1, 3), (1, 4), (2, 3) and (3, 4) type partially null slant helix in R^4_2 can be proved similarly.

Now we will show the existence of (2, 4) type slant helix.
Let ξ represent a partially null slant helix of (2, 4) type in R^4_2.Then we have

$$< N, U >= \alpha \quad and \quad < B_2, U >= \beta \tag{3.4}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.
Let U be a fixed vector,then it can be written as

$$U = u_1T + \alpha N + u_3B_1 + \beta B_2 \tag{3.5}$$

Differentiating with respect to t, we get

$$0 = u_1T' + u_1'T + \alpha N' + u_3B_1' + u_3'B_1 + \beta B_2' \tag{3.6}$$

Using equation 2.2 in the above equation, we obtain

$$0 = u_1k_1N + u_1'T + \alpha(-k_1T + k_2B_1) + u_3k_3B_1 + u_3'B_1 + \beta(-k_2N - k_3B_2) \tag{3.7}$$

which on simplification gives

$$u_1' - \alpha k_1 = 0, u_1k_1 - \beta k_2 = 0, \alpha K_2 + u_3k_3 + u_3' = 0, \beta k_3 = 0. \tag{3.8}$$

Now,differentiating second equation of (3.8), we get

$$u_1k_1' + u_1'k_1 - \beta k_2' = 0, \tag{3.9}$$

Solving 3.8 and 3.9, we get

$$\beta\frac{k_2}{k_1}k_1' - \alpha k_1^2 - \beta k_2' = 0, \tag{3.10}$$

Hence proved. □

Corollary 3.2. There does not exist a partially null slant helix of (1, k) type in R^4_2, where k $\in$ {2,3,4}.

Corollary 3.3. ξ represents a partially null slant helix of (2, 4) type in R^4_2 if there exist a fixed vector U given by $U = \beta\frac{k_2}{k_1}T + \alpha N - \alpha\int_0^s k_2B_1 + \beta B_2$

Conclusion. We conclude the existence and non-existence of different (k,m) type partially null slant helices in R^4_2 from the following table.

Existence and non-existence of partially null slant helices	
Type of BNS helix	Existence/Non-existence
(1,2)-type	does not exist
(1,3)-type	does not exist
(1,4)-type	does not exist
(2,3)-type	does not exist
(2,4)-type	exists if $\beta\frac{k_2}{k_1}k_1' - \alpha k_1^2 - \beta k_2' = 0$
(3,4)-type	does not exist

4 (k,m)-type slant helices for pseudo null curves in semi Euclidean space

In this section, we discuss the existence and non existence of various (k, m) type slant helices for Pseudo null curves R_2^4. First we see the case of (1, k) type slant helices and prove the following.From 2.3 we have $k_1 = 1$. We also suppose that $k_2, k_3 \neq 0$.

Theorem 4.1. There does not exist (1, 2) and (1,4) type pseudo null slant helix in R_2^4.

Proof. Let ξ represents a pseudo null slant helix of (1, 2) type in R_2^4.
Then by definition, for any fixed vector U, we have

$$< T, U >= \alpha \quad and \quad < N, U >= \beta \tag{4.1}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.
Differentiating with respect to t and Using equation 2.3, we get

$$< N, U >= 0 \quad and \quad < B_1, U >= 0 \tag{4.2}$$

which contradicts our assumption. Hence there does not exist a pseudo null slant helix of (1, 2) type in R_2^4.
Similarly we can prove the non existence of (1, 4) type pseudo null slant helix in R_2^4. Hence proved. □

Theorem 4.2. ξ represents a pseudo null slant helix of (1, 3) type in R_2^4,if third curvatures satisfies the following relation:

$$\alpha + k_3\beta = 0$$

Proof. Let ξ represents a pseudo null slant helix of (1, 3) type in R_2^4. Let U be a fixed vector,then from definition we have

$$< T, U >= \alpha \quad and \quad < B_1, U >= \beta \tag{4.3}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.
Differentiating with respect to t and using equation 2.3, we get

$$< N, U >= 0 \quad and \quad < -k_3N - k_2B_2, U >= 0 \quad respectively \tag{4.4}$$

which on simplification gives the following

$$< N, U >= 0 \quad and \quad < B_2, U >= 0 \quad respectively \tag{4.5}$$

Differentiating second equation of (4.5), we obtain

$$< T, U > -k_3 < B_1, U >= 0. \tag{4.6}$$

Using equation (4.3) in above relation , we arrived at

$$\alpha + k_3\beta = 0 \tag{4.7}$$

Hence proved. □

Theorem 4.3. ξ represents a pseudo null slant helix of (2, 3) type in R_2^4 if the following curvature differential equations holds:

$$\beta k_3 k_3' - \alpha k_1 k_2 = 0, \beta k_2 k_3^2 - \alpha k_3 k_2' + \alpha k_2 k_3' = 0$$

Proof. Let ξ represents a pseudo null slant helix of (2, 3) type in R_2^4. Let U a fixed vector,then we get

$$< N, U >= \alpha \quad and \quad < B_1, U >= \beta \tag{4.8}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.
then we can write U as

$$U = u_1 T + \alpha N + \beta B_1 + u_4 B_2 \tag{4.9}$$

Differentiating with respect to t, we get

$$0 = u_1 T' + u_1' T + \alpha N' + \beta B_1' + u_4 B_2' + u_4' B_2 \tag{4.10}$$

Using equation 2.3, we get

$$u_1 k_1 N + u_1' T + \alpha k_2 B_1 + \beta(k_3 N - k_2 B_2) + u_4(-k_1 T - k_3 B_1) + u_4' B_2 \tag{4.11}$$

which on simplification gives

$$u_1' - k_1 u_4 = 0, u_1 k_1 + \beta k_3 = 0, \alpha k_2 - u_4 k_3 = 0, -k_2 \beta + u_4' = 0 \tag{4.12}$$

from equations 4.12 we conclude that

$$\beta k_3 k_3' - \alpha k_1 k_2 = 0, \beta k_2 k_3^2 - \alpha k_3 k_2' + \alpha k_2 k_3' = 0 \tag{4.13}$$

Hence proved. □

Theorem 4.4. ξ represents a pseudo null slant helix of (2, 4) type in R_2^4 if the third curvatures satisfies the following equation:

$$\beta t + k_3 \beta \int k_3 dt = C$$

Proof. Let ξ represents a pseudo null slant helix of (2, 4) type in R_2^4.

$$< N, U >= \alpha \quad and \quad < B_2, U >= \beta \tag{4.14}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.
Let U a fixed vector,then we get

$$U = u_1 T + \alpha N + u_3 B_1 + \beta B_2 \tag{4.15}$$

Differentiating with respect to t, we get

$$0 = u_1 T' + u_1' T + \alpha N' + u_3 B_1' + u_3' B_1 + \beta B_2' \tag{4.16}$$

Using equation 2.3, we get

$$0 = u_1 k_1 N + u_1' T + \alpha k_2 B_1 + u_3(k_3 N - k_2 B_2) + u_3' B_1 + \beta(-k_1 T - k_3 B_1) \tag{4.17}$$

which on simplification gives

$$u_1 - \beta = 0, u_1 + u_3 k_3 = 0, \alpha K_2 + u_3' - _3 = 0, u_3 k_2 = 0. \tag{4.18}$$

From fourth relation of the above system of equation we arrive at 2 cases.

Case 1:-When $u_3 = 0$ then relation 1 and 2 of equation 4.18 gives $\beta = 0$, which is a contradiction.

Case 2:- When $k_2 = 0$ then relation 1 and 2 of equation 4.18 gives $\beta t + k_3 \beta \int k_3 dt = C$
Hence proved

Corollary 4.5. ξ represents a pseudo null slant helix of (2, 4) type in R_2^4 if there exist a fixed vector U given by:

$$U = \beta t T + \alpha N + \beta \int_0^s k_3 dt B_1 + \beta B_2$$

Theorem 4.6. ξ represents a pseudo null slant helix of (3, 4) type in R_2^4 if:

$$\beta k_3 k_3' = \alpha k_2$$

Proof. Let ξ represents a pseudo null slant helix of (3, 4) type in R_2^4. Then for any fixed vector U we have

$$< B_1, U >= \alpha \quad and \quad < B_2, U >= \beta \tag{4.19}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are constants.
Differentiating with respect to t and Using equation 2.3, we get

$$< k_2 N - k_2 B_2, U >= 0 \quad and \quad < -k_1 T - k_3 B_1, U >= 0 \quad respectively \tag{4.20}$$

Then on simplification we get

$$\beta k_3 k_3' = \alpha k_2 \tag{4.21}$$

Hence proved. □

Conclusion. We conclude the existence and non-existence for different (k,m) type pseudo null salnt helices in R_2^4 from the following table.

Existence and non-existence of pseudo null slant helices	
Type of BNS helix	Existence/Non-existence
(1,2)-type	does not exist
(1,3)-type	exists if $\alpha + k_3\beta = 0$
(1,4)-type	does not exist
(2,3)-type	exists if $\beta k_3 k_3' - \alpha k_1 k_2 = 0, \beta k_2 k_3^2 - \alpha k_3 k_2' + \alpha k_2 k_3' = 0$
(2,4)-type	exists if $\beta t + k_3 \beta \int k_3 dt = C$
(3,4)-type	exists if $\beta k_3 k_3' = \alpha k_2$

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