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Ordered Weak Idempotent Rings

T. Abera, Y.Yitayew, D. Wasihun and Venkateswarlu Kolluru

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Corresponding Author: Venkateswarlu Kolluru

Abstract. In this paper, we introduce ordering in a commutative weak idempotent ring R with unity. We obtain certain properties on the poset $(R, <)$. We establish that every nonzero finite commutative weak idempotent ring with unity admits an atomic basis. Further, we prove that every nonzero nilpotent element is an atom whenever Ann(N) is a prime ideal of a commutative weak idempotent ring with unity. Also, we prove N_I is an R_B submodule of R and every nonzero element of N_I is an atom. We obtain a relation between a submodule N_I of R and atoms of N. Finally, we establish certain results concerning nilpotent atoms.

1 Introduction

Foster [2], gave the definition of Boolean-like ring (BLR, for short) as a commutative ring with unity R in which $ab(1-a)(1-b) = 0$ for all $a, b \in R$ and $a + a = 0$ for all $a \in R$. A ring $(R, +, \cdot)$ is weak idempotent if it is of characteristic 2 and $a^4 = a^2$ for each $a \in R$. It is clear that every Boolean-like ring is weak idempotent but not conversely. In [6],[7] Venkateswarlu et al have studied the structure and submaximal ideals of a weak idempotent ring (WI-ring, for short). For an element a in a WI-ring R: $a^n = a,\ a^2$ or a^3 for any positive integer n, $a = a^2 + (a^2 + a)$ and if $0 \neq a$ is a nilpotent element, then $a^2 = 0$, see [6, 4, 3]. Also, every completely prime ideal of a WI-ring with unity is maximal. Further, an ideal I of a WI-ring with unity is left completely primary if and only if for any idempotent element $b \in R$, either $b \in I$ or $1 + b \in I$ (See [[6], Theorem 2.5]). Observe that every prime ideal is a completely prime ideal.

This paper is a continuation of the work on commutative WI-rings (cWI-ring, for short). In section 2, we introduce the ordering on a cWI-ring with unity and obtain the natural properties of the ordering. We also prove that any cyclic module over the set of Boolean elements of a cWI-ring with unity is a Boolean lattice. In the last section, we obtain that Ann(N) is a prime ideal of a cWI-ring R with unity if and only if every nonzero nilpotent element of R is an atom. Further, we establish that every finite nonzero cWI-ring R with unity admits an atomic basis.

In the entire of this paper R, R_B and N will denote a cWI-ring with unity, the set of all idempotent and the set of all nilpotent elements of R respectively.

2 Partial ordering

We now begin with the following Lemma.

Lemma 2.1. *For $x,\ y \in R$, define $y < x$ if and only if there exists $b \in R_B$ such that $bx = y$. Then $(R, <)$ is a poset. If the relation is restricted to the set of all idempotent elements, the partial ordering " $<$ " coincides with the natural partial ordering over the Boolean ring.*

Proof. Clearly $(R, <)$ is a poset. Let $a, b \in R_B$. If $a < b$, then there exists $x \in R_B$ such that $bx = a$. So, $ab = b^2x = bx = a$. If $ab = a$, then $a < b$ since $b \in R_B$. Therefore, the partial ordering " $<$ " is restricted to R_B coincides with the natural partial ordering over the Boolean ring. □

Example 2.2. Let R be a cWI-ring with unity. Then, the direct product $R \times R$ is also a cWI-ring with unity $(1, 1)$. Clearly, $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$ are idempotent elements of $R \times R$. Consider $(1, 0)(a, b) = (a, 0)$ and $(0, 1)(a, b) = (0, b)$ for $a, b \in R$. Thus, $(a, 0) < (a, b)$ and $(0, b) < (a, b)$.

Remark 2.3. R can be regarded as $R_B$module with the ordering defined in Lemma 2.1 above.

Definition 2.4. $(R, +)$ is a Boolean group. As a module over $\{0, 1\}$, R has a basis. If every element of a basis of R is an atom, then R is said to have an atomic basis.

Definition 2.5. Let S be an additive subgroup of R. If there exists $a \in R$ such that $S = R_B a$, then we said that S is a cyclic R_B submodule of R.

Theorem 2.6. *Any cyclic R_B submodule of a cWI-ring with unity is a Boolean lattice under the ordering defined in Lemma 2.1. Furthermore, the map $b \mapsto bx$ of $R_B \to R_B x$ is an isotone module epimorphism.*

Proof. Let $R_B x = \{bx : b \in R_B\}$ be a cyclic R_B submodule of a cWI-ring with unity. Let $R_B x$ be partially ordered as in Lemma 2.1. Let $b_1 x,\ b_2 x \in R_B x$. Then $b_1(b_1 + b_2 + b_1 b_2)x = b_1 x$ and $b_2(b_1 + b_2 + b_1 b_2)x = b_2 x$. Hence, $b_1 x < (b_1 + b_2 + b_1 b_2)x$ and $b_2 x < (b_1 + b_2 + b_1 b_2)x$. Suppose $bx \in R_B x$ be an upper bound of $b_1 x$ and $b_2 x$, that is, $b_1 x < bx$ and $b_2 x < bx$. There exist $c, d \in R_B$ such that $cbx = b_1 x$ and $dbx = b_2 x$. Since $(c + d + cb_2)bx = (b_1 + b_2 + b_1 b_2)x$, $(b_1 + b_2 + b_1 b_2)x < bx$. Therefore, $(b_1 + b_2 + b_1 b_2)x$ is the least upper bound of $b_1 x$ and $b_2 x$. Obviously, $b_1 b_2 x < b_1 x$ and $b_1 b_2 x < b_2 x$. Suppose $bx \in R_B x$ be a lower bound of $b_1 x$ and $b_2 x$, that is, $bx < b_1 x$ and $bx < b_2 x$. So, there exist $c, d \in R_B$ such that $bx = cb_1 x$ and $bx = db_2 x$. Thus, $cb_1 x = db_2 x$ and hence $cdb_1 b_2 x = db_2 x = bx$. Thus, $bx < b_1 b_2 x$ and hence $b_1 b_2 x$ is the greatest lower bound of $b_1 x$ and $b_2 x$. Therefore, $(R_B x, <)$ is a lattice. Define the map $\phi : R_B \to R_B x$ by $\phi(b) = bx$ for each $b \in R_B$. ϕ is a module epimorphism. Let $b_1,\ b_2 \in R_B$ and $b_1 < b_2$. Then $b_1 b_2 = b_1$. Thus, $\phi(b_1) = b_1 x = b_1 b_2 x = b_1 \phi(b_2)$. Therefore, $\phi(b_1) < \phi(b_2)$ and hence ϕ is an isotone. □

Definition 2.7. A nonzero element $m \in (R, <)$ is called an atom of R if for every $x \in R$, $x < m$ implies $x = m$ or $x = 0$. Furthermore, R is called atomic if, for every nonzero element x of R, there exists an atom $m \in R$ such that $m < x$.

Remark 2.8. For any atom $x \in R$ and $b \in R_B$, $bx = bx$ implies $bx < x$. Hence, $bx = 0$ or $bx = x$.

Definition 2.9. An element $m \in R$ is called simple if $m \neq 0$ and there exists an atom $b \in R_B$ such that $bm = m$. An element $m \in R$ is called nil if $bm = 0$ for all atoms $b \in R_B$.

Lemma 2.10. *Let R has idempotent atoms. Then, the following holds.*

i. Every simple element of R is an atom of R.

ii. Every atom of R is either nil or simple.

Proof. i. Let $r \in R$ be a simple element of R. Then there exists an atom $b \in R_B$ such that $br = r$. Suppose $x < r$, where $x \in R$. We get $b_1 r = x$ for some $b_1 \in R_B$ and hence $bx = b_1 br = b_1 r = x$. Thus, $x = b_1 br = br = r$ or $x = rb_1 b = 0$ since $b_1 b = b$ or 0 for the atom b. Therefore, r is an atom of R.

ii. Let $r \in R$ be an atom of R. $br < r$ for any $b \in R_B$. Since r is an atom of R, $br = r$ or $br = 0$ for any $b \in R_B$. If $br = 0$ for all $b \in R_B$, then r is a nil element of R. Otherwise, there exists some $b \in R_B$ such that $br = r$. Hence, r is either nil or a simple element of R. □

Theorem 2.11. *The following properties hold in R.*

(i) $0 < a$ *for all* $a \in R$.

(ii) $b < a$ *implies that* $bc < ac$ *for any* $a,\ b,\ c \in R$.

(iii) *For any* $a \in R$, $a^3 < a$ *and* $a_N(1 + a_B) < a$.

(iv) $a,\ b \in R,\ b < a$ *implies*

a. $ab = b_B = a_B b_B$.

b. $a_B b_N = a_N b_B = b_B b_N$.

c. $ab_B = a_B b = bb_B$.

d. $xa = xb = b$ *for some* $x \in R_B$.

e. $b_B < a_B$ *and* $b_N < a_N$.

(v) $a < 1$ *if and only if* $a \in R_B$.

(vi) $a \in R,\ b \in R_B,\ b < a$ *if and only if* $ab = b$.

(vii) *Let* a *be a unit in* $R,\ b \in R$. *Then* $b < a$ *if and only if* $ab = b_B$.

(viii) $a \in R_B,\ b \in R,\ b < a$ *implies* $b \in R_B$.

(ix) $a \in N,\ b \in R,\ b < a$ *implies* $b \in N$.

(x) *Units in* R *are maximal elements in* $(R, <)$.

Proof. (i) Since $a0 = 0$ for all $a \in R,\ 0 < a$ for all $a \in R$.

(ii) Suppose $b < a$. Then $ax = b$ for some $x \in R_B$. Thus, $acx = bc$. That is, $bc < ac$ for any $c \in R$.

(iii) For any $a \in R$, $aa^2 = a^3$ and $a^2 \in R_B$. Hence, $a^3 < a$. We have $a(1 + a^2) = a(1 + a^2)$ and $(1 + a^2) \in R_B$ that gives $a(1 + a^2) < a$. Thus, $a + a^3 + a^2 + a^2 < a$ and hence $a_N(1 + a_B) < a$.

(iv) $a,\ b \in R,\ b < a$ implies

a. $ax = b$ for some $x \in R_B$. This gives us $a^2x = b^2 = b_B$. Thus, $ab = b_B$ and hence $b_B = a_B b_B$.

b. $a_B b_N = a^2(b + b^2) = a^2b + a^2b^2 = ab^2 + a^2b^2 = a_N b_B$ and $a_N b_B = (a + a^2)b_B = ab^2 + a^2b^2 = b^2b + b_B^2 = b^2(b + b^2) = b_B b_N$.

c. $ab_B = ab^2 = (ab)b = b^2b = bb_B$ and $a_B b = a^2b = ab^2$.

d. $xa = b$ implies that $x^2a = xa = xb = b$ for some $x \in R_B$.

e. $b < a$ implies $ax = b$ for some $x \in R_B$. Since $a^2x = ab = b^2,\ b_B < a_B$. The result $ax = b$ and $a^2x = b^2$ implies $ax + a^2x = b + b^2$. Hence, $b_N < a_N$.

(v) If $a < 1$, then $1x = a$ for some $x \in R_B$. Hence, $a \in R_B$. If $a \in R_B$, then $1.a = a$ which implies $a < 1$.

(vi) Let $a \in R$ and $b \in R_B$. If $b < a$, then $ax = b$ for some $x \in R_B$. Thus, $a^2x = b$ and hence $ab = b$. If $ab = b$, then $b < a$ as $b \in R_B$.

(vii) Let a be a unit in $R,\ b \in R$. If $b < a$, then $ab = b_B$ by $4(a)$. If $ab = b_B$, then $b = ab_B$ and hence $b < a$.

(viii) Let $a \in R_B,\ b \in R$. If $b < a$, then $ax = b$ and $b^2 = (ax)^2 = ax = b$ which implies $b \in R_B$.

(ix) Let $a \in N,\ b \in R$. If $b < a$, then $ax = b$ and $b^2 = (ax)^2 = a^2x^2 = 0$ which implies $b \in N$.

(x) Let $a \in R$ be a unit and suppose $a < b$. Then, $bx = a$ for some $x \in R_B$ and it gives us $b^2x = 1$. Thus, x is a unit. So, $x = 1$ since it is both unit and idempotent. Hence, $a = b$. That is, a is a maximal element in $(R, <)$.

Note that from (8) and (9) above, we observe that any nonzero idempotent and nonzero nilpotent elements are incomparable. □

Remark 2.12. The unit element in R is maximal in the poset $(R, <)$. However, the converse fails. Consider the following example.

+	0	1	p	1+p
0	0	1	p	1+p
1	1	0	1+p	p
p	p	1+p	0	1
1+p	1+p	p	1	0

·	0	1	p	1+p
0	0	0	0	0
1	0	1	p	1+p
p	0	p	0	p
1+p	0	1+p	p	1

Example 2.13. Let $H_4 = \{0, 1, p, 1+p\}$ where $+$ and $\cdot$ are defined by the following tables and $B = \{0, a, b, a+b\}$ be a Boolean group of 4 elements. Define a unitary H_4-module structure on B by the multiplication generated from the following: $pa = a$ and $pb = 0$. Consider the ring $R = (H_4 \times B, +, .)$ where the operations are defined as: $(a_1, b_1)+(a_2, b_2) = (a_1+a_2, b_1+b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, a_2b_1 + a_1b_2)$. Then, R is a cWI-ring with unity. The element $(p, a+b)$ is a maximal element with the above order relation but it is not a unit element.

Theorem 2.14. *For $a, b, x \in R$, $b < a$ and $b + x < a + x$ implies $ax = bx$.*

Proof. Suppose $b < a$ and $b + x < a + x$. By Theorem 2.11(4a), $ab = b_B$ and $(a + x)(b + x) = (b + x)_B = b_B + x_B$. Simplifying the second equation gives us $ab + x^2 + ax + bx = b_B + x_B$ which implies $ax = bx$. □

But $b < a,\ ax = bx$ may not imply $b + x < a + x$ in general.
In Example 2.13, $(0, 0) < (p, 0)$ and $(0, 0)(p, 0) = (p, 0)(p, 0)$ but $(p, 0) \not< (0, 0)$.

3 Atoms and Atomic basis

Definition 3.1. An atom is a nonzero element $m \in R$ such that if for every $x \in R$, $x < m$ implies $x = m$ or $x = 0$.

Theorem 3.2. *Let N be the nil-radical of R. $Ann(N)$ is a prime ideal of R if and only if every nonzero nilpotent element is an atom of R.*

Proof. Suppose $Ann(N)$ is a prime ideal of R. Then, for any $b \in R_B,\ b \in Ann(N)$ or $1 + b \in Ann(N)$. That is, $bn = 0$ or $(1 + b)n = 0$ for any $n \in N$. Thus, $bn = n$ or 0 for any $n \in N$. Hence, every nonzero nilpotent element is an atom of R. Conversely, suppose every nonzero nilpotent element is an atom of R. Assume $Ann(N)$ is not prime. Then, there exists $b \in R_B$ such that $b \notin Ann(N)$ and $1 + b \notin Ann(N)$. That is, there exists nonzero nilpotent elements $n_1,\ n_2$ such that $bn_1 \neq 0$ and $(1+b)n_2 \neq 0$. Since $n_1,\ n_2$ are atoms, $bn_1 = n_1$ and $(1+b)n_2 = n_2$. Obviously $b(n_1 + n_2) = n_1$. Thus, $n_1 + n_2$ is not an atom which is a contradiction. Therefore, $Ann(N)$ is a prime ideal of R. □

Theorem 3.3. *Every finite nonzero cWI-ring with unity has an atomic basis.*

Proof. Let R be a finite nonzero cWI-ring with unity. Consider the partial ordering given by Lemma 2.1 over R. Let $0 \neq r \in R$ and consider the Boolean lattice $(R_Br, <)$. Assume br is an atom of R_Br. Let $x \in R$ and $x < br$. Then, $b_1br = x$ for some $b \in R_B$ and hence $x \in R_Br$. Since br is an atom of R_Br, $x = br$ or $x = 0$. Thus, br is an atom of R. Since R_Br is a finite Boolean lattice it is atomic. Hence, r is the join of all atoms of R_Br. Let b_1r and b_2r be two distinct atoms of R_Br. We know that $b_1b_2r < b_1r$ and $b_1b_2r < b_2r$ and hence $b_1b_2r = 0$ because $b_1r \neq b_2r$. Thus, $b_1r \vee b_2r = (b_1 \vee b_2)r = (b_1 + b_2)r$. Let $b_1r, b_2r, b_3r, ..., b_kr$ be distinct atoms of R_Br. Then, $b_1r \vee b_2r \vee b_3r \vee ... \vee b_kr = (b_1 + b_2 + b_3 + ... + b_k)r$. Therefore, $r = \sum_{i=1}^{n} b_ir$, where $\{b_ir\}_{1 \leq i \leq n}$ are all atoms of R_Br. That is, r is the sum of atoms of R. Let A be the set of all atoms of R. A is non-empty since R is finite. As we proved above, every nonzero element of R is the sum of elements of A. Thus, A generates R as a vector space over $\{0, 1\}$. That is, R has a basis contained in A. Hence, R has an atomic basis. □

Theorem 3.4. *Let n be a nonzero nilpotent element of R. n is an atom of R if and only if $Ann(n) \cap R_B$ is a prime ideal of R_B, where $Ann(n) = \{x \in R : \ xn = 0\}$.*

Proof. Let n be a nonzero nilpotent element of R. Suppose n is an atom of R. For every $b \in R_B$, $bn = 0$ or $bn = n$. That is, $b \in Ann(n)$ or $1 + b \in Ann(n)$. Thus, $b \in Ann(n) \cap R_B$ or $1 + b \in Ann(n) \cap R_B$. Let $a,\ b \in R_B$ and $ab \in Ann(n) \cap R_B$. Assume $a \notin Ann(n) \cap R_B$. Then, $1 + a \in Ann(n) \cap R_B$. We have that $Ann(n) \cap R_B$ is an ideal of R_B and $b \in R_B$ which implies $b(1 + a) \in Ann(n) \cap R_B$ and hence $b + ab \in Ann(n) \cap R_B$. Thus, $b \in Ann(n) \cap R_B$. Hence $Ann(n) \cap R_B$ is a prime ideal of R_B. Conversely, suppose $Ann(n) \cap R_B$ is a prime ideal of R_B. Let $x < n$. Then, $bn = x$ for some $b \in R_B$. If $bn = x \neq 0$, then $b \notin Ann(n) \cap R_B$. It implies that $1 + b \in Ann(n) \cap R_B$ and $Ann(n) \cap R_B$ is a prime ideal of R_B. Thus, $(1 + b)n = 0$ which implies $bn = n$. Hence, n is an atom of R. □

Theorem 3.5. *Let n be a nonzero nilpotent element of R. n is an atom of R if and only if $Ann(n)$ is a prime ideal of R.*

Proof. Suppose n is an atom of R. For every $b \in R_B,\ bn = 0$ or $bn = n$. That is, $b \in Ann(n)$ or $1 + b \in Ann(n)$. Thus, $Ann(n)$ is a prime ideal of R. Conversely, suppose $Ann(n)$ is a prime ideal of R. Let $x < n$. Then, $bn = x$ for some $b \in R_B$. If $bn = x \neq 0$, then $b \notin Ann(n)$ which implies $1 + b \in Ann(n)$. Thus, $(1 + b)n = 0$ and it implies $bn = n$. Hence, n is an atom of R. □

Theorem 3.6. *For every nilpotent atom n of R, if $n < x$ for some $x \in R$, then $xn = 0$.*

Proof. Let n be a nilpotent atom of R and $n < x$ for $x \in R$. By Theorem 2.11 4(a), $xn = n_B = 0$. □

Theorem 3.7. *Let I be a prime ideal of R_B. Define $N_I = \{n \in N : bn = 0 \text{ for all } b \in I\}$. Then:*

i. N_I is an R_B submodule of R.

ii. Every nonzero element of N_I is an atom.

iii. For every $0 \neq n \in N_I,\ Ann(n) \cap R_B = I$.

Proof. i. Obvious.

ii. For every $b \in R_B$, $b \in I$ or $1 + b \in I$ since I is a prime ideal of R_B. Thus, $bn = 0$ or $bn = n$ for every $0 \neq n \in N_I$ and every $b \in R_B$. Hence, every nonzero element of N_I is an atom.

iii. For every $0 \neq n \in N_I$, $x \in I$ implies that $xn = 0$. Thus, $x \in Ann(n)$ and hence $x \in Ann(n) \cap R_B$. Thus, $I \subseteq Ann(n) \cap R_B$. We know that every prime ideal is a maximal ideal in a Boolean ring and I and $Ann(n) \cap R_B$ are prime ideals of R_B. Hence, $Ann(n) \cap R_B = I$ for every $0 \neq n \in N_I$.

□

Theorem 3.8. *Let I be a prime ideal of R_B. There is no nilpotent atom if and only if $N_I = \{0\}$ for every prime ideal I of R_B.*

Proof. Suppose $N_I \neq \{0\}$ for some prime ideal I of R_B. By Theorem 3.7(ii), N has a nonzero atom. Hence, N has no atom if $N_I = \{0\}$ for every prime ideal I of R_B. Conversely, suppose $N_I = \{0\}$ for every prime ideal I of R_B. Assume that $0 \neq n \in N$ is an atom. Then, $Ann(n) \cap R_B$ is a prime ideal of R_B by Theorem 3.4. Let $I = Ann(n) \cap R_B$. Then, $0 \neq n \in N_I$ which is a contradiction. Hence, N has no atom. □

Corollary 3.9. *Let I be a prime ideal of R_B and $n_1, n_2 \in N_I$, where $n_1 \neq n_2$. Then $n_1, n_2, n_1 + n_2$ are all atoms of R and Ann(n_1) $\cap R_B$=Ann(n_2) $\cap R_B$=Ann($n_1 + n_2$) $\cap R_B = I$.*

Corollary 3.10. *If I and J are two distinct prime ideals of R_B, then $N_I \cap N_J = \{0\}$.*

Proof. Suppose I and J are two distinct prime ideals of R_B. Let $0 \neq n \in N_I \cap N_J$. Then $I = Ann(n) \cap R_B = J$ which is a contradiction. Therefore, $N_I \cap N_J = \{0\}$. □

Lemma 3.11. *If b is an idempotent atom of R and $n \in N$ where $bn \neq 0$, then bn is a nilpotent atom.*

Proof. Assume that b is an idempotent atom of R and $n \in N$ where $bn \neq 0$. bn is a nilpotent element. Let $x < bn$. Then, $cbn = x$ for some $c \in R_B$. We know that $cb = 0$ or $cb = b$ since b is an atom. If $x \neq 0$, then $x = cbn = bn$. Hence, bn is a nilpotent atom. □

Lemma 3.12. *If b_1 and b_2 are two distinct idempotent atoms of R, then $b_1b_2 = 0$.*

Proof. Assume that $b_1b_2 \neq 0$. Then, $b_1b_2 = b_1$ as b_1 is an atom and $b_1b_2 = b_2$ as b_2 is an atom. Thus, $b_1 = b_2$ which is a contradiction. Therefore, $b_1b_2 = 0$. □

Theorem 3.13. *Let N_I have only one nonzero element for each prime ideal I of R_B. Then, Ann(N)$\cap R_B = \{0\}$. Additionally, N is finite if and only if R_B is so.*

Proof. Let $b \in Ann(N) \cap R_B$. Then, $bn = 0$ for every $n \in N$. Thus, $b \in Ann(n) \cap R_B$ for every $n \in N$. Let n_I be the only nonzero element of N_I. Then, $b \in Ann(n_I) \cap R_B = I$ for each prime ideal of R_B. Therefore, $b = 0$ since the intersection of all prime ideals of a Boolean ring is $\{0\}$. Hence, $Ann(N) \cap R_B = \{0\}$. Suppose R_B is infinite. Then, the number of distinct prime ideals of R_B is infinite. Define a map $\phi : Spec(R_B) \to N_A$ by $\phi(I) = n_I$, where n_I is the nonzero element in N_I, N_A is the set of all atoms of N and Spec(R_B) is the set of all prime ideals of R_B. ϕ is a one-to-one mapping by Corollary 3.10. Thus, N_A is infinite and hence N is so. Conversely, suppose that R_B is finite and n_1, n_2 are two distinct nilpotent atoms. Then, $Ann(n_1) \cap R_B = I$ and $Ann(n_2) \cap R_B = J$ are prime ideals of R_B. Since N_I has only one nonzero element, $I \neq J$. Define a map $\theta : N_A \to Spec(R_B)$ by $\theta(n_i) = Ann(n_i) \cap R_B$ for every $n_i \in N_A$. We obtain that θ is a one-to-one mapping. Thus, N_A is finite as Spec(R_B) is so. Since R_B is finite, there exists distinct atoms $b_1, b_2, ..., b_k$ in R_B such that $\bigvee_{i=1}^{k} b_i = 1$. Consider the set $\{b_in\}_{n\in N, b_in\neq 0, 1\leq i\leq k}$. It is a collection of distinct atoms and hence it is finite. For any $n \in N$, $n = (b_1 + b_2 + ... + b_k)n$. Hence, $\{b_in\}_{n\in N, b_in\neq 0, 1\leq i\leq k}$ generates N as a vector space over $\{0, 1\}$. That means N has a finite atomic base. Therefore, N is finite. □

Theorem 3.14. *For any two distinct nonzero elements n_1 and n_2 of N, $n_2 < n_1$ implies $n_1 + n_2 < n_1$ but n_2 and $n_1 + n_2$ are incomparable.*

Proof. Suppose $n_2 < n_1$. Then, $bn_1 = n_2$ for some $b \in R_B$ which gives $(b + 1)n_1 = n_1 + n_2$. Hence, $n_1 + n_2 < n_1$. Assume that n_2, $n_1 + n_2$ are comparable. If $n_2 < n_1 + n_2$, then $b(n_1 + n_2) = n_2$ for some $b \in R_B$. It gives us $(b + 1)(n_1 + n_2) = n_1$ and hence $n_1 < n_1 + n_2$. Thus, $n_1 + n_2 = n_1$ which is a contradiction. If $n_1 + n_2 < n_2$, then $bn_2 = n_1 + n_2$ for some $b \in R_B$. It gives us $(b + 1)n_2 = n_1$ and hence $n_1 < n_2$. Thus, $n_1 = n_2$ which is a contradiction. Therefore, n_2 and $n_1 + n_2$ are incomparable. □

Theorem 3.15. *Let n_1, n_2 be two distinct atoms of N. $n_1 + n_2$ is an atom if and only if $bn_1 = 0 \Leftrightarrow bn_2 = 0$ for all $b \in R_B$.*

Proof. Let n_1 and n_2 be two distinct atoms of N. Suppose $n_1 + n_2$ is an atom. Assume $bn_1 = 0$. Then, $b(n_1 + n_2) = bn_1 + bn_2 = bn_2$. We have that $bn_2 = 0$ or $bn_2 = n_2$ and $b(n_1 + n_2) = 0$ or $b(n_1 + n_2) = n_1 + n_2$ since n_2 and $n_1 + n_2$ are atoms. If $bn_2 \neq 0$, then $n_1 + n_2 = n_2$ which is a contradiction. Hence, $bn_2 = 0$. Similarly, we can show $bn_2 = 0 \Rightarrow bn_1 = 0$ for all $b \in R_B$. Conversely, suppose that $bn_1 = 0 \Leftrightarrow bn_2 = 0$ for all $b \in R_B$. Let $b \in R_B$. Then, $b(n_1 + n_2) = bn_1 + bn_2 = n_1 + n_2$ or 0 since $bn_1 = 0 \Leftrightarrow bn_2 = 0$ and n_1, n_2 are atoms. Hence, $n_1 + n_2$ is an atom. □

Corollary 3.16. *Let n_1, n_2 be two distinct atoms of N. $n_1 + n_2$ is an atom if and only if $Ann(n_1) \cap R_B = Ann(n_2) \cap R_B = Ann(n_1 + n_2) \cap R_B$.*

Theorem 3.17. *Any two distinct atoms n_1, n_2 of N have an upper bound in R if and only if the sum of the two distinct atoms n_1, n_2 of N is not an atom.*

Proof. Let n_1 and n_2 be two distinct atoms of N and have an upper bound in R. Assume that $n_1 + n_2$ is an atom of N. Let $x \in R$ be the upper bound of n_1 and n_2. By Theorem 2.11(9), $x \in N$. So, $n_1 < x$ and $n_2 < x$ implies $b_1x = n_1$ and $b_2x = n_2$ for some b_1, $b_2 \in R_B$. Thus, $b_1x = b_1n_1 = n_1$ and $b_2x = b_2n_2 = n_2$. By theorem 3.15, $b_2n_2 = n_2 \neq 0$ implies that $b_2n_1 = n_1 \neq 0$. Now, we get $b_2b_1x = b_2n_1 = n_1$ and $b_2b_1x = b_1n_2 = n_2$ and hence

$n_1 = n_2$ which is a contradiction. Therefore, $n_1 + n_2$ is not an atom. Conversely, suppose that $n_1 + n_2$ is not an atom of N. By Theorem 3.15, $bn_1 = 0$ but $bn_2 \neq 0$ for some $b \in R_B$. Then, $b(n_1 + n_2) = n_2$ which implies $n_2 < n_1 + n_2$ and $(1 + b)(n_1 + n_2) = n_1$ which implies $n_1 < n_1 + n_2$. Therefore, n_1 and n_2 have an upper bound. □

Theorem 3.18. *For any two distinct atoms $n_1,\ n_2$ of N, if $n_1 + n_2$ is not an atom, then it is the least upper bound of n_1 and n_2.*

Proof. Let $n_1,\ n_2$ of N be any two distinct atoms. Suppose that $n_1 + n_2$ is not an atom. Then, $n_1 + n_2$ is an upper bound of n_1 and n_2 by theorem 3.17. Let $n_1 < x < n_1 + n_2$ and $n_2 < x < n_1 + n_2$. If $n_1 = x$, then $n_2 < n_1$ which is a contradiction. Thus, $n_1 \neq x$ and similarly $n_2 \neq x$. We have $n_1 = b_1x$ and $n_2 = b_2x$ for some $b_1, b_2 \in R_B$. Then, $b_1x + b_2x = n_1 + n_2$ which implies $(b_1 + b_2)x = n_1 + n_2$ and hence $n_1 + n_2 < x$. Thus, $x = n_1 + n_2$. Therefore, $n_1 + n_2$ is the least upper bound of n_1 and n_2. □

Theorem 3.19. *Let $n_1,\ n_2,\ n_3, ..., n_k$ be a finite number of distinct atoms of N. If any two elements of $\{n_1,\ n_2,\ n_3, ..., n_k\}$ have an upper bound, then $n_1 + n_2 + n_3 + ... + n_k$ is the least upper bound of $\{n_1,\ n_2,\ n_3, ..., n_k\}$.*

Proof. Suppose any two elements of $\{n_1,\ n_2,\ n_3, ..., n_k\}$ have an upper bound. Then, $Ann(n_i) \cap R_B \neq Ann(n_j) \cap R_B$ for each $i \neq j$ and $i, j \in \{1, 2, ..., k\}$. Thus, there exists $b_i \in R_B$ such that $b_i \in Ann(n_i)$ but $b_i \notin Ann(n_j)$ for $i \neq j$ and $i, j \in \{1, 2, ..., k\}$. Consider, $b_2b_3...b_k \in Ann(n_j) \cap R_B$ for all $j \in \{2, ..., k\}$. $b_2b_3...b_k \notin Ann(n_1) \cap R_B$ since $Ann(n_1) \cap R_B$ is a prime ideal of R_B. Hence, $b_2b_3...b_kn_1 = n_1$ as n_1 is an atom. We obtain that $b_2b_3...b_k(n_1 + n_2 + n_3 + ... + n_k) = n_1$ and hence $n_1 < n_1 + n_2 + n_3 + ... + n_k$. Similarly, we can show that $n_1 + n_2 + n_3 + ... + n_k$ is an upper bound of n_i for all $i \in \{2, ..., k\}$. Let n_o be an upper bound of $\{n_1, n_2, n_3, ..., n_k\}$. Then, for each $i = 1, 2, 3, ..., k$, there exists $c_i \in R_B$ such that $c_in_o = n_i$. Thus, $(\sum_{i=1}^{k} c_i)n_o = \sum_{i=1}^{k} n_i$ and hence $n_1 + n_2 + n_3 + ... + n_k < n_o$. Therefore, $n_1 + n_2 + n_3 + ... + n_k$ is the least upper bound of $\{n_1,\ n_2,\ n_3, ..., n_k\}$. □

4 Conclusion.

This paper intends to obtain ordering in a commutative weak idempotent ring R with unity. Also, certain properties on the poset $(R, <)$ have been discussed. Further, we establish that every nonzero finite commutative weak idempotent ring with unity admits an atomic basis. Consequently, the outcomes of this work are noteworthy and stimulating to advance its further study in the future.

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Author information

T. Abera, Department of Mathematics, Addis Ababa University, Ethiopia.
E-mail: tameabe2@yahoo.com

Y.Yitayew, Department of Mathematics, Addis Ababa University, Ethiopia.
E-mail: yibeltal.yitayew@aau.edu.et

D. Wasihun, Division of Mathematics, Physics and Statistics, Addis Ababa Science and Technology University, Artificial Intelligence and Robotics Center of Excellence, Ethiopia.
E-mail: dereje.wasihun@aastu.edu.et

Venkateswarlu Kolluru, Department of Computer Science and Systems Engineering, College of Engineering, Andhra University, India.
E-mail: drkvenkateswarlu@gmail.com

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